MA4J0 Advanced Real Analysis

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Contents

These notes are based on the 2013 `MA4J0 Advanced Real Analysis` course, taught by Jose Rodrigo, typeset by Matthew Egginton.
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Good books for the course are Folland-Real Analysis for Distributions, L^p and functional Analysis. Grafakos -Classical Fourier Analysis, and Stein- Singular Integrals.

1 Fourier Transform

We have a typical setting of $\mathbb{R}$ with the Lebesgue measure, written dx.

Definition 1.1 *We define, for $f \in L^1(\mathbb{R}^n)$ the* **Fourier Transform** *to be*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx$$

We now motivate this definition, from the Fourier series. We restrict ourselves to $\mathbb{R}$ for simplicity. Suppose that $f(x)$ is periodic of period L. We take $\{K_n e^{c_n inx}\}$ and that for specific K_n, c_n that they are orthonormal. With period L, we have $e_n = \frac{1}{\sqrt{L}} e^{\frac{2\pi i}{L} nx}$ and so $(e_n, e_m) = \delta_{mn}$. Then given any $f \in L^2([\frac{-L}{2}, \frac{L}{2}])$ define

$$\hat{f}(n) = (f, e_n) = \frac{1}{\sqrt{L}} \int_{\frac{-L}{2}}^{\frac{L}{2}} f(x) \frac{1}{\sqrt{L}} e^{-\frac{2\pi i}{L} nx} dx$$

Since $\{e_n\}$ is an orthonormal basis, we have $f(x) = \sum \hat{f}(n) e_n$. We now want to send L to ∞. There is a well known formula called Plancherel:

$$\int |f|^2 dx = \sum |\hat{f}(n)|^2$$

We build a step function as follows:

$$g_L(\xi) = \sqrt{L}\hat{f}(n) \text{ if } \xi \in \left[\frac{2\pi n}{L}, \frac{2\pi(n+1)}{L}\right)$$

and then $\int |f|^2 dx = \sum |\hat{f}(n)|^2 = \frac{1}{2\pi} \int |g_L(\xi)|^2 d\xi$ and the limit of g_L gives what we want. Explicitly,

$$\lim_{L\to\infty} g_L(\xi) = \hat{f}(\xi) = \hat{f}(\xi)$$

and

$$g_L(\xi) = \sqrt{L}\hat{f}(n) = \frac{\sqrt{L}}{\sqrt{L}} \int_{\frac{-L}{2}}^{\frac{L}{2}} f(x) \frac{1}{\sqrt{L}} e^{-\frac{2\pi i}{L} nx} dx$$

Now $\frac{2\pi n}{L}$ is the left endpoint of $\left[\frac{2\pi n}{L}, \frac{2\pi(n+1)}{L}\right)$ and we think of $\xi = \frac{2\pi n}{L}$ and take limits when $L \to \infty$ keeping ξ "fixed".

1.1 Properties of the Fourier Transform

We think of $\wedge : L^1 \to ?$ but for sure ? is not L^1, but what is it?.

Lemma 1.2 *Let $f, g, h \in L^1$ and $\alpha, \beta \in \mathbb{R}$. Then*

1. *$\wedge$ is a linear operator, i.e. $(\widehat{\alpha f + \beta g})(\xi) = \alpha\hat{f}(\xi) + \beta\hat{g}(\xi)$.*
2. *$||\hat{f}||_{L^\infty} \leq ||f||_{L^1}$*
3. *$f \in L^1$ then $\hat{f}(\xi)$ is continuous. Moreover, $\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0$. This is called Riemann-Lebesgue.*

4. *Convolution. If* $f, g \in L^1$ *define* $f \star g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$ *and then* $\widehat{(f \star g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$
5. *Define* $\tau_h f(x) = f(x+h)$. *Then* $\widehat{\tau_h f}(\xi) = \hat{f}(\xi)e^{2\pi i x \cdot h}$ *and* $\widehat{f(x)e^{2\pi i x\cdot h}} = \hat{f}(\xi - h)$
6. *If* $\theta \in SO(n)$ *,i.e.* θ *a rotation matrix, then* $\widehat{f(\theta x)}(\xi) = \hat{f}(\theta\xi)$
7. *If we define* $g(x) = \frac{1}{\lambda^n} f(\frac{x}{\lambda})$ *for* $\lambda > 0$ *then we have* $\hat{g}(\xi) = \hat{f}(\lambda\xi)$.
8. *If* $f \in C^1, f \in L^1, \frac{\partial f}{\partial x_j} \in L^1$ *then* $\widehat{\frac{\partial f}{\partial x_j}}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$
9. $(\widehat{-2\pi i x_j f(x)})(\xi) = \frac{\partial}{\partial \xi_j}(\hat{f}(\xi))$.

Proof

1. Obvious
2. Fix ξ and then $|\hat{f}(\xi)| \leq \int |f(x)||e^{-2\pi i x\cdot\xi}|dx = ||f||_{L^1}$ and so
$$||\hat{f}||_{L^\infty} = \sup_{\xi} |\hat{f}(\xi)| \leq ||f||_{L^1}$$
3. Pick $e_n = (0, ..., 0, 1)$ and so we have
$$\hat{f}(\xi) = -\int f(x)e^{-2\pi i(x + \frac{1}{\xi_n}e_n)\cdot\xi}dx = -\int f(x - \frac{1}{\xi_n}e_n)e^{-2\pi i x\cdot\xi}dx$$
and so
$$\hat{f}(\xi) = \frac{1}{2}\int (f(x) - f(x - \frac{1}{\xi_n}e_n))e^{-2\pi i x\cdot\xi}dx$$
and if $|\xi_n| \to \infty$ then the dominated convergence theorem implies that $|\hat{f}(\xi)| \to 0$. It is clear that this doesn't depend on e_n, and so this shows the result for $|\xi| \to \infty$ along any axis. Property 6 then gives any direction.
4. $$\widehat{(f \star g)}(\xi) = \int \left(\int f(y)g(x-y)dy\right) e^{-2\pi i x\cdot\xi}dx$$
and then Fubini gives
$$\begin{aligned}\int \left(\int f(y)g(x-y)dy\right) e^{-2\pi i x\cdot\xi}dx &= \int f(y)e^{-2\pi i y\cdot\xi}\left(\int g(x-y)e^{-2\pi i(x-y)\cdot\xi}dx\right)dy \\ &= \hat{f}(\xi)\hat{g}(\xi)\end{aligned}$$
5. $$\widehat{\tau_h f}(\xi) = \int f(x+h)e^{-2\pi i x\cdot\xi}dx = \int f(y)e^{-2\pi i(y-h)\cdot\xi}dy = e^{2\pi i h\cdot\xi}\hat{f}(\xi)$$
The other part is left as an exercise.
6. If $\theta \in SO(n)$ then $\theta^{-1} = \theta^T$ and $\det\theta = 1$. Then
$$\begin{aligned}\int f(\theta x)e^{-2\pi i x\cdot\xi}dx &= \int f(\theta x)e^{-2\pi i x\cdot\xi}dx \\ &= \int f(\theta x)e^{-2\pi i\theta^{-1}\theta x\cdot\xi}dx \\ &= \int f(\theta x)e^{-2\pi i\theta x\cdot\theta\xi}dx \\ &= \hat{f}(\theta\xi)\end{aligned}$$

7. $$\hat{g}(\xi) = \int \frac{1}{\lambda^n} f\left(\frac{x}{\lambda}\right) e^{-2\pi i x\cdot\xi} dx = \int f(y) e^{-2\pi i y\cdot(\lambda\xi)} dy = \hat{f}(\lambda\xi)$$

8. $$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = \int \frac{\partial f}{\partial x_j}(x) e^{-2\pi i x\cdot\xi} dx = \int f(x) 2\pi i \xi_j e^{-2\pi i x\cdot\xi} dx + \overset{0}{\cancel{BT}} = 2\pi i \xi_j \hat{f}(\xi)$$

9. exercise

Q.E.D.

The main problem with this definition of the Fourier transform is that $\wedge : L^1 \to L^\infty$ but L^∞ is not contained in L^1. In an interval however, $L^\infty \subset L^1$. If one is doing the Fourier series, we can define the inverse of the Fourier transform, and it should be

$$f(x) = \int \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

but since $\hat{f}(\xi)$ is not necessarily in L^1 the right hand side of the above does not necessarily make sense. We thus have a goal to change L^1 into something else so that it is somehow true. It turns out that the correct place is L^2. However, for $f \in L^2$ the definition of Fourier transform doesn't necessarily make sense.

1.2 Schwartz space

This is intuitively the space of C^∞ functions that decay faster than any polynomial. We first introduce some notation.

A point in space is denoted $x = (x_1, ..., x_n)$ and a multiindex is denoted $\alpha = (\alpha_1, ..., \alpha_n)$ for $\alpha_i \in \mathbb{N}$. $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\alpha! = \alpha_1!...\alpha_n!$. We have $\partial^\alpha f = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n} f$ and $x^\alpha = (x_1^{\alpha_1}, ..., x_1^{\alpha_n})$ and they satisfy the Leibniz rule

$$\frac{d^m}{dt^m}(fg) = \sum_{k=0}^m \binom{m}{k} \frac{d^k f}{dt^k} \frac{d^{m-k} g}{dt^{m-k}}$$

or more generally

$$\partial^\alpha(fg) = \sum_{\beta\leq\alpha} \binom{\alpha_1}{\beta_1} ... \binom{\alpha_n}{\beta_n} \partial^\beta f \partial^{\alpha-\beta} g$$

where $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for all i.

Definition 1.3 $f : \mathbb{R}^n \to \mathbb{R}$ *is* **Schwartz ($\mathcal{S}$)** *if for all α, β multiindices, there exists $C_{\alpha,\beta}$ such that*

$$\rho_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| \leq C_{\alpha,\beta}$$

We make the following observations

1. $C_c^\infty \subset \mathcal{S}$ and $e^{-c|x|^2} \in \mathcal{S}$ for $c > 0$ but $\frac{1}{1+|x|^a}$ is not in $\mathcal{S}$.

2. $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^m)$ then $h(x_1, ..., x_{n+m}) = f(x_1, ..., x_n) g(x_{n+1}, ..., x_{n+m})$ is in $\mathcal{S}(\mathbb{R}^{n+m})$.

3. If $P(x)$ is any polynomial and $f \in \mathcal{S}(\mathbb{R}^n)$ then $P(x)f(x) \in \mathcal{S}(\mathbb{R}^n)$

4. If $f \in \mathcal{S}$ and α is any multiindex then $\partial^\alpha f \in \mathcal{S}$.

Remark $f \in \mathcal{S}$ if and only if for all N there exists $C_{\alpha,N}$ such that $|\partial^\alpha f| \leq \frac{C_{\alpha,N}}{1+|x|^N}$

1.3 Convergence in $\mathcal{S}$.

Definition 1.4 *$\{f_n\}$ for $f_n \in \mathcal{S}$* **converges** *to $f \in \mathcal{S}$ in $\mathcal{S}$ if and only if for all α, β multiindices,*

$$\rho_{\alpha,\beta}(f_k - f) = \sup_x |x^\alpha \partial^\beta (f_k - f)| \to 0$$

This is a very demanding definition. Note that if $\alpha = 0$ then $\sup_x |\partial^\beta (f_k - f)| \to 0$.

This definition generates a topology on $\mathcal{S}$ and with respect to that topology the operators $+, a\cdot, \partial^\alpha$ are continuous functions.

The objects $\rho_{\alpha,\beta}(f)$ are seminorms. They satisfy all properties of norms except $\rho_{\alpha,\beta}(f) = 0$ does not imply that $f = 0$.

One can construct a distance function in $\mathcal{S}$ which generates the same topology as follows:

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(f-g)}{1+\rho_j(f-g)}$$

where ρ_j are any enumeration of ρ_{α_β}.

Theorem 1.5 *Suppose $\{f_k\}$ and $f \in \mathcal{S}(\mathbb{R}^n)$ and $f_k \to f$ in $\mathcal{S}$ then $f_k \to f$ in L^p for $1 < p \le \infty$ (why not $p = 1$?). Moreover, there exists $C_{n,p}$ such that*

$$\|\partial^\beta f\|_{L^p} \le C_{n,p} \sum_{|\alpha| \le N+1} \rho_{\alpha,\beta}(f)$$

where $N = \frac{n+1}{p}$

Proof

$$\begin{aligned}
\|\partial^\beta f\|_{L^p}^p &= \int_{\mathbb{R}^n} |\partial^\beta f|^p dx \\
&= \int_{|x|<1} |\partial^\beta f|^p dx + \int_{|x|>1} |\partial^\beta f|^p dx \\
&\le C_n \|\partial^\beta f\|_{L^\infty}^p + \int_{|x|>1} \frac{1}{|x|^{n+1}} \left[|x|^{\frac{n+1}{p}} |\partial^\beta f| \right]^p dx \\
&\le C_n \|\partial^\beta f\|_{L^\infty}^p + \int_{|x|>1} \frac{1}{|x|^{n+1}} \sup_x \left[|x|^{\frac{n+1}{p}} |\partial^\beta f| \right]^p dx
\end{aligned}$$

and so (using different constants)

$$\|\partial^\beta f\|_{L^p} \le C_n \|\partial^\beta f\|_{L^\infty}^p + \rho_{\frac{N+1}{p},\beta}(f) \int_{|x|>1} \frac{1}{|x|^{n+1}} dx \le C_{n,p} \sum_{|\alpha| \le N+1} \rho_{\alpha,\beta}(f)$$

To prove convergence part, use the estimate with f replaced by $f_k - f$ and $\beta = 0$. Thus

$$\|f_k - f\|_{L^p} \le C \sum_{|\alpha| \le N+1} \rho_{\alpha,0}(f_k - f) \to 0$$

Q.E.D.

Theorem 1.6 *The Fourier transform is a continuous map from $\mathcal{S}$ to $\mathcal{S}$ such that*

$$\int f\hat{g}dx = \int \hat{f}gdx$$

Moreover

$$f(x) = \int \hat{f}(\xi)e^{2\pi ix\cdot\xi}d\xi$$

for $f, g \in \mathcal{S}$.

Proof If $f \in \mathcal{S}$ then $f \in L^1$. Thus we can define $\hat{f}(\xi)$. We claim that $\wedge : \mathcal{S} \to \mathcal{S}$ is well defined on $\mathcal{S}$ by the previous comment. Thus it is left to show that the range of $\wedge$ lies in $\mathcal{S}$. We need $\sup|\xi^\alpha\partial^\beta\hat{f}(\xi)| \leq a_{\alpha\beta}$. Recall rules on $\wedge$ from before. We ignore factors of $2\pi i$ for simplicity.

$$\xi^\alpha\partial^\beta\hat{f}(\xi) = C(\widehat{\partial^\alpha(x^\beta f)})(\xi)$$

and so

$$\sup|\xi^\alpha\partial^\beta\hat{f}(\xi)| \leq C\sup|(\widehat{\partial^\alpha(x^\beta f)})(\xi)|$$

We've seen that $g \in L^1 \implies g \in L^\infty$. We want $\|(\widehat{\partial^\alpha(x^\beta f)})(\xi)\|_{L^\infty} \leq a_{\alpha\beta}$ It is enough to show that $(\widehat{\partial^\alpha(x^\beta f)})(\xi) \in L^1$. Notice that when you expand it you get factors of the form $x^a\partial^b f$ for various a and b. and each one of these is in $\mathcal{S}$ and so is in L^1. Thus it is bounded. We have

$$\sup|\xi^\alpha\partial^\beta\hat{f}(\xi)| = \|C(\widehat{\partial^\alpha(x^\beta f)})(\xi)\|_{L^\infty} \leq C\|(\partial^\alpha(x^\beta f))\|_{L^1} \tag{1.1}$$

To prove continuity, we show that $\wedge$ is sequentially continuous, i.e. if $f_n \to f$ in $\mathcal{S}$ then $\hat{f}_n \to \hat{f}$ in $\mathcal{S}$.

Convergence in $\mathcal{S}$ is defined in terms of the seminorms. Thus we need

$$\rho_{\alpha,\beta}(f_n - f) \to 0 \implies \rho_{\alpha,\beta}(\hat{f}_n - \hat{f}) \to 0$$

We have from (1.1) that

$$\rho_{\alpha,\beta}(\hat{f}_n - \hat{f}) = \sup_\xi|\xi^\alpha D^\beta(\hat{f}_n - \hat{f})| \leq C\|\partial_x^\alpha(x^\beta(f_n - f)\|_{L^1}$$

If we now apply the Leibniz rule, we get

$$C\|\partial_x^\alpha(x^\beta(f_n - f)\|_{L^1} \leq \|\sum_{a,b} Cx^a\partial^b(f_n - f)\|_{L^1} \leq \sum\rho_{\alpha\beta}(x^a\partial^b(f_n - f))$$

and since $\rho_{\alpha\beta}(f_n - f) \to 0$ for fixed α and β then

$$\rho_{\alpha\beta}(x^a\partial^b(f_n - f)) \to 0$$

Thus we have shown continuity.

Now for the first equality.

$$\begin{aligned}\int f(x)\hat{g}(x)dx &= \int f(x)\int g(y)e^{-2\pi ix\cdot y}dydx\\ &\overset{Fubini}{=} \int g(y)\int f(x)e^{-2\pi ix\cdot y}dxdy\\ &= \int \hat{f}(x)g(x)dx\end{aligned}$$

Remember if $h(x) \in L^1$ then $h_\lambda(x) = \frac{1}{\lambda^n} h(\frac{x}{\lambda})$ has $\hat{h_\lambda}(\xi) = \hat{f}(\lambda \xi)$. If $g \in \mathcal{S}$ then $g_\lambda \in \mathcal{S}$ for all λ. Then

$$\int f(x)\hat{g}(\lambda x)dx = \int f(x)\hat{g_\lambda}(x)dx = \int \hat{f}(x)g_\lambda(x)dx = \int \hat{f}(x)\frac{1}{\lambda^n}g(\frac{x}{\lambda})dx$$

and so

$$\lambda^n \int f(x)\hat{g}(\lambda x)dx = \int \hat{f}(x)g(\frac{x}{\lambda})dx$$

and then changing variables in the right hand side by $y = \lambda x$ we get

$$\int f(\frac{y}{\lambda})\hat{f}(y)dy = \int \hat{f}(x)g(\frac{x}{\lambda})dx$$

which is true for all $\lambda > 0$ and thus

$$\lim_{\lambda\to\infty} \int f(\frac{y}{\lambda})\hat{f}(y)dy = \lim_{\lambda\to\infty} \int \hat{f}(x)g(\frac{x}{\lambda})dx$$

We can use the DCT here (CHECK) to get

$$f(0)\int \hat{g}(x)dx = g(0)\int \hat{f}(x)dx$$

for all $f, g \in \mathcal{S}$.

We claim that if $g(x) = e^{-\pi|x|^2}$ then $\hat{g}(x) = e^{-\pi|x|^2}$. Then using this g we get that

$$f(0) = \int \hat{f}(x)dx$$

which is what we want with $x = 0$. Recall that $\widehat{\tau_h f}(\xi) = \hat{f}(\xi)e^{2\pi i h\cdot\xi}$. We work with a function $f(y)$. Then $f(x) = \tau_x(f)(0)$ and $\tau_x f(y) = f(x+y)$. Then

$$f(x) = (\tau_x f(0)) = \int \widehat{\tau_x f(\cdot)}(\xi)d\xi = \int \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

as required. *Q.E.D.*

Lemma 1.7 *If $f(x) = e^{-\pi|x|^2}$ then $\hat{f}(\xi) = e^{-\pi|\xi|^2}$*

Proof f is the unique solution of the ODE $u' + 2\pi x u = 0$ with $u(0) = 1$. If we Fourier transform both sides we get $\hat{u}' + 2\pi\xi\hat{u} = 0$ and this is an ODE for $\hat{u}$, with $\hat{u}(0) = 1$. This is the same ODE as before, and so $\hat{u}(\xi) = e^{-\pi|\xi|^2}$. *Q.E.D.*

Proposition 1.8 *If $f, g \in \mathcal{S}$ then $\partial^\alpha(f \star g) = (\partial^\alpha f) \star g = f \star (\partial^\alpha g)$*

Definition 1.9 *For f, define $\check{f}(x) = \int f(\xi)e^{2\pi i x\cdot\xi}d\xi$.*

1.4 Fourier Transform in L^p

Observe that $\check{f}(x) = \hat{f}(-x)$. Also $\check{\hat{f}} = f(x)$ and $\hat{\check{f}} = f(x)$ and also from above $\int f\bar{h} = \int \hat{f}\bar{\hat{h}}$.

So far we have $f \in L^1$ and a Fourier transform, but the Fourier transform is not necessarily in L^1, and so we cant define $\int \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$. Then we had $f \in \mathcal{S}$ and $\hat{f} \in L^1$ and so we could define the inverse and $\wedge : \mathcal{S} \to \mathcal{S}$ as an isometry in L^2.

There exists a unique extension of $\wedge$ from $\mathcal{S}$ to L^2. The reason is that $\mathcal{S}$ is dense in L^2. After all C_c^∞ is dense in L^p for $p \neq \infty$ and $C_c^\infty \subset \mathcal{S}$. To define $\wedge$ for $f \in L^2$, take $\{f_n\}$ in $\mathcal{S}$ with $f_n \to f$ in L^2. Then define $\mathcal{F}(f) = \lim \hat{f}_n$ understood as a limit in L^2.

We should be clear that we are not claiming that $\hat{f}(\xi)$ has any pointwise limit. We are claiming that $\{f_n\}$ converges, and so $\{f_n\}$ is Cauchy, so $\|f_n - f\|_2 = \|\hat{f}_n - \hat{f}\|_2$ and so $\{\hat{f}\}$ is Cauchy, and take $\mathcal{F}(f)$ to be the limit in L^2 of that Cauchy sequence. This works because L^2 is complete.

About the unique extension. By contradiction, suppose $f_n \to f$ in L^2 and $g_n \to f$ in L^2 for $f_n, g_n \in \mathcal{S}$, but $\hat{f}_n \to F$ and $\hat{g}_n \to G$ in L^2, with $G \neq F$. Then

$$0 \neq \|G - F\|_2 = \lim \|\hat{g}_n - \hat{f}_n\|_2 = \lim \|g_n - f_n\|_2 = 0$$

and clearly this cannot be the case. Using the density of $\mathcal{S}$ in L^2, coupled with $\wedge : \mathcal{S} \to \mathcal{S}$ is an isometry in L^2 we can define the Fourier transform for $f \in L^2$.

We get that $\mathcal{F}$ is an isometry in L^2 since $\|\mathcal{F}(f)\|_2 = \lim \|\hat{f}_n\|_2 = \lim \|f_n\|_2 = \|f\|_2$. We can do the same thing for $\vee$, and get that $\|\check{f}\|_2 = \|f\|_2$. We can define $\mathcal{F}'(f) = \lim_{L^2} \check{f}_n$ for $f_n \in \mathcal{S}$ where $f_n \to f$ in L^2. It is then straightforward to see that $\mathcal{F}\mathcal{F}'(f) = f = \mathcal{F}'\mathcal{F}(f)$.

How though do we compute $\mathcal{F}(f)$ for $f \in L^2$. If $f \in L^1$ then $\hat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} dx$. If $f \in L^1 \cap L^2 \cap \mathcal{S}$ then $\hat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} dx$. If $f \in L^1 \cap L^2$ then the same thing. How about taking $f_n \in L^1 \cap L^2$ such that $f_n \to f$ in L^2. How about taking $f_n = f(x) \chi_{B_n(0)}$ for $f \in L^2$. Then claim that $f_n \in L^1$, and to show this uses Holder.

Define $\mathcal{F}(f) = \lim_{L^2} \int f(x) \chi_{B_n(0)} e^{-2\pi i x \cdot \xi} dx$.

Does $\int f(x) \chi_{B_n(0)} e^{-2\pi i x \cdot \xi} dx$ converge pointwise to anything? We know that $\hat{f}_n \to \mathcal{F}(f)$ in L^2. It is an open question whether $\int f(x) \chi_{B_n(0)} e^{-2\pi i x \cdot \xi} dx$ converges pointwise to $\mathcal{F}(f)$. We do know L^2 convergence though. From measure theory, convergence in L^2 implies that there exists a subsequence that converges pointwise. Thus we know there is a $\{n_j\}$ such that $\int f(x) \chi_{B_{n_j}(0)} e^{-2\pi i x \cdot \xi} dx \to \mathcal{F}(f)$ pointwise.

What now about the Fourier transform in L^p for $1 \leq p \leq 2$.

Theorem 1.10 *Suppose* $1 \leq p < q < r \leq \infty$. *Then* $L^q \subset L^p + L^r = \{f + g : f \in L^p, g \in L^r\}$

Proof If $f \in L^q$ then write $f(x) := f_{<M} + f_{>M}$ where $f_{<M} = f \chi_{\{x:|f(x)|<M\}}$ and $f_{>M} = f \chi_{\{x:|f(x)| \geq M\}}$. We take $M = 1$ here and claim that $f_{<1} \in L^r$ and $f_{>1} \in L^p$.

$$\int |f_{<1}|^r \leq \int |f_{<1}|^q \leq \int |f|^q < \infty$$

and the other one is proved similarly. *Q.E.D.*

We hope to define $\mathcal{F}(f)$ for $f \in L^p$ for $1 < p < 2$ by $\mathcal{F}(f) := \mathcal{F}(f_{<1}) + \hat{f}_{>1}$. In fact, one can use any decomposition. If $f \in L^p$ write $f = g_1 + g_2 = h_1 + h_2$. with the ones in L^1 and the twos in L^2. Define $\mathcal{F}(f) = \hat{g_1} + \mathcal{F}(g_2) = \hat{h}_1 + \mathcal{F}(h_2)$ and this is independent of the choice because of the following: We have $g_1 - h_1 = h_2 - g_1$ adn the LHS is in L^1 and the RHS is in L^2, so they both have a $\wedge$ and it agrees with $\mathcal{F}$.

Proposition 1.11 *For all* $f \in L^p$ *for* $1 \leq p \leq 2$ *then* $\|\mathcal{F}(f)\|_{p'} \leq \|f\|_p$ *where* $\frac{1}{p} + \frac{1}{p'} = 1$.

We introduce a bit of an abuse of notation. We use $\wedge$ always with the understanding that we need to take limits if we are not in $\mathcal{S}$.

This above proposition is a consequence of Riesz Thorin interpolation.

Proof We know two cases of the inequality in the above proposition, namely for $p = 1$ and $p = 2$. Applying Riesz-Thorin gives the result. We spill some more of the details below. *Q.E.D.*

Lemma 1.12 ((3 line lemma) Stein 1960s) *Suppose F is a bounded and continuous complex valued function and $S = \{x+iy : x,y \in \mathbb{R}, 0 \le x \le 1\}$ that is analytic in the interior of S. If $|F(iy)| \le m_0$ for $y \in \mathbb{R}$ and $|F(1+iy)| \le m_1$ for $y \in \mathbb{R}$ then for fixed x,*

$$|F(x+iy)| \le m_0^{1-x} m_1^x$$

Proof See Duoandikoetchea *Q.E.D.*

Theorem 1.13 (Riesz-Thorin Interpolation) *Suppose that $1 \le p_0, q_0, p_1, q_1 \le \infty$ and define for $0 < \theta < 1$ the numbers p, q by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{p_1} + \frac{\theta}{q_1}$$

Let T be a linear operator from L^{p_0} into L^{q_0} and L^{p_1} into L^{q_1} that satisfies

$$||Tf||_{q_0} \le M_0 ||f||_{p_0}$$

$$||Tf||_{q_1} \le M_1 ||f||_{p_1}$$

and also suppose that T is linear from $L^{p_0} + L^{p_1}$ into $L^{q_0} + L^{q_1}$. Then

$$||Tf||_q \le M_0^{1-\theta} M_1^{\theta} ||f||_p$$

A proof is omitted. Its too hard for this course. The following is an application of Riesz-Thorin.

Lemma 1.14 (Young's Inequality) *Suppose that $f \in L^p$ and $g \in L^q$. Then*

$$||f \star g||_r \le ||f||_p ||g||_q$$

where $\frac{1}{r} = 1 + \frac{1}{p} + \frac{1}{p'}$

Proof There are two easy cases, namely

$$||f \star g||_\infty \le ||f||_p ||g||_{p'}$$

and

$$||f \star g||_p \le ||f||_p ||g||_1$$

The former is essentially Holder's inequality:

$$|f \star g(x)| \le \int |f(x-y)||g(y)| dy \le ||f(x-\cdot)||_p ||g||_{p'} = ||f||_p ||g||_{p'}$$

and taking the supremum gives the required result. The latter uses Minkowski's inequality, which is stated below the proof.

If we fix $f \in L^p$ and define $T_f(g) = f \star g$ then T_f is linear, $T_f : L^{p'} \to L^\infty$ with $||T_f g||_\infty \le M_0 ||g||_{p'}$ with $M_0 = ||f||_p$ and $T_f : L^1 \to L^p$ with $||T_f g||_p \le M_1 ||g||_1$ with $M_1 = ||f||_p$. Then by the Riesz-Thorin interpolation, we get $T_f : L^q \to L^r$ and

$$||T_f g||_r \le M_0^{1-\theta} M_1^{\theta} ||g||_q$$

where $\frac{1}{q} = \frac{1-\theta}{p'} + \frac{\theta}{1}$ and $\frac{1}{r} = \frac{1-\theta}{\infty} + \frac{\theta}{p}$ and so

$$\frac{1}{q} = (1 - \frac{p}{r})\frac{p-1}{p} + \frac{p}{r} = \frac{p-1}{p} - \frac{p-1}{r} + \frac{p}{r} = 1 - \frac{1}{p} + \frac{1}{r}$$

as required. *Q.E.D.*

Take $g \in L^q$ and define $\phi_g(f) = \int fg d\mu$. By Holder this is well defined for $f \in L^p$ since $|\int fg| \le ||f||_p ||g||_q$ and so $\phi_g : L^p \to \mathbb{R}$ and $\phi_g \in (L^p)^\star$.

Theorem 1.15 *Suppose* $1 < p < \infty$. *Then* $(L^p)^\star = L^q$ *where* $\frac{1}{p} + \frac{1}{q} = 1$. *If* μ *is* σ *finite then* $(L^1)^\star = L^\infty$.

Proposition 1.16 (Duality) *Suppose* $1 \le q < \infty$ *and* $g \in L^q$. *Then*

$$||g||_q = ||\phi_g|| := \sup\{\int fg d\mu : ||f||_p = 1\}$$

Lemma 1.17 (Minkowski Inequality) *Suppose* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. *Then*

$$||\int f(\cdot, y)dy||_p \le \int ||f(\cdot, y)||_p dy$$

Proof This is easy for $p = 1$ by Fubini and similarly $p = \infty$ is easy. Then for rest,

$$\begin{aligned}
||\int f(\cdot, y)dy||_p &= \sup_{\substack{h \in L^{p'} \\ ||h||_{p'}=1}} \{\iint f(x,y)dyh(x)dx\} \\
&= \sup_h \iint f(x,y)h(x)dxdy \\
&\le \sup_h \int ||f(x,y)||_{L^p_x} ||h||_{L^{p'}} dy \\
&= \sup_h \int ||f(x,y)||_{L^p_x} dy \\
&= \int ||f(x,y)||_{L^p_x} dy
\end{aligned}$$

. *Q.E.D.*

We justify the Fourier transform in L^p for $1 < p < 2$. We have inequalities at the end bounds. From R-T we get bounds for p and q where

$$\frac{1}{p} = \frac{1-\theta}{1} - \frac{\theta}{2} \qquad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$$

and so we get $\frac{1}{p} = 1 - \frac{1}{q}$ as required.

1.4.1 Scaling analysis

We ask the question, is $||f \star g||_r \le ||f||_p ||g||_q$ ever goint to be true, for any choice of r, p, q. If so, then it must also be true for $f(\lambda x)$ and $g(\lambda x)$. We get

$$\left[\int \left[\int f(\lambda(x-y))g(\lambda y)dy\right]^r dx\right]^{1/r} \le \left(\int |f(\lambda x)|^p dx\right)^{1/p} \left(\int |f(\lambda x)|^q dx\right)^{1/q}$$

and then if we change variables by $\lambda x = \bar{x}$ and $\lambda y = \bar{y}$ we get

$$\left[\int \left[\int f((x-y))g(y)\lambda^{-n}dy\right]^r \lambda^{-n}dx\right]^{1/r} \le \left(\int |f(x)|^p \lambda^{-n}dx\right)^{1/p} \left(\int |g(x)|^q \lambda^{-n}dx\right)^{1/q}$$

and so we get

$$\lambda^{-n(1+\frac{1}{r})}||f \star g||_r \le \lambda^{-n(\frac{1}{p}+\frac{1}{q})}||f||_p ||g||_q$$

and so if the inequality is true, then $-n(1+\frac{1}{r}) = -n(\frac{1}{p}+\frac{1}{q})$

1.5 Fourier Series

Take 2π periodic functions on $\mathbb{R}$ and define $\hat{f}(n) = \int_0^{2\pi} f(x)e^{-inx}dx$. Then are you able to recover f from $\{\hat{f}(n)\}$. We would like it to be $f(x) = \sum_{-\infty}^{\infty} \hat{f}(n)e^{inx}$. We define $S_N f(x) = \sum_{-N}^{N} \hat{f}(n)e^{inx}$ and wonder whether $\|S_N f - f\|_p \to 0$ or $S_N f(x) \to f(x)$ a.e.x.

Theorem 1.18 $\|S_N f - f\| \to 0 \iff \|S_N f\|_p \le c_p \|f\|_p$ *where* $1 \le p < \infty$ *and* $f \in L^p[0, 2\pi]$.

Proof "$\Longleftarrow$" For $g \in C^\infty$ we have

$$\|S_N f - f\|_p \le \|S_N f - S_N g\|_p + \|S_N g - g\|_p + \|g - f\|_p$$

and since C^∞ is dense in L^p, for all $\varepsilon > 0$ there is a g such that $\|g - f\|_p < \varepsilon$. Since S_N is linear and by the assumptions in the theorem we have $\|S_N f - S_N g\|_p \le c_p \|f - g\|_p$ and thus

$$\|S_N f - f\|_p \le 2\varepsilon + \|S_N g - g\|_p < \varepsilon'$$

where we have assumed the result for C^∞ functions.

"$\Longrightarrow$" We use the alternative statement to the UBP below. Suppose that $X = L^p = Y$ and $T_\alpha = S_N$. If we work by contradiction then there exists $f \in L^p$ such that $\sup_N \|S_N f\|_p = \infty$. However, $\|S_N f\|_p \le \|S_N f - f\|_p + \|f\|_p$ and the $\|S_N f - f\|_p$ is bounded and so this is less than $M + \|f\|_p < \infty$ and this is a contradiction. *Q.E.D.*

Theorem 1.19 (Uniform Boundedness principle) *Suppose that X and Y are normed spaces. A denotes a subset of $L(X, Y)$, the linear bounded maps $X \to Y$. Then*

1. *If* $\sup_{T \in A} \|Tx\|_Y < \infty$ *for all x then* $\sup_{T \in A} \|T\| < \infty$.
2. *If furthermore X is a Banach space and* $\sup_{T \in A} \|Tx\| < \infty$ *for all x then* $\sup_{T \in A} \|T\| < \infty$.

Theorem 1.20 (Alternative Statement) *Suppose that X is a Banach space and Y is a normed space. Suppose $\{T_\alpha\}_{\alpha \in A}$ is a set of linear and bounded functionals $T_\alpha : X \to Y$. Then either*

$$\sup_{\alpha \in A} \|T_\alpha\| < \infty$$

or

$$\exists x \in X \text{ such that } \sup \|Tx\|_Y = \infty$$

In one dimension, we have $\|S_N f\| \le c_p \|f\|$ implies that we have L^p convergence of $S_N f$ to f. In two or more dimensions, this is true for $p = 2$ but false for all other p. In $n = 1$ a.e. convergence is also true (Carlesson for p=2 Hunt did rest). All hell breaks loose in $n = 2$ and above.

We go back to $n = 1$. Take $f : \mathbb{R} \to \mathbb{R}$, $f \in L^p$, for $1 < p < 2$, and define $S_R f(x) = \int_{-R}^{R} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$. We have seen $f \in \mathcal{S}$ then $f(x) = \int \hat{f}(\xi)e^{2\pi i x\xi}d\xi$. and so we ask does $S_R f(x) \to f(x)$ or $\|S_R f - f\|_p \to 0$. Then

$$\begin{aligned} S_R f(x) &= \int_{-R}^{R} \int_{\mathbb{R}} f(y)e^{-2\pi i y\xi}dy e^{2\pi i x\xi}d\xi \\ &= \int_{\mathbb{R}} f(y) \int_{-R}^{R} e^{2\pi i(x-y)\xi}d\xi dy \\ &= \int f(y)D_R(x-y)dy \end{aligned}$$

where $D_R z = \int_{-R}^{R} e^{2\pi i(x-y)\xi}d\xi = \frac{1}{2\pi z}e^{2\pi i z\xi}|_{-R}^{R} = \frac{\sin(2\pi Rz)}{\pi z}$ and so we have $S_R f = f \star D_R$.

Theorem 1.21 $||S_R f - f||_p \to 0$ *if and only if* $||S_R f||_p \le c_p ||f||_p$

Proof The forwards direction is the Uniform boundedness principle. The backwards direction is adding in a $g \in \mathcal{S}$ and the same as the above theorem. *Q.E.D.*

We hope that $||S_R f - f||_p \to 0$. This is by the above, equivalent to boundedness. We thus need $||S_R f||_p \le c_p ||f||_p$ or $||f \star D_R||_p \le c_p ||f||_p$. We get from Young's that $||f \star D_R||_p \le ||D_R||_q ||f||_p$ with $q = 1$. The problem is that $\int |D_R| = \infty$. This is somewhat unhelpful. It turns out that boundedness is true, but just that Young's inequality is too wasteful. In $n \ge 2$ Fefferman showed that $||S_R f||_p \le c_p ||f||_p$ is only true for $p = 2$.

Note that Young's also applies for $|f|$ and $|g|$ if it applies for f and g, and as the moduli are in general larger, it doesnt see the cancellations involved.

We ask the question, does $S_R f \to f$ a.e. for $n \ge 1$. We prove this for $n = 1$ and $1 < p < \infty$ and $f \in L^p$. Carleson proved this by proving the a.e. convergence when $n = 1$ and $1 < p \le 2$ by proving the following:

$$||\sup_R |S_R f(x)|||_p \le c_p ||f||_p$$

This is where we fix x, compute $S_R f(x)$ and then supremum over all $R > 0$. This gives an example of a maximal function.

The goal we now have is to recover f from $\hat{f}$. So far we know that if $f \in L^2$ then $\hat{f} \in L^2$ and in that case there exists a functional $\vee$ such that $\check{\hat{f}} = f$

In history, people gave up on the idea of defining $\vee$ for $f \in L^p$ in the sense of hoping for $\check{\hat{f}} = f$.

1.5.1 Summability in Fourier series

Suppose we have a function $f : [-\pi, \pi] \to \mathbb{R}$ and then define $S_N f = \sum_{-N}^{N} \hat{f}(n) e^{inx}$, an effort to reconstruct f out of $\{\hat{f}(n)\}$. This convergence though fails sometimes.

If we define $F_M f = \frac{S_0 f + \ldots + S_{M-1} f}{M}$ then this gives a notion of Cesaro convergence, if this sum converges. We can write this sum as $\sum_{-M}^{M} c_n e^{inx}$ and note that $F_M f \to f$ a.e. for $1 < p < 2$ and they converge much faster.

1.5.2 Summability of the Fourier Transform

We have a Cesaro summation formula for the Fourier transform:

$$S_R f(x) = \int_{|\xi| < R} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

and then we define

$$\sigma_R f = \frac{1}{R} \int_0^R S_t f(x) dt$$

We can then write this as follows

$$\sigma_R f = \frac{1}{R} \int_0^R D_t \star f(x) dt = \left(\frac{1}{R} \int_0^R D_t dt \right) \star f(x) =: F_R \star f(x)$$

and it can be computed that $F_R(z) = \frac{\sin^2(z\pi R)}{R(\pi z)^2}$. Now note that F_R is greater than or equal to zero and $\int F_R(x) dx = 1$.

We claim that $||\sigma_R f||_p \le c_p ||f||_p$ and $\sigma_R f \to f$ in L^p for $1 < p < 2$.

1.6 Approximations to the identity

The Abel-Poisson method

$$u(x,t) = \int_{\mathbb{R}^n} e^{-2\pi t|\xi|}\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

Gauss-Weierstrass method.

$$w(x,t) = \int_{\mathbb{R}^n} e^{-4\pi t^2|\xi|^2}\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

We have that, for $f \in L^1$, the Fourier transform $\hat{f} \in L^\infty$, which does not guarantee that $\int \hat{f}e^{2\pi i x\cdot\xi}d\xi$ makes sense. Then since the above u and w do make sense, do they converge to f as $t \to 0$.

Fix $\phi \in C_c^\infty$ or $\mathcal{S}$ such that $\int \phi = 1$. Then define $\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{x}{t}\right)$.

Theorem 1.22 *$\phi_t \star f(x) \to f(x)$ in L^p as $t \to 0$ for $1 \le p < \infty$. Moreover $\phi_t \star f(x) \in C^\infty$ and $\phi_t \star f(x) \to f(x)$ uniformly if $f \in C_c$.*

Note we are trying to show $\phi_t \to \delta$ in distribution.

Proof

$$\phi_t \star g(x) = \int \phi_t(y)g(x-y)dy = \frac{1}{t^n}\int \phi\left(\frac{y}{t}\right)g(x-y)dy = \int \phi(z)g(x-tz)dz$$

and then

$$\phi_t \star g(x) - g(x) = \int \phi(z)[g(x-tz)-g(x)]dz$$

and thus

$$||\phi_t \star g(x) - g(x)||_p = ||\int \phi(z)[g(x-tz)-g(x)]dz||_p$$

and then using the Minkowski inequality we get

$$||\int \phi(z)[g(\cdot - tz) - g(\cdot)]dz||_p \le \int ||\phi(z)[g(\cdot - tz)-g(\cdot)]||_p dz = \int |\phi(z)|||[g(\cdot - tz)-g(\cdot)]||_p dz$$

We cannot move a limit inside the integral here. We thus make two claims to get around this

<u>Claim 1</u> $\forall \varepsilon > 0$ there exists a h_0 such that

$$||g(\cdot + h) - g(\cdot)||_p \le \frac{\varepsilon}{100\int|\phi(x)|dx}$$

<u>Claim 2</u> there exists a δ and t_0 such that for $t \le t_0$ we have

$$\int_{|y|>\delta/t}|\phi(y)|dy \le \frac{\varepsilon}{100||f||_p}$$

Then

$$\int |\phi(z)|||[g(\cdot - tz) - g(\cdot)]||_p dz = \int_{|tz|>\delta} + \int_{|tz|\le\delta}(|\phi(z)|||[g(\cdot - tz) - g(\cdot)]||_p)dz =: I + II$$

We first consider II:

$$II = \int_{|tz|\le\delta}|\phi(z)|||[g(\cdot - tz)-g(\cdot)]||_p dz \le \int |\phi(z)|\frac{\varepsilon}{100\int|\phi(x)|dx}dz \le \frac{\varepsilon}{100}$$

if $\delta < h_0$.

We now consider I:

$$\begin{aligned}\int_{|tz|>\delta} |\phi(z)| \|[g(\cdot - tz) - g(\cdot)]\|_p dz &\leq \int_{|tz|>\delta} |\phi(z)| 2\|g\|_p dz \\ &= 2\|g\|_p \int_{|z|\geq\delta/t} |\phi(z)| dz \\ &\leq 2\|g\|_p \frac{\varepsilon}{100\|g\|_p} \\ &\leq \frac{\varepsilon}{50}\end{aligned}$$

Q.E.D.

We now prove the claims we made

Proof We take $g \in C_c^\infty$ and then for such a g we can use the DCT. We then get that there exists an h such that

$$\int |g(x+h) - g(x)|^p dx \leq \left(\frac{\varepsilon}{100 \int |\phi(x)| dx} \right)^p$$

Thus we have the result for $g \in C_c^\infty$, and since this is dense in L^p, given $\delta > 0$, and fixing g there exists a $W \in C_c^\infty$ such that $\|g - W\|_p < \delta$. Then

$$\begin{aligned}\|g(\cdot + h) - g(\cdot)\|_p &\leq \|g(\cdot + h) - W(\cdot + h)\|_p + \|W(\cdot + h) - W(\cdot)\|_p + \|W(\cdot) - g(\cdot)\|_p \\ &\leq 2\delta + \frac{\varepsilon}{100 \int |\phi(x)| dx} \\ &\leq \bar{\varepsilon}\end{aligned}$$

Q.E.D.

Proof

$$\int_{|y|>\delta/t} |\phi(y)| dy = \int \chi_{\{|y|>\delta/t\}} |\phi(y)| dy = \int f_t dy$$

and note that $f_{t_1}(y) \leq f_{t_2}(y)$ for $t_1 \leq t_2$ and this is less than or equal to $|\phi(y)|$. Then the MCT or the DCT means you can exchange limit and integral, and the limit is zero. Thus you can make it as small as you like. *Q.E.D.* Back to Cesaro summation, if we set $R = \frac{1}{t}$ then $F_R(z) = \frac{\sin^2(\frac{\pi z}{t})}{t(\frac{\pi z}{t})^2}$ and then define $\phi(z) = \frac{\sin^2 \pi z}{(\pi z)^2}$ and then $F_R \star f = \phi_t \star f$. However, ϕ_t is not C_c^∞ or $\mathcal{S}$. However looking at the above proof, we did not need this assumption. We only needed $\phi \in L^1$ and $\int \phi(z) dz = 1$.

Lemma 1.23

$$\int \frac{\sin^2 \pi z}{(\pi z)^2} dz = 1$$

1.6.1 Abel-Poisson

$$u(x,t) = \int_{\mathbb{R}^n} e^{-2\pi t|\xi|} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

We hope to prove that $e^{-2\pi t|\xi|}$ is the Fourier transform of some function h.

Lemma 1.24

$$\int e^{-2\pi t|\xi|} e^{2\pi i x\cdot\xi} d\xi = c_n \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}} =: P_t(x)$$

which is called the Poisson kernel.

This uses the subordination principle, namely that

$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\beta^2/4u} du$$

Theorem 1.25 $u(x,t) = P_t \star f(x)$

Thus now to check that $u(x,t) \to f(x)$ we need only check that P_t has integral 1 and is of the form in theorem 1.22.

$$P_t(x) = c_n \frac{t}{(t^2 + t^2 \left|\frac{x}{t}\right|^2)^{\frac{n+1}{2}}} = \frac{c_n}{t^n} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}} = \frac{1}{t^n} P\left(\frac{x}{t}\right)$$

where $P(x) = \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}$ and also by a miracle $\int P(x)dx = 1$. Thus we have convergence because P is an approximation to the identity.

If we are trying to solve $\Delta u = 0$ in $\mathbb{R}^n \times \mathbb{R}^+$ with $u(x_1, ..., x_n, 0) = f(x)$ where $f(x)$ is given then a solution is $u(x,t) = P_t \star f(x)$.

1.6.2 Gauss-Weierstrass

$$w(x,t) = \int_{\mathbb{R}^n} e^{-4\pi t^2 |\xi|^2} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

We try to perform a similar argument to the above, i.e. we try to find an h such that $\hat{h} = e^{-4\pi t^2|\xi|^2}$. To compute this we simply take the inverse Fourier transform, as $e^{-4\pi t^2|\xi|^2} \in \mathcal{S}$. Thus

$$h(x,t) = \int e^{-4\pi t^2|\xi|^2} e^{2\pi i x\cdot\xi} d\xi = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/2t}$$

If we define $W(x) = \frac{1}{(4\pi)^{n/2}} e^{-|x|^2/2}$ then note that

$$W_{\sqrt{t}}(x) = \frac{1}{\sqrt{t}^n} W(\frac{x}{\sqrt{t}}) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/2t}$$

We then know that $w(x,t) = h \star f(x) = W_{\sqrt{t}} \star f(x)$ and we remark that $\int W(x)dx = 1$. We then have $w(x,t) \to f(x)$ because we invoke theorem 1.22.

1.6.3 Heat Equation

The equation $w_t - \Delta_x w = 0$ with $w(x,0) = f(x)$ where $w(x_1, ..., x_n, t)$ with time t and f is the given initial data. We proceed heuristically, and take the Fourier transform. For fixed t we get $\hat{w}(\xi, t) = \int w(x,t) e^{-2\pi i x\xi} dx$ and then

$$\frac{\partial}{\partial t} \hat{w} = \hat{w}_t$$

and also since $\widehat{\partial_{x_j} f}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$ and $\widehat{\partial_{x_j}^2 f}(\xi) = -4\pi^2 \xi_j^2 \hat{f}(\xi)$ and so we have

$$\widehat{\Delta f}(\xi) = -4\pi^2 |\xi|^2 \hat{f}(\xi)$$

and so the heat equation becomes, since $\hat{w}_t - \widehat{\Delta w} = 0$ and then this is

$$\partial_t \hat{w}(\xi, t) + 4\pi^2 |\xi|^2 \hat{w}(\xi, t) = 0$$

which is an ODE if we fix ξ!!!!!! This then gives

$$\hat{w}(\xi,t) = e^{-4\pi^2|\xi|^2}\hat{f}(\xi)$$

and thus this suggest that a solution is of the form

$$w(x,t) = \int e^{-4\pi^2|\xi|^2}\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi = W_{\sqrt{t}} \star f(x)$$

It is an easy exercise to check that this does indeed solve the Heat equation. It also gives the initial date, as it is an approximation to the identity.

2 Almost everywhere convergence, Weak type inequalities and Maximal functions

Definition 2.1 (X, M, μ) *a measure space. Suppose* $f : X \to \mathbb{R}$ *then*

$$\lambda_f(\alpha) = \mu\{x \in X : |f(x)| > \alpha\}$$

is called the **distribution function** *of* f*.* $\lambda_f : [0,\infty) \to \mathbb{R}^+$

Proposition 2.2 *1.* λ_f *is decreasing, and right continuous.* $\lambda_f(\alpha+\varepsilon) \to \lambda_f(\alpha)$

2. $|f| \le |g|$ *then* $\lambda_f(\alpha) \le \lambda_f(\alpha)$

3. $|f_n| \to |f|$ *in an increasing manner then* $\lambda_{f_n}(\alpha) \to \lambda_f(\alpha)$

4. $f = g + h)$ *then* $\lambda_f(\alpha) \le \lambda_g(\alpha/2) + \lambda_h(\alpha/2)$*.*

Proof We only prove 4. It is enough to show that

$$\{x \in X : |g+h| > \alpha\} \subset \{x \in X : |g| > \alpha/2\} \cup \{x \in X : |h| > \alpha/2\}$$

which should be clear *Q.E.D.*

Proposition 2.3 *Suppose that* ϕ *is Borel measurable, and* $\phi \ge 0$*. Let* $f : X \to \mathbb{R}$ *such that* $\lambda_f(\alpha) < \infty$*. Then*

$$\int \phi(|f(x)|)d\mu = -\int_0^\infty \phi(\alpha)d\lambda_f(\alpha)$$

Corollary 2.4

$$\int |f(x)|^p d\mu = -\int_0^\infty \alpha^p d\lambda_f(\alpha) = \int_0^\infty p\alpha^{p-1}\lambda_f(\alpha)d\alpha$$

2.0.4 Weak L^p spaces

Definition 2.5 $f \in L^p_W$ *the Weak* L^p *space, if and only if*

$$\mu\{x \in X : |f(x)| > \alpha\} \le \frac{C^p}{\alpha^p}$$

For example $f \in L^1_W$ if and only if $\mu\{x \in X : |f(x)| > \alpha\} \le C/\alpha$ and so for example $\frac{1}{x} \notin L^1$ since $\int_{|x|<1} |\frac{1}{x}|dx = \infty = \int_{|x|>1} |\frac{1}{x}|dx$ but $\frac{1}{x} \in L^1_W$ since $\{x : |\frac{1}{x}| > \alpha\} = \{x : \frac{1}{\alpha} > |x|\}$ and the Lebesgue measure of this set is $\frac{2}{\alpha}$

For example, in $\mathbb{R}^n$, $\frac{1}{|x|^n} \in L^1_W$ and $\frac{1}{|x|^{n/p}} \in L^p_W$.

Lemma 2.6 $L^p \subset L^p_W$

Proof We show that if $f \in L^p$ then $f \in L^p_W$. Chebyshev's inequality.

$$\int |f|^p d\mu \geq \int_{\{x:|f|>\alpha\}} |f|^p d\mu \geq \int_{\{x:|f|>\alpha\}} \alpha^p d\mu = \alpha^p \mu\{x : |f| > \alpha\}$$

and so

$$\mu\{x : |f| > \alpha\} \leq \left(\frac{||f||}{\alpha}\right)^p$$

Q.E.D.

Observe that the smallest C for a given f such that $\mu\{x : |f| > \alpha\} \leq \left(\frac{C}{\alpha}\right)^p$ can be taken as a semi norm for L^p_W.

2.1 Strong-(p,q) operators

Suppose that $T : L^p \to L^q$. Then we say that T is strong-(p,q) if and only if there exists C_{pq} such that $||Tf||_q \leq C_{pq}||f||_p$ for all $f \in L^p$.

In other words it is a bounded operator $L^p \to L^q$.

When proving convergence of $S_R f \to f$ in L^p we saw it was equivalent to $||S_R f||_p \leq C_p||f||_p$ i.e. equivalent to being strong (p,p). Also, independently of the dimension, S_R is never strong $(1,1)$ but it turns out that it is weak $(1,1)$.

Definition 2.7 *T is* **weak-**(p,q) *if and only if*

$$\mu\{x : |Tf| > \alpha\} \leq \left(\frac{C||f||_p}{\alpha}\right)^q$$

Lemma 2.8 *T is strong (p,q) implies that it is weak (p,q).*

Proof

$$\infty > C^q_{pq}||f||^q_p \geq ||Tf||^q_q = \int |Tf|^q d\mu \geq \int_{\{x:|Tf|>\alpha\}} |Tf|^q d\mu \geq \alpha^q \mu\{x : Tf > \alpha\}$$

Q.E.D.

Theorem 2.9 *Suppose that T_t is a fanily of operators indexed by t. T_t operators in L^p, we are interested in $\lim_{t\to t_0} T_t f$. Define the maximal operator by $T^\star f(x) = \sup_t |T_t f(x)|$. If $T^\star$ is weak-$(pq,)$ then the set $\{f \in L^p : \lim_{t\to t_0} T_t f(x) = f(x) a.e.\}$ is closed in L^p.*

Carlesson showed that $S_R f \to f$ by showing $S^\star$ is weak (p,q) and that the result is true for the Schwartz functions.

Proof Take a sequence $\{f_n\}$ with $f_n \in L^p$. Assume that $T_t f_n(x) \to f_n(x)$ a.e. Assume also that $f_n \to f$ in L^p. We need to show that $T_t f(x) \to f(x)$ a.e. We look at $\{x \in X : \limsup_{t\to t_0} |T_t f(x) - f(x)| > \lambda\}$ and we want to show that the measure of this set is zero for all $\lambda > 0$. This suffices since

$$\{x \in X : \lim T_t f(x) - f(x) \neq 0\} \subset \cup_1^\infty \{x \in X : \limsup_{t\to t_0} |T_t f(x) - f(x)| > \frac{1}{n}\}$$

and the right hand side has measure 0. Now

$$\begin{aligned}\mu\{x \in X : \limsup_{t\to t_0} |T_t f(x) - f(x)| > \lambda\} \\ &\leq \mu\{x : \limsup |T_t(f(x) - f_n(x)) + (f_n(x) - f(x))| > \lambda\} \\ &\leq \mu\{x : \limsup |T_t(f(x) - f_n(x))| > \frac{\lambda}{2}\} + \\ &\qquad + \mu\{x : \limsup (f_n(x) - f(x))| > \frac{\lambda}{2}\} \\ &\leq \mu\{x : |T^\star(f(x) - f_n(x))| > \frac{\lambda}{2}\} + \mu\{x : |f_n - f| > \frac{\lambda}{2}\} \\ &\leq \left(\frac{C\|f_n - f\|_p}{\lambda}\right)^q + \left(\frac{\|f_n - f\|_p}{\lambda}\right)^p\end{aligned}$$

and this is true for all n, and so the LHS is less than $\lim RHS = 0$ *Q.E.D.*

Corollary 2.10 *$\{T_t\}$ in L^p , $T^\star$ as above, and $T^\star$ weak (p,q) then*

$$\{f \in L^p : \lim T_t f(x) \text{ exists }\}$$

is closed

Proof Consider the set $\{x : |\limsup T_t f(x) - \liminf T_t f(x)| > \lambda\}$ and the part in the modulus is $2T^\star f(x)$. *Q.E.D.*

2.2 Marcinkiewicz Interpolation

Proposition 2.11 *Suppose that $\phi : [0,\infty) \to [0,\infty)$ is differentiable and increasing and $\phi(0) = 0$ Furthermore suppose $f : X \to \mathbb{R}$. Then*

$$\int_X \phi(|f(x)|)d\mu = \int_0^\infty \phi'(\lambda)\mu\{x \in X : |f(x)| > \lambda\}d\lambda$$

Proof

$$\begin{aligned}\int_X \phi(|f(x) &= \int_X \int_0^{|f(x)|} \phi'(\lambda)d\lambda d\mu \\ &= \int_X \int_0^\infty \phi'(\lambda)\chi_{\{0\leq\lambda\leq|f(x)|\}}(\lambda)d\lambda d\mu \\ &= \int_0^\infty \phi'(\lambda) \int_X \chi_{\{0\leq\lambda\leq|f(x)|\}}(\lambda)d\mu d\lambda \\ &= \int_0^\infty \phi'(\lambda)\mu\{x : |f(x)| \geq \lambda\}(\lambda)d\lambda\end{aligned}$$

Q.E.D.

Corollary 2.12 *$\phi(x) = x^p$ for $p \geq 1$ then $\int |f(x)|^p d\mu = \int_0^\infty p\lambda^{p-1}\mu\{|f| > \lambda\}d\lambda$*

Definition 2.13 *An operator T is* **sublinear** *if $|T(f+g)| \leq |Tf| + |Tg|$ and $|T(\alpha g)| = |\alpha||Tg|$ for $\alpha \in \mathbb{R}$.*

Theorem 2.14 (Marcinkiewicz) *Let (X, M, μ) be a measure space. Let T be a sublinear operator from $L^{p_0} + L^{p_1} \to L^{p_0} + L^{p_1}$ such that T is weak (p_0, p_0) and weak (p_1, p_1). Then T is strong (p,p) for $p_0 < p < p_1$.*

We have seen before another interpolation theorem, Riesz Thorin. This gives us boundedness from other boundedness. However, here for Marcinkiewicz you have much weaker assumptions.

Definition 2.15 $L^\infty_W = L^\infty$.

Proof We can assume that $p_0 < p_1$. We have two cases, $p_1 = \infty$ and $p_1 < \infty$. We consider the former:

We know that $||Tf||_\infty \le A_1||f||_\infty$ from weak (p_1, p_1) and we know from weak (p_0, p_0) that

$$\mu\{x : |Tf| > \lambda\} \le \left(\frac{C||f||_{p_0}}{\lambda}\right)^{p_0}$$

and we want to show that $||Tf||_p \le C||f||_p$. Recall the facts in above corollary 2.12 and the last property of proposition 2.2, as these will come in handy.

We split up f as follows, for a c to be chosen later.

$$f(x) = f(x)\chi_{\{x:|f(x)|\le c\lambda\}} + f(x)\chi_{\{x:|f(x)|>c\lambda\}} =: f_1(x) + f_0(x)$$

The former is clearly in L^∞ and the latter is in L^{p_0}, since

$$\begin{aligned}\int |f(x)\chi_{\{x:|f(x)|>c\lambda\}}|^{p_0} d\mu &= \int |\frac{f(x)}{c\lambda} c\lambda \chi_{\{x:|f(x)|>c\lambda\}}|^{p_0} d\mu \\ &= |c\lambda|^{p_0} \int \left(\frac{f(x)}{c\lambda}\right)^{p_0} \chi_{\{x:|f(x)|>c\lambda\}} d\mu \\ &\le |c\lambda|^{p_0} \int \left(\frac{f(x)}{c\lambda}\right)^{p} d\mu \\ &< \infty\end{aligned}$$

Now

$$\int |Tf|^p d\mu = \int_0^\infty p\lambda^{p-1}\mu\{|Tf| > \lambda\} d\lambda$$

and consider the set $\mu\{|Tf| > \lambda\}$. We have

$$\mu\{|Tf| > \lambda\} = \mu\{|T(f_0 + f_1)| > \lambda\} \le \mu\{|Tf_0| > \frac{\lambda}{2}\} + \mu\{|Tf_1| > \frac{\lambda}{2}\}$$

We claim that if $c = \frac{1}{2A_1}$ then $\mu\{|Tf_1| > \frac{\lambda}{2}\} = 0$. To show this, we know that $Tf_1(x) \le A_1||f_1||_\infty$ for a.e. x since $||Tf||_\infty \le A_1||f||_\infty$. Thus

$$Tf_1(x) \le A_1||f\chi_{\{x:|f(x)|\le c\lambda\}}||_\infty \le A_1 c\lambda \le \frac{\lambda}{2}$$

a.e. and the claim is shown.

Now

$$\begin{aligned}
\int |Tf|^p d\mu &= \int_0^\infty p\lambda^{p-1}\mu\{|Tf| > \lambda\}d\lambda \\
&\le \int_0^\infty p\lambda^{p-1}\mu\{|Tf_0| > \frac{\lambda}{2}\}d\lambda \\
&\le \int_0^\infty p\lambda^{p-1}\left(\frac{C||f_0||_{p_0}}{\lambda}\right)^{p_0} d\lambda \\
&= pC^{p_0}\int_0^\infty ||f_0||_{p_0}^{p_0}\lambda^{p-1-p_0}d\lambda \\
& pC^{p_0}\int_0^\infty \lambda^{p-1-p_0}\int_X |f|^{p_0}\chi_{\{|f|>\frac{\lambda}{2A_1}\}}d\mu d\lambda \\
&= pC^{p_0}\int_X |f(x)|^{p_0}\int_0^\infty \lambda^{p-1-p_0}\chi_{\{|f|>\frac{\lambda}{2A_1}\}}d\lambda d\mu \\
&= pC^{p_0}\int_X |f(x)|^{p_0}\int_0^{2A_1|f(x)|} \lambda^{p-1-p_0}d\lambda d\mu \\
& pC^{p_0}\int_X |f(x)|^{p_0}\frac{1}{p-p_0}(2A_1|f(x)|)^{p-p_0}d\mu \\
&= \frac{pC^{p_0}(2A_1)^{p-p_0}}{p-p_0}\int_X |f(x)|^p d\mu
\end{aligned}$$

as we wanted.

We now take $p_1 < \infty$. We take the same decomposition of f as before, namely

$$f(x) = f(x)\chi_{\{x:|f(x)|\le c\lambda\}} + f(x)\chi_{\{x:|f(x)|>c\lambda\}} =: f_1(x) + f_0(x)$$

and then we claim that $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$ and the proof of this is left to the reader (it is essentially the same as before).

Then

$$\begin{aligned}
\int |Tf|^p d\mu &= \int_0^\infty p\lambda^{p-1}\mu\{|Tf| > \lambda\}d\lambda \\
&= \int_0^\infty p\lambda^{p-1}\mu\{|T(f_0+f_1)| > \lambda\}d\lambda \\
&\le \int_0^\infty p\lambda^{p-1}\mu\{|T(f_0)| > \frac{\lambda}{2}\}d\lambda + \int_0^\infty p\lambda^{p-1}\mu\{|T(f_1)| > \frac{\lambda}{2}\}d\lambda \\
&= \int_0^\infty p\lambda^{p-1}\left(\frac{C_0||f_0||_{p_0}}{\lambda}\right)^{p_0} d\lambda + \int_0^\infty p\lambda^{p-1}\left(\frac{C_1||f_1||_{p_1}}{\lambda}\right)^{p_1} d\lambda \\
&= \int_0^\infty pC_0^{p_0}\lambda^{p-1-p_0}\int |f_0|^{p_0}d\mu d\lambda + \int_0^\infty pC_1^{p_1}\lambda^{p-1-p_1}\int |f_1|^{p_1}d\mu d\lambda \\
&= \int_0^\infty pC_0^{p_0}\lambda^{p-1-p_0}\int |f(x)\chi_{\{x:|f(x)|>c\lambda\}}|^{p_0}d\mu d\lambda + \\
&\qquad + \int_0^\infty pc_1^{p_1}\lambda^{p-1-p_1}\int_X |f(x)\chi_{\{x:|f(x)|\le c\lambda\}}|^{p_1}d\mu d\lambda
\end{aligned}$$

$$= \int_X pC_0^{p_0} \int_0^\infty \lambda^{p-1-p_0} |f|^{p_0} \chi_{\{x:|f(x)|>c\lambda\}} d\lambda d\mu +$$
$$+ \int_X pC_1^{p_1} \int_0^\infty \lambda^{p-1-p_1} |f|^{p_1} \chi_{\{x:|f(x)|\leq c\lambda\}} d\lambda d\mu$$
$$\leq \int_X pC_0^{p_0} |f|^{p_0} \int_0^{|f|/c} \lambda^{p-1-p_0} d\lambda d\mu + \int_X pC_1^{p_1} |f|^{p_1} \int_{|f|/c}^\infty \lambda^{p-p_1-1} d\lambda d\mu$$
$$= \frac{pC_0^{p_0}}{(p-p_0)c^{p-p_0}} \int_X |f|^p d\mu + |\frac{C_1^{p_1}}{(p-p_1)c^{p-p_1}}| \int_X |f|^p d\mu$$
$$\leq K \int_X |f|^p d$$

where K is some crazy ugly constant. *Q.E.D.*

2.3 Hardy Littlewood Maximal Functions

We first set some notation. B_r is the ball centred at 0 with radius r, and $B_r(x)$ is the ball centred at x with radius r. $|B_r|$ is the volume of the ball.

Definition 2.16 *The H-L maximal function is defined to be*

$$Mf(x) := \sup_r \frac{1}{|B_r|} \int_{B_r} |f(x-y)| dy$$

The following are equivalent ways to write it:

$$\begin{aligned} Mf(x) :&= \sup_r \frac{1}{|B_r|} \int_{B_r} |f(x-y)| dy \\ &= \sup_r \frac{1}{|B_r|} \int_{B_r(x)} |f(y)| dy \\ &= \sup_{r>0} \int |f(x-y)| \frac{1}{|B_r|} \chi_{B_r}(y) dy \\ &= \sup_{r>0} \int |f(x-y)| \frac{1}{cr^n} \chi_{B_r}(y) dy \\ &= \sup_{r>0} \int |f(x-y)| \frac{1}{cr^n} \chi_{B_1}(\frac{y}{r}) dy \end{aligned}$$

and then if we define $\phi(y) = \frac{1}{|B_1|}\chi_{B_1}(y)$ we have that

$$Mf(x) = \sup_{r>0} \int |f(x-y)| \phi_r(y) dy = \sup_{r>0} |f| \star \phi_r(x)$$

Observe that it is possible to replace B_r by cubes Q_r (centre zero, of side 2r). Then we could define

$$M_Q f(x) = \sup_{r>0} \frac{1}{|Q_r|} \int_{Q_r} |f(x-y)| dy$$

and we claim that there exist $a, A \geq 0$ such that

$$aM_Q f(x) \leq Mf(x) \leq AM_Q f(x)$$

with a and A independent of f but dependent on the dimension. If we let $|B_r| = cr^n$ and $|Q_R| = qR^n$ then we have

$$\frac{1}{cr^n} \int_{B_r} |f(x-y)| dy \leq \frac{1}{cr^n} \int_{Q_R} |f(x-y)| dy \leq \frac{qR^n}{cr^n} \frac{1}{qR^n} \int_{Q_R} |f(x-y)| dy$$

where we choose Q_R containing B_r. In $\mathbb{R}^n$ with the Lebesgue measure, for a given r we can take R to be a multiple of r, and so $\frac{qR^n}{cr^n}$ is independent of r.

We observe that the cubes or balls need not be centred at the origin.

Definition 2.17

$$\tilde{M}f(x) = \sup_{\textit{all balls s.t. } x\in B} \frac{1}{|B|}\int_B |f(y)|dy$$

We then claim that there are $\tilde{a}, \tilde{A}$ such that $\tilde{a}\tilde{M}f(x) \leq Mf(x) \leq \tilde{A}\tilde{M}f(x)$

This doesnt work for any sets, note the Architects paradox. Suppose we are in $[0,1]\times[0,1]$. Then there exists a set $A\subset[0,1]\times[0,1]$ such that for all $x\in A$, there exists a ray emanating from x which does not belong to A and A has measure 1.

Proposition 2.18 *Let ϕ be a positive radial and decreasing function, and $\phi\in L^1$. Then*

$$\sup_{t>0}|\phi_t \star f(x)| \leq \|\phi\|_1 Mf(x)$$

Note that this is pointwise.

Proof Write $\phi(x) = f(|x|)$ and approximate by simple functions, so define

$$\phi_n(x) = \sum_{j=1}^n a_j \chi_{B_{r_j}}(x)$$

with $a_j > 0$.

We claim that given ϕ there exists a sequence ϕ_n of the above form so that $\phi_n \to \phi$ a.e.

We then show that $|\phi_t \star f(x)| \leq \|\phi_t\|_1 Mf(x) \leq \|\phi\|_1 Mf(x)$ and it is enough to show for some t due to scaling. We take $t = 1$ for simplicity. Then

$$\phi_n \star f(x) = \left(\sum_1^n a_j \chi_{B_{r_j}}(\cdot)\star f\right)(x)$$

and also

$$Mf(x) = \sup_{r>0}\left(\frac{1}{|B_1|}\chi_{B_1}\right)\star|f|(x)$$

and we have

$$\phi_n \star f(x) = \left(\sum_1^n a_j \frac{|B_{r_j}|}{|B_{r_j}|}\chi_{B_{r_j}}(\cdot)\star f\right)(x)$$

and so

$$\begin{aligned}
|\phi_n \star f(x)| &\leq \left(\sum_1^n a_j \frac{|B_{r_j}|}{|B_{r_j}|}\chi_{B_{r_j}}(\cdot)\star |f|\right)(x)\\
&\leq \left(\sum_1^n a_j |B_{r_j}| \sup_r \frac{1}{|B_r|}\chi_{B_r}(\cdot)\star|f|\right)(x)\\
&= \left(\sum_1^n a_j|B_{r_j}|\right)Mf(x)\\
&= Mf(x)\|\phi_n\|_1\\
&\leq Mf(x)\|\phi\|_1
\end{aligned}$$

and sending $n\to\infty$ we get $|\phi\star f(x)| \leq Mf(x)\|\phi\|_1$ as required. *Q.E.D.*

Observe that the given ϕ needs almost no restrictions. It suffices that if there exists a ψ that it positive, radial and decreasing such that $|\phi(x)| \leq \psi(x)$ then the result from the previous proposition holds.

Theorem 2.19 *Mf is weak-(1,1) and strong-(p,p) for $1 < p \leq \infty$*

Proof Trivially true for strong (∞, ∞), and we show later for weak $(1,1)$. Then Marcinkiewicz implies the rest. *Q.E.D.*

Lemma 2.20 (Vitali Covering) *Let $E \subset \mathbb{R}^n$ be measurable wrt the Lebesgue measure m. Assume that E is covered by a family of balls $\{B_\alpha\}_{\alpha \in \Lambda}$ (of bounded diameter) and Λ not necessarily countable. Then there exists a pairwise disjoint subset $\{B_j\}$ and a constant c such that*

$$\sum_{j=1}^{\infty} m(B_j) \geq cm(E)$$

Proof Let $R = \sup_{\alpha \in \Lambda}\{\text{diam}(B_\alpha)\}$. Choose B_1 to be any ball such that $\text{diam}(B_1) > \frac{1}{2}R$.

Assume that we have chosen $B_1, ..., B_k$. Then choose B_{k+1} to be any ball disjoint with $B_1, ..., B_k$ such that

$$\text{diam}(B_{k+1}) \geq \frac{1}{2}\sup\{\text{diam}(B_\alpha) : B_\alpha \text{ disjoint with } B_1, ..., B_k\}$$

if this is possible.

We thus obtain $\{B_j\}$ which is countable or finite. We have two possibilities

$$\sum m(B_j) = \infty \qquad\qquad \sum m(B_j) < \infty$$

If the former, then there is nothing to prove. We thus assume the latter. Then

$$\sum_1^\infty m(B_j) < \infty \implies \text{diam}(B_j) \to 0$$

We define $B_j^\star$ to be the ball with the same centre as B_j but five times the diameter. Thus $m(B_j^\star) = 5^n m(B_j)$.

We want to show that $\cup B_j^\star \supset E$ because then

$$5^n \sum m(B_j) = \sum m(B_j^\star) \geq m(E)$$

and this would conclude the proof. We show this by showing that $B_\alpha \subset \cup B_j^\star$ for all $\alpha \in \Lambda$, which gives the result.

We argue by contradiction, and so we assume that there exists an α such that B_α is not contained in $\cup B_j^\star$. Then pick k to be the first integer such that $\text{diam}(B_{k+1}) < \frac{1}{2}\text{diam}(B_\alpha)$. Then B_α must intersect at least one of the B_js, else we would have it in the collection.

Let B_{j_0} be the first one which it intersects. Now $j_0 \leq k$ because if not then B_α is disjoint with $B_1, ..., B_k$ and so it is the suitable candidate when we choose B_{k+1}, in other words we would have chosen B_α instead of B_{k+1}, due to our assumption $\text{diam}(B_{k+1}) < \frac{1}{2}\text{diam}(B_\alpha)$.

We claim that $B_\alpha \subset B_{j_0}^\star$. We have that $\text{diam}(B_\alpha)$ and $\text{diam}(B_{j_0})$ are comparable. When we chose B_{j_0} we made sure that $\text{diam}(B_{j_0})$ was greater than or equal to half the supremum of the diameters of the remaining disjoint balls. In particular $\frac{1}{2}\text{diam}(B_\alpha) \leq \text{diam}(B_{j_0})$ and so $B_\alpha \subset B_{j_0}^\star$. *Q.E.D.*

The proof that Mf is weak-$(1,1)$ is still to do, and we do so below. It is an application of Vitali's theorem, although initially you wouldn't expect that; at the least I didn't.

Proof (Mf **is weak-**$(1,1)$) We want $m\{x : Mf(x) > \alpha\} \leq C/\alpha$. Define $E_\alpha = \{x : Mf(x) > \alpha\}$. If $x \in E_\alpha$ then

$$Mf = \sup_r \frac{1}{|B_r|}\int_{B_r(x)} |f(y)|dy > \alpha \implies \exists r(x) \text{ such that } \frac{1}{|B_{r(x)}|}\int_{B_{r(x)}(x)} |f(y)|dy > \alpha$$

and thus $E_\alpha \subset \cup_{x\in E_\alpha} B_{r(x)}(x)$.

We can apply Vitali's lemma to find $\{B_j\}$ such that $\sum m(B_j) \geq cm(E_\alpha)$ and $\{B_j\}$ pairwise disjoint. Notice that

$$|B_{r(x)}| \leq \frac{1}{\alpha}\int_{B_{r(x)}(x)} |f(y)|dy$$

In general, for any disjoint collection of balls $B_{r(x)}(x)$ we have

$$\left|\dot{\bigcup_{k=1}^{\infty}} B_{r(x_k)}(x_k)\right| \leq \sum_{k=1}^{\infty}\frac{1}{\alpha}\int_{B_{r(x_k)}(x_k)}|f(y)|dy = \frac{1}{\alpha}\int_{\dot\cup B_{r(x_k)}(x_k)}|f(y)|dy \leq \frac{\|f\|_1}{\alpha}$$

Then we have that

$$m(E_\alpha) \leq 5^n\sum_1^\infty m(B_j) \leq 5^n m(\dot\bigcup B_{r(x)}(x)) \leq 5^n\frac{\|f\|_1}{\alpha}$$

Q.E.D.

Corollary 2.21 *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $|\phi(x)| \leq \psi(x)$ for some positive ψ that is radial and decreasing. Then*

$$\lim_{t\to 0}\phi_t \star f(x) = \left(\int \phi(x)dx\right) f(x) \ \ a.e.$$

Proof We have seen that if $\sup_t \phi_t \star f$ is weak (p,p) then the set

$$\{f \in L^p : \lim_{t\to 0}\phi_t \star f = f \text{ a.e. }\}$$

is closed. Also we know that

$$\sup_{t>0}|\phi_t \star f(x)| \leq \|\phi_1\| Mf(x)$$

and also if $Mf(x)$ is weak (p,p) for all p then we have $\sup_{t>0}|\phi_t \star f(x)|$ is weak (p,p) for $f \in \mathcal{S}$. Since $\mathcal{S}$ is dense in L^p and the set of functions $\{f \in L^p : \lim_{t\to 0}\phi_t \star f = f \text{ a.e. }\}$ is closed and contains $\mathcal{S}$ it must be L^p *Q.E.D.*

Corollary 2.22 *This applies to Gauss-Weierstrass, Abel-Poisson and Cesaro, but not to $S_R f$*

Proof The functions ϕ in GW and AP in the expressions are

$$P_t = \frac{ct}{(t^2+|x|^2)^{\frac{n+1}{2}}}$$

and

$$W_t = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$$

and these are radial, decreasing and positive. For Cesaro, the function ϕ is less that the function $\psi(x) = \begin{cases} 1 & x \leq 1 \\ \frac{1}{\pi^2 z^2} & x \geq 1 \end{cases}$ *Q.E.D.*

2.4 Dyadic Maximal functions

This is an effort to make everything more computable, or manageable.

In $\mathbb{R}$, if O is an open set then $O = \dot\bigcup I_\alpha$, a union of disjoint open intervals, but in $\mathbb{R}^n$ one cannot write an open set in this manner.

We denote by Q the set $[0,1)^n = \underbrace{[0,1) \times ... \times [0,1)}_{n \text{ times}}$ and $\mathcal{Q}_0$ is the set of all cubes congruent to Q with vertices in $\mathbb{Z}^n$. We let $Q_k = \frac{1}{2^k}Q$ the "shrunk " cube for $k \in \mathbb{Z}$ and $\mathcal{Q}_k$ the set of all cubes congruent to Q_k with vertices on $(\frac{1}{2^k}\mathbb{Z})^n$

We remark that for all $x \in \mathbb{R}^n$ there is a unique cube in $\mathcal{Q}_k$ that contains x, i.e. $\mathbb{R}^n = \dot\bigcup_{Q_k \in \mathcal{Q}_k} Q_k$

Also note that any two dyadic cubes (allowing different generations) are either disjoint or one is contained in the other. Also every dyadic cube in $\mathcal{Q}_k$ is contained in a unique cube of the previous generation and itself contains 2^n cubes of the next.

Definition 2.23

$$E_k f(x) = \sum_{Q \in \mathcal{Q}_k} \frac{1}{|Q|} \int_Q f(y) dy \chi_Q(x)$$

Observe that

$$\int E_k f(x) dx = \int f(x) dx$$

and also that

$$\int \sum_{\mathcal{Q}_k} \frac{1}{|Q|} \int f(y) dy \chi_Q(x) dx = \sum_{\mathcal{Q}_k} \int \frac{1}{|Q|} \chi_Q(x) dx \int_Q f(y) dy = \int_{\bigcup Q} f(x) dx$$

Definition 2.24 *The* **Dyadic maximal function** *is defined to be*

$$M_d f(x) = \sup_k |E_k f(x)| = \sup_{\substack{Q_k \in \mathcal{Q}_k \\ x \in Q_k \\ k \in \mathbb{Z}}} \left| \frac{1}{|Q_k|} \int_{Q_k} f(y) dy \right|$$

Observe that in the above supremum there is only one Q_k for each generation.

Theorem 2.25 *1. $M_d f$ is weak-(1,1) and strong-(p,p) for $1 < p \le \infty$*

2. if $f \in L^1_{loc}$ then

$$\lim_{k \to \infty} E_k f(x) = f(x) \ a.e.$$

Proof (an example of Calderon-Zygmund decomposition) Without loss of generality we can assume that $f \ge 0$. We then need to show that

$$|\{x \in \mathbb{R}^n : M_d f(x) > \alpha\}| \le \frac{c}{\alpha}$$

We first claim that

$$\{x \in \mathbb{R}^n : M_d f(x) > \alpha\} = \cup_{k \in \mathbb{Z}} \Omega_k$$

for some sets Ω_k. Define $\Omega_k = \{x \in \mathbb{R}^n : E_k f(x) > \alpha \text{ and } E_j f(x) \le \alpha \forall j < k\}$. Suppose that $x \in \{x \in \mathbb{R}^n : M_d f(x) > \alpha\}$ for fixed α. Then $f \in L^1$ implies that

$$\frac{1}{|Q_k|} \int_{Q_k} f dx \to 0 \text{ as } k \to -\infty$$

as it is bounded above by $\frac{1}{|Q_k|}\|f\|_1$. Thus for a α fixed there exists a K_0 such that

$$\frac{1}{|Q_j|}\int_{Q_j} f dx \leq \alpha$$

for all $j < K_0$. This implies that $\Omega_j = \emptyset$ for all $j \leq K$.

Once we have defined Ω_j for $j \leq K_0$ we define the rest inductively.

For the claim, if $x \in \{x \in \mathbb{R}^n : M_d f(x) > \alpha\}$ then there exists an index k such that $E_k f(x) > \alpha$. We know this set of indices is bounded below by K_0. Then trivially $x \in \Omega_{K_0}$. This shows the inclusion

$$\{x \in \mathbb{R}^n : M_d f(x) > \alpha\} \subset \cup_{k\in\mathbb{Z}}\Omega_k$$

To show the other inclusion, if $x \in \Omega_k$ then $E_k f(x) > \alpha$ and so $M_d f(x) > \alpha$ so $x \in \{x \in \mathbb{R}^n : M_d f(x) > \alpha\}$

Observe that Ω_k are pairwise disjoint.

Now if $x \in \Omega_k$ we have $E_k f(x) > \alpha$ and if $x \in Q_k$ for Q_k a dyadic cube then

$$E_k f(x) = \frac{1}{|Q_k|}\int_{Q_k} f(y)dy > \alpha$$

and so

$$|Q_k| \leq \frac{\int f(y)dy}{\alpha}$$

Each Ω_k is a union of dyadic cubes and so

$$|\Omega_k| \leq \frac{\int_{Q_k} f(y)dy}{\alpha}$$

To conclude

$$|\{x \in \mathbb{R}^n : M_d f(x) > \alpha\}| = |\dot{\bigcup} Q_k| = \sum_{k\geq K_0} |\Omega_k| \leq \frac{1}{\alpha}\sum\int_{\Omega_k} f(y)dy \leq \frac{1}{\alpha}\int_{\mathbb{R}^n} f(y)dy \leq \frac{\|f\|_1}{\alpha}$$

For the second part, if $f \in L^1_{loc}$ then $\lim_{k\to\infty} E_k f(x) = f(x)$ and if $x \in Q_k$ then

$$E_k f(x) = \frac{1}{|Q_k|}\int_{Q_k} f(y)dy$$

and given $\{E_k f(x)\}$ we have $M_d f$ is the maximal operator associated to them by definition. We have seen that if $M_d f(x)$ is weak (p, q) then the following set

$$\{f \in L^p : \lim_{k\to\infty} E_k f(x) = f(x) \text{ a.e. }\}$$

is a closed set. Moreover the result is trivially true for $\mathcal{S}$ and so the result is true for L^p. *Q.E.D.*

Note that if $f \geq 0$ then

$$|\{x : M_Q f(x) > 4^n\alpha\}| \leq 2^n|\{x : M_d f(x) > \alpha\}|$$

$$|\{x : M_Q f(x) > \lambda\}| \leq 2^n|\{x : M_d f(x) > \frac{\lambda}{4^n}\}|$$

Corollary 2.26 (Lebesgue Differentiation Theorem) *Suppose that* $f \in L^1_{loc}$ *then*

$$\lim_{|B_r| \to 0} \frac{1}{|B_r|} \int_{B_r} f(y) dy = f(x) \ \textit{a.e.}$$

Corollary 2.27 *Suppose that* $f \in L^1_{loc}$ *then*

$$\lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r} |f(x-y) - f(x)| dy = 0 \ \textit{a.e.}$$

Proof (sketch) Let $T_r f(x) = \frac{1}{|B_r|} \int_{B_r} |f(x-y) - f(x)| dy$ and also $T^\star = \sup_r T_r$. Then $\{f \in L^p : \lim_{r \to r_0} T_r f(x)\}$ is closed provided $T^\star$ is weak (p,q) for some p and q. The limit exists and is trivially zero for $f \in C^0$ or $\mathcal{S}$. Then $T^\star$ is weak $(1,1)$ and

$$\begin{aligned}
|T_r f(x)| &\le \frac{1}{|B_r|} \int_{B_r} |f(x-y)| dy + |f(x)| \\
&\le \sup_r \frac{1}{|B_r|} \int_{B_r} |f(x-y)| dy + |Mf(x)| \\
&\le 2Mf(x)
\end{aligned}$$

and so $T^\star f \le 2Mf$. *Q.E.D.*

3 Hilbert Transform

We saw before the Poisson kernel $u(x,t) = P_t \star f(x)$ where $P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}$ in $\mathbb{R}^n$ with $c_1 = \frac{1}{\pi}$. This function has fourier transform $\hat{u} = e^{-2\pi t|\xi|} \hat{f}(\xi)$. Now for $n = 1$ we have

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}} e^{-2\pi t|\xi|} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \\
&= \int_0^\infty e^{-2\pi t \xi} \hat{f}(\xi) e^{2\pi i x \xi} d\xi + \int_{-\infty}^0 e^{2\pi t \xi} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \\
&= \int_0^\infty \hat{f}(\xi) e^{2\pi i (x+it)\xi} d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i (x-it)\xi} d\xi
\end{aligned}$$

and if we rewrite this with $z = x + it$ we get

$$u(z) = \int_0^\infty \hat{f}(\xi) e^{2\pi i z \xi} d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \xi} d\xi$$

and then if we define the function $v(z)$ by

$$iv(z) = \int_0^\infty \hat{f}(\xi) e^{2\pi i z \xi} d\xi - \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \xi} d\xi$$

and we can then write

$$v(x,t) = -i \int \operatorname{sgn}(\xi) e^{-2\pi t|\xi|} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

In exactly the same form that u defined above can be written as a convolution with P_t we can also write v as a convolution with Q_t where

$$Q_t(x) = \frac{1}{\pi} \frac{x}{x^2 + t^2}$$

and note that $\hat{Q}_t(\xi) = -i\mathrm{sgn}(\xi)e^{-2\pi t|\xi|}$. Q_t is called the conjugate Poisson kernel. Observe that

$$P_t(x) + iQ_t(x) = \frac{1}{\pi}\left(\frac{t}{x^2+t^2} + \frac{x}{x^2+t^2}\right) = \frac{1}{\pi}\frac{i\bar{z}}{z\bar{z}} = \frac{i}{\pi}\frac{1}{z}$$

which is an analytic function for $\mathrm{Im}z > 0$.

Now $P_t f \to f$ because P_t is an approximation to the identity. But what happens to $Q_t \star f(x)$? If we write formally, we see that

$$Q_0 \star f(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{y} f(x-y)dy$$

and this expression is not defined, even for $f \in \mathcal{S}$. The problem is not at ∞, it is at the origin. In some sense, $\lim_{t\to 0} Q_t = \frac{1}{\pi x}$ and is it possible to make sense of $\frac{1}{\pi x} \star f$?

Definition 3.1 *We define the* **principle value** *to be, for $\phi \in \mathcal{S}$, (or $\phi \in C_c^\infty$)*

$$\text{p.v.}\frac{1}{x}(\phi) = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\phi(x)}{x}dx$$

and we claim that this is well defined. Note first that $\int_{1>|x|>\varepsilon}\frac{1}{x}dx = 0$, since it is symmetric about the origin. Then

$$\int_{|x|>\varepsilon}\frac{\phi(x)}{x}dx = \int_{1>|x|>\varepsilon}\frac{\phi(x)}{x}dx + \int_{|x|>1}\frac{\phi(x)}{x}dx$$

The latter is independent of 1 and $|\phi| \le \frac{1}{1+|x|^{1000000}}$ as it is Schwartz or with compact support. The former is

$$\int_{1>|x|>\varepsilon}\frac{\phi(x)}{x}dx = \int_{1>|x|>\varepsilon}\frac{\phi(x)}{x}dx - \phi(0)\int_{1>|x|>\varepsilon}\frac{1}{x}dx = \int_{1>|x|>\varepsilon}\frac{\phi(x)-\phi(0)}{x}dx$$

and if ϕ has one derivative then the integrand is less than the infinity norm of the derivative, and since the domain of integration is compact, we have it bounded.

Proposition 3.2

$$\lim_{t\to 0} Q_t = \frac{1}{\pi}\text{p.v.}\frac{1}{x}$$

where the limit is understood in the sense of distributions.

Proof For all $\varepsilon > 0$ let $\psi_\varepsilon(x) = \frac{1}{x}\chi_{\{|x|>\varepsilon\}}(x)$ and then $\text{p.v.}\frac{1}{x}(\phi) = \lim_{\varepsilon\to 0}\psi_\varepsilon(x)(\phi)$ where we think of the function ψ_ε as a distribution by

$$\psi_\varepsilon(x)(\phi) = \int \psi_\varepsilon(x)\phi(x)dx$$

Then

$$\text{p.v.}\frac{1}{x}(\phi) = \lim_{\varepsilon\to 0}\int \psi_\varepsilon(x)\phi(x)dx = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\phi(x)}{x}dx$$

and we want to show that $\lim_{t\to 0}[Q_t(\phi) - \frac{1}{\pi}\text{p.v.}\frac{1}{x}(\phi)] = 0$. We have

$$\begin{aligned}
\lim_{t\to 0}[Q_t(\phi) - \frac{1}{\pi}\text{p.v.}\frac{1}{x}(\phi)] &= \lim_{t\to 0}[Q_t(\phi) - \frac{1}{\pi}\int \psi_t(x)\phi(x)dx] \\
&= \lim_{t\to 0}\left[\int \frac{x}{x^2+t^2}\phi(x)dx - \frac{1}{\pi}\int \psi_t(x)\phi(x)dx\right] \\
&= \lim_{t\to 0}\left[\int \frac{x}{x^2+t^2}\phi(x)dx - \frac{1}{\pi}\int_{|x|>t} \frac{1}{x}\phi(x)dx\right] \\
&= \lim_{t\to 0}\left[\int_{|x|<t} \frac{x}{x^2+t^2}\phi(x)dx + \frac{1}{\pi}\int_{|x|>t} \left(\frac{x}{x^2+t^2} - \frac{1}{x}\right)\phi(x)dx\right] \\
&= \lim_{t\to 0}\left[\int_{|y|<1} t\frac{ty}{t^2y+t^2}\phi(ty)dy + \int_{|y|>1} \left(\frac{ty}{t^2y+t^2} - \frac{1}{ty}\right)\phi(ty)tdy\right] \\
&= \lim_{t\to 0}\left[\int_{|y|<1} \frac{y}{y+1}\phi(ty)dy + \int_{|y|>1} \left(\frac{y}{y+1} - \frac{1}{y}\right)\phi(ty)dy\right] \\
&= 0
\end{aligned}$$

since we can take the limit inside using DCT and then we have integrals of odd functions over symmetric domains. *Q.E.D.*

Note that $e^{inx} \to 0$ in the same sense. This is Riemann-Lebesgue.

Corollary 3.3

$$\lim_{t\to 0} Q_t \star f(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}dy$$

Corollary 3.4

$$\widehat{(\text{p.v.}\frac{1}{x})}(\xi) = -i\text{sgn}(\xi)$$

Proof We only give a sketch

$$(\widehat{Q_t \star \phi})(\xi) = -i\text{sgn}(\xi)e^{-2\pi t|\xi|}\hat{\phi}(\xi)$$

and also

$$\frac{1}{\pi}(\widehat{\text{p.v.}\frac{1}{\cdot} \star \phi})(\xi) = \widehat{(\text{p.v.}\frac{1}{x})}(\xi)\hat{\phi}(\xi)$$

and since

$$\lim_{t\to 0}(\widehat{Q_t \star \phi})(\xi) = \frac{1}{\pi}(\widehat{\text{p.v.}\frac{1}{\cdot} \star \phi})(\xi)$$

we get

$$-i\text{sgn}(\xi)\hat{\phi}(\xi) = \widehat{(\text{p.v.}\frac{1}{x})}(\xi)\hat{\phi}(\xi)$$

as required *Q.E.D.*

Definition 3.5 (Hilbert Transform)

$$Hf(x) := \frac{1}{\pi}\text{p.v.}\int \frac{f(x-y)}{y}dy = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}dy$$

We could have alternatively defined this as

$$Hf(x) = \lim_{t\to 0} Q_t \star f(x)$$

or as

$$\widehat{Hf}(\xi) = -i\text{sgn}(\xi)\hat{f}(\xi)$$

Proposition 3.6

$$||Hf||_2 = ||f||_2$$

Proof

$$||Hf||_2 = ||\hat{Hf}||_2 = ||-i\text{sgn}(\xi)\hat{f}(\xi)||_{L^2_\xi} = ||\hat{f}||_2 = ||f||_2$$

Q.E.D.

and thus we have that H is strong (2,2).

Proposition 3.7

$$H(Hf) = -f$$

and also

$$\int fHg = -\int (Hf)g$$

The proof of this uses the Fourier transform definition. Due to the propositions here, H can be extended to L^2. We will see that H can be extended to L^p for $1 \le p < \infty$.

Theorem 3.8 *H can be extended to $f \in L^p$ for $1 \le p < \infty$ and furthermore*

1. *(Kolmogorov) H is weak (1,1), that is*

$$|\{x \in \mathbb{R} : |Hf(x)| > \alpha\}| \le \frac{C||f||_1}{\alpha}$$

2. *(M. Riesz) H is strong (p,p) for $1 < p < \infty$, i.e. there exists a c_p such that*

$$||Hf||_p \le c_p||f||_p$$

Theorem 3.9 (Calderón-Zygmund Decomposition) *Let $f \in L^1(\mathbb{R}^n)$ and $f \ge 0$ and fix $\alpha > 0$. Then*

1. *$\mathbb{R}^n = F \cup \Omega$*
2. *$f(x) \le \alpha$ for a.e. $x \in F$*
3. *Ω is a union of cubes $\Omega = \cup Q_k$ where Q_k have disjoint interior and*

$$\alpha < \frac{1}{|Q_k|}\int_{Q_k} f(y)dy \le 2^n\alpha$$

Observe that $|Q_k| < \frac{1}{\alpha}\int_{Q_k} f(y)dy$ and

$$|\Omega| = |\cup Q_k| < \frac{1}{\alpha}\int_{\cup Q_k} f(y)dy \le \frac{||f||_1}{\alpha}$$

Proof Given α, since $f \in L^1$ there exists m such that

$$\frac{1}{(2^m)^n}\int_{\mathbb{R}^n} f(y)dy < \alpha$$

and this implies that if Q is a dyadic cube of side 2^m then $\frac{1}{|Q|}\int_Q f dy < \alpha$.

Consider the family of dyadic cubes of side 2^m. Take a cube in this collection, and bisect every side. Let Q' be one of the resulting 2^n cubes. We have two options

$$\frac{1}{|Q'|}\int_{Q'} f(y)dy \leq \alpha \qquad\qquad \frac{1}{|Q'|}\int_{Q'} f(y)dy > \alpha$$

If in the latter then we keep Q' for our collection. Then we have

$$\alpha < \frac{1}{|Q'|}\int_{Q'} f(y)dy \leq \frac{|Q|}{|Q'|}\frac{1}{|Q'|}\int_{Q} f(y)dy = 2^n\alpha$$

If Q' satisfies the latter, we bisect every side and look at every sub-cube. Iterate this procedure. We thus gain Ω as a disjoint union of such cubes. We define $F := \Omega^c$. We are left to check property 2.

Let $x \in F$ so that x is in one cube from every generation of dyadic cubes. Then we have $\frac{1}{|Q|}\int_Q f(y)dy \leq \alpha$ for every dyadic cube containing x. We have a family of cubes $\{R_j\}$ such that if $x \in R_j$ then $\frac{1}{|R_j|}\int_{R_j} f(y)dy \leq \alpha$. Lebesgue's differentiation thereom gives that

$$\lim_{r\to 0}\frac{1}{|Q_r|}\int_{Q_r} f(y)dy = f(x) \text{ a.e.}$$

and if $x \in Q_r$ for all r and $|Q_r| = cr^n$ then we have

$$f(x) = \lim_{j\to\infty}\frac{1}{|R_j|}\int_{R_j} f(y)dy \leq \alpha \text{ a.e.}$$

Q.E.D.

Proof (of theorem 3.8) We prove this in $\mathbb{R}$. Suppose $f \geq 0$. From Calderon-Zygmund decomposition we write $\mathbb{R} = F \cup \Omega$ where $\Omega = \cup I_j$ with disjoint interior and

$$\alpha < \frac{1}{|I_j|}\int_{I_j} f dx \leq 2\alpha$$

and $|\Omega| \leq \frac{1}{\alpha}||f||_1$.

Decompose f into a good part and a bad part, so $f = g + b$ where

$$g(x) = \begin{cases} f(x) & x \in \Omega^C \\ \frac{1}{|I_j|}\int_{I_j} f dx & x \in I_j \end{cases}$$

and $b(x)$ is defined as whatever it has to be, i.e. $b(x) = f(x) - g(x)$. We think of b as $b(x) = \sum_{j=1}^{\infty} b_j(x)$ with

$$b_j(x) = \left(f(x) - \frac{1}{|I_j|}\int_{I_j} f dx\right)\chi_{I_j}(x)$$

We need to study $|\{x : |Hf| > \alpha\}|$ and show it is $\leq \frac{c}{\alpha}||f||_1$. We have

$$\{x : |Hf| > \alpha\} \subset \{x : |Hg| > \frac{\alpha}{2}\} \cup \{x : |Hb| > \frac{\alpha}{2}\}$$

Now

$$\left(\frac{2}{\alpha}\right)^2\int |Hg|^2 \geq \left(\frac{2}{\alpha}\right)^2\int_{\{x:|Hg|>\frac{\alpha}{2}\}} |Hg|^2 \geq \left(\frac{2}{\alpha}\right)^2\left(\frac{\alpha}{2}\right)^2\int_{\{x:|Hg|>\frac{\alpha}{2}\}} = |\{x : |Hg| > \alpha\}|$$

From the Fourier transform definition of Hg, we immediately get that $\|Hg\|_2^2 = \|g\|_2^2$ and so

$$\left(\frac{2}{\alpha}\right)^2 \int |Hg|^2 = \left(\frac{2}{\alpha}\right)^2 \|Hg\|_2^2 = \left(\frac{2}{\alpha}\right)^2 \|g\|_2^2 = \left(\frac{2}{\alpha}\right)^2 \int |g|^2 dx \leq \left(\frac{2}{\alpha}\right)^2 \int |g| 2\alpha dx \leq \frac{8}{\alpha}\|f\|_1$$

since $|g| \leq 2\alpha$ by definition and

$$\int g = \int_F g + \int_\Omega g = \int_F f + \sum_j \int_{I_j} \frac{1}{|I_j|} \int_{I_j} f(y) dy dx = \int_F f + \sum_j \int_{I_j} f(y) dy = \int f$$

For $\{x : |Hb| > \alpha\}$ take $\Omega = \cup_{j=1}^\infty I_j$ and define $\Omega^\star := \cup_{j=1}^\infty 2I_j$ with $2I_j$ meaning the interval with the same centre but double length. Observe that

$$|\Omega^\star| \leq 2|\Omega| \leq \frac{2}{\alpha}\|f\|_1$$

and then

$$|\{x : |Hb| > \alpha\}| \leq |\Omega^\star| + |\{x \in (\Omega^\star)^C : |Hb| \geq \frac{\alpha}{2}\}| \leq \frac{2}{\alpha}\|f\|_1 + |\{x \in (\Omega^\star)^C : |Hb| \geq \frac{\alpha}{2}\}|$$

To finish we need

$$|\{x \in (\Omega^\star)^C : |Hb| \geq \frac{\alpha}{2}\}| \leq \frac{C}{\alpha}\|f\|_1$$

We have

$$|\{x \in (\Omega^\star)^C : |Hb| \geq \frac{\alpha}{2}\}| \leq \frac{2}{\alpha} \int_{(\Omega^\star)^C} |Hb| dx \leq \sum_j \frac{2}{\alpha} \int_{(\Omega^\star)^C} |Hb_j| dx$$

and it is enough to show that $\sum_j \int_{(\Omega^\star)^C} |Hb_j| dx \leq \|f\|_1$. Observe that if $2I_j \subset \Omega^\star$ then $(2I_j)^C \subset (\Omega^\star)^C$. Also observe that

$$\int b_j(x) dx = \int_{I_j} b_j(x) dx = \int_{I_j} f(x) dx - \int_{I_j} \frac{1}{|I_j|} \int_{I_j} f(y) dy dx = \int_{I_j} f(x) dx - \int_{I_j} f(x) dx = 0$$

We then have

$$\begin{aligned}
\frac{2}{\alpha} \sum_j \int_{(\Omega^\star)^C} |Hb_j| dx &= \sum_j \int_{(2I_j)^C} |Hb_j| dx \\
&= \frac{2}{\alpha} \sum_j \int_{(2I_j)^C} \left| \text{p.v.} \int_{\mathbb{R}} \frac{b_j(y)}{x - y} dy \right| dx \\
&= \frac{2}{\alpha} \sum_j \int_{(2I_j)^C} \left| \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{b_j(y)}{x - y} dy \right| dx \\
&= \frac{2}{\alpha} \sum_j \int_{(2I_j)^C} \left| \lim_{\varepsilon \to 0} \int\limits_{\substack{|x-y|>\varepsilon \\ y \in I_j}} \frac{b_j(y)}{x - y} dy \right| dx \\
&= \frac{2}{\alpha} \sum_j \int_{(2I_j)^C} \left| \int\limits_{I_j} \frac{b_j(y)}{x - y} dy \right| dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\alpha}\sum_j \int_{(2I_j)^C} \left| \int_{I_j} b_j(y)\left[\frac{1}{x-y} - \frac{1}{x-c_j}\right]dy\right| dx \\
&\leq \frac{2}{\alpha}\sum_j \int_{(2I_j)^C} \int_{I_j} |b_j(y)| \frac{|y-c_j|}{|x-y||x-c_j|} dydx \\
&\leq \frac{2}{\alpha}\sum_j \int_{(2I_j)^C} \int_{I_j} |b_j(y)| \frac{|I_j|}{|x-c_j|^2} dydx \\
&\leq \sum_j \frac{4}{\alpha}\int_{I_j} |b_j(y)| dy \\
&\leq \sum_j \frac{4}{\alpha}\left[\int_{I_j} |f(x)| + \frac{1}{|I_j|}\left|\int_{I_j} f(y)dy\right| dx\right] \\
&\leq \frac{4}{\alpha}\sum_j \left[\int_{I_j} |f(y)|dy + \int_{I_j} |f(y)|dy\right] \\
&\leq \frac{8}{\alpha}\int_{\cup I_j} |f(y)|dy \\
&\leq \frac{8}{\alpha}||f||_1
\end{aligned}$$

since $|y-c_j| \leq \frac{1}{2}|I_j|$ and $|x-y| \geq \frac{|x-c_j|}{2}$. The above is true, because, if $I_j = (c_j - a, c_j + a)$ then

$$\int_{(2I_j)^C} \frac{|I_j|}{|x-c_j|^2} dx = \int_{|x-c_j|>2a} \frac{2a}{|x-c_j|^2} dx = \int_{|y|>2a} \frac{2a}{|y|^2} dy = 2$$

Now for strong (p,p). We know that H is weak (1,1) and strong (2,2) and so by Marcinkiewicz H is strong (p,p) for $1 < p < 2$. We now use duality to deduce for $p > 2$. We have that

$$||f||_p = \sup_{\substack{g\in L^q \\ ||g||_q = 1}} \{\int fg\}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If $p > 2$ then

$$||Hf||_p = \sup_{\substack{g\in L^q \\ ||g||_q = 1}} \{\int (Hf)g\} = \sup_{\substack{g\in L^q \\ ||g||_q = 1}} \{-\int f(Hg)\}$$

Now

$$\int fHgdx \leq ||f||_p ||Hg||_q \leq ||f||_p c_q ||g||_q$$

and so

$$||Hf||_p = \sup\{-\int fHg\} \leq c_q ||f||_p$$

Q.E.D.

The moral of this is to work out the result for (2,2) and (1,1) and then use duality. The reason is that (∞, ∞) is most of the time false.

3.1 Natural Generalisations

We look for maps

$$Tf(x) = \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} f(x-y)dy$$

where $y' = \frac{y}{\|y\|} \in \mathbb{S}^{n-1}$. A necessary condition is that $\int_{\mathbb{S}^{n-1}} \Omega(y')d\sigma = 0$.

Suppose we are solving $-\Delta u = f$ and $f \in L^p$. Then

$$u = c_n \int \frac{1}{|y|^{n-2}} f(x-y)dy = c_n \int \frac{1}{|x-y|^{n-2}} f(y)dy$$

and then

$$\partial_{x_i}^2 u = c \int \frac{g}{|x-y|^n} f(y)dy + \int \frac{h}{|x-y|^{n+1}} f(y)dy$$

where g and h are polynomials and are essentially constants, and g happens to satisfy the property of Ω above.

A more useful generalisation is

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

Theorem 3.10 *Suppose that K is a tempered distribution, that agrees with a function on $\mathbb{R}^n \setminus \{0\}$, and is in $L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ such that*

1. $|\hat{K}(\xi)| \leq A$ *for some A*
2. $\int_{|x|>2|y|} |K(x-y) - K(x)|dx \leq B$ *for some B. This is called Hormander's condition.*

Then $Tf = \int K(x-y)f(y)dy$ is weak (1,1) and strong (p,p) for $1 < p < \infty$.

The next stage would be $\tilde{T}f(x) = \int K(x,y)f(y)dy$ which is used for Green's functions. The proof of the above is very similar to the proof that the Hilbert transform is weak (1,1) and strong (p,p).

Proof We first show that Tf is strong (2,2).

$$\|Tf\|_2 = \|\hat{Tf}\|_2 = \|\hat{K}(\xi)\hat{f}(\xi)\|_2 \leq \|\hat{K}\|_\infty \|\hat{f}\|_2 \leq A\|f\|_2$$

where we have used Plancherel twice. We now show that Tf is weak (1,1) and then use Marcinkiewicz and the duality argument to conclude the result.

Without loss of generality we can assume $f \geq 0$. If it isnt, then we can decompose $f = f^+ - f^-$ and then look at $Tf^+ - Tf^-$.

We use a Calderon-Zygmund decomposition for $f \geq 0$ and $\alpha > 0$ fixed in $\mathbb{R}^n$ so that $\mathbb{R}^n = F \cup \Omega$ where $f(x) \leq \alpha$ for $x \in \Omega^C$ and

$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} f(y)dy \leq 2^n \alpha$$

where $\Omega = \cup_j Q_j$ and Q_j have disjoint interior.

We construct good and bad functions such that

$$g(x) = \begin{cases} f(x) & x \in \Omega^C \\ \frac{1}{|Q_k|}\int_{Q_k} f(y)dy & x \in Q_k \end{cases}$$

and $b(x) = \sum_k b_k$ where

$$b_k(x) = \left(f(x) - \frac{1}{|Q_k|}\int_{Q_k} f(y)dy \right) \chi_{Q_k}(x)$$

Then

$$|\{x : |Tf| > \alpha\}| \leq |\{x : |Tg| > \frac{\alpha}{2}\}| + |\{x : |Tb| > \frac{\alpha}{2}\}|$$

and we want each one less than or equal to $\frac{C}{\alpha}\|f\|_1$. Then

$$|\{|Tg| > \frac{\alpha}{2}\}| \le \left(\frac{2}{\alpha}\right)^2 \int |Tg|^2 = \left(\frac{2}{\alpha}\right)^2 \|Tg\|_2^2 \le \left(\frac{2}{\alpha}\right)^2 A^2\|g\|_2^2$$

and notice that $g(x) \le 2^n\alpha$ and thus

$$\left(\frac{2}{\alpha}\right)^2 A^2\|g\|_2^2 \le \left(\frac{2}{\alpha}\right)^2 A^2\|g\|_\infty\|g\|_1 \le \frac{2^{n+2}}{\alpha}A^2\|g\|_1 = \frac{2^{n+2}}{\alpha}A^2\|f\|_1$$

since $\int b_k = 0$ so $\int g = \int f$ like before. This concludes bounding the good part.

Call $Q_k^\star$ the cube centre c_k with side length $2\sqrt{n}$ times the side length of Q_k. Then let $\Omega^\star := \cup_k Q_k^\star$. Then

$$|\Omega^\star| \le C|\Omega| \le C\sum|Q_k| \le \frac{C}{\alpha}\sum\int_{Q_k} f(y)dy \le \frac{C}{\alpha}\|f\|_1$$

and then

$$|\{x : |Tb| > \frac{\alpha}{2}\}| \le |\Omega^\star| + |\{x \in (\Omega^\star)^C : |Tb| > \frac{\alpha}{2}\}| \le \frac{C}{\alpha}\|f\|_1 + |\{x \in (\Omega^\star)^C : |Tb| > \frac{\alpha}{2}\}|$$

Then

$$\begin{aligned}
|\{x \in (\Omega^\star)^C : |Tb| > \frac{\alpha}{2}\}| &\le \frac{2}{\alpha}\int_{(\Omega^\star)^C}|Tb|dx \\
&\le \frac{2}{\alpha}\sum_k\int_{(\Omega^\star)^C}|Tb_k|dx \\
&\le \frac{2}{\alpha}\sum_k\int_{(Q_k^\star)^C}|Tb_k|dx \\
&\le \frac{2}{\alpha}\sum_k\int_{(Q_k^\star)^C}\left|\int_{\mathbb{R}^n}K(x-y)b_k(y)dy\right|dx \\
&\le \frac{2}{\alpha}\sum_k\int_{(Q_k^\star)^C}\left|\int_{Q_k}K(x-y)b_k(y)dy\right|dx \\
&\le \frac{2}{\alpha}\sum_k\int_{(Q_k^\star)^C}\left|\int_{Q_k}\left[K(x-y)-K(x-c_k)\right]b_k(y)dy\right|dx \\
&\le \frac{2}{\alpha}\sum_k\int_{Q_k}\underbrace{\int_{(Q_k^\star)^C}|K(x-y)-K(x-c_k)|\,|b_k(y)|dy\,dx}_{\le C} \\
&\le \frac{2C}{\alpha}\sum_k\int_{Q_k}|b_k(y)|dy \\
&\le \frac{2C}{\alpha}\sum_k\int_{Q_k}\left[|f(y)| + \frac{1}{|Q_k|}\int_{Q_k}|f(z)|dz\right]dy \\
&\le \frac{4C}{\alpha}\sum_k\int_{Q_k}|f(y)|dy \\
&\le \frac{4C}{\alpha}\|f\|_1
\end{aligned}$$

and now for $\int_{(Q_k^\star)^C}|K(x-y)-K(x-c_k)|\,|b_k(y)|dy \le C$ note that $\mathbb{R}^n \setminus Q_k^\star \subset \{x : |x-c_k| >$

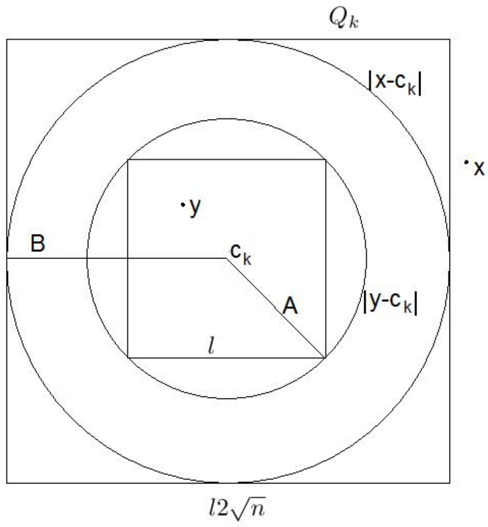

Figure 1: Explaining last part of proof above

$2|y - c_k|\}$ and so

$$\int_{(Q_k^\star)^C} |K(x-y) - K(x-c_k)|\,|b_k(y)|dy \le \int_{|x-c_k|>2|y-c_k|} |K(x-y) - K(x-c_k)|\,|b_k(y)|dy$$
$$\le B$$

and see figure 1. We thus need $B > 2A$ from the picture below. Then if we suppose l is the side length of Q_k we have $l2\sqrt{n}$ as the side length of of $Q_k^\star$ so $2B = l2\sqrt{n} > 4\sqrt{l^2 n/4}$ as required. *Q.E.D.*

4 Bounded Mean Oscillation (BMO)

Recall that $f \in L^1_{loc}$ if and only if for all $K \subset \mathbb{R}^n$ compact we have $\int_K |f|dx < \infty$.

Definition 4.1

$$f_Q = \frac{1}{|Q|}\int_Q f(x)dx$$

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f - f_Q|$$

BMO is the space

$$\{f \in L^1_{loc} : M^\sharp f \in L^\infty\}$$

For an $f \in BMO$ we write $\|f\|_\star = \|M^\sharp f\|_\infty$. The reason for the strange notation is that this object is not quite a norm, as if f is constant then it is zero. There is a way to construct a norm on BMO, which is usually denoted by $\|\cdot\|_{BMO}$ by taking equivalence classes. We do not do that here.

Observe that

$$|M^\sharp f(x)| \le c_n Mf(x)$$

where the M is the Hardy Littlewood maximal function and c_n depends only on the dimension.

Proposition 4.2 *1.*

$$\frac{1}{2}||f||_\star \le \sup_Q \inf_{a\in\mathbb{C}} \frac{1}{|Q|}\int_Q |f(y)-a|dy \le ||f||_\star$$

for a given $f \in BMO$

2.

$$M^\sharp|f|(x) \le 2M^\sharp f(x)$$

Proof

1. $$\sup_Q \inf_a \frac{1}{|Q|}\int_Q |f(y)-a|dy \le \sup_Q \frac{1}{|Q|}\int_Q |f(y)-f_Q|dy \le ||M^\sharp f(x)||_\infty = ||f||_\star$$

   which shows the second inequality. For the first, observe that

   $$\int_Q |f-f_Q| = \int_Q |f-a+a-f_Q| \le \int_Q |f-a| + \int_Q |a-f_Q| \le 2\int_Q |f-a|$$

   for all Q and a. Then

   $$\frac{1}{2}\int_Q |f(y)-f_Q|dy \le \inf_a \int_Q |f(y)-a|dy$$

   since the left hand side doesn't depend on a. Now divide by $|Q|$ and take the supremum over $Q \ni x$ to get that

   $$\frac{1}{2}||M^\sharp f||_\infty \le \sup_{Q\ni x}\inf_a \frac{1}{|Q|}\int_Q |f(y)-a|dy$$

   To prove that $\int_Q |f_Q - a| \le \int_Q |f(y)-a|dy$ we note that we can assume that $a=0$, as if not, consider $g = f-a$. Then

   $$\begin{aligned}\int_Q |g_Q|dx &= \int_Q \frac{1}{|Q|}\left|\int_Q g(y)dy\right|dx \\ &= \left|\int_Q g(y)dy\right| \int_Q \frac{1}{|Q|}dx \\ &= \left|\int_Q g(y)dy\right| \\ &\le \int_Q |g(y)|dy\end{aligned}$$

   as required.

2. We have

   $$\begin{aligned}M^\sharp|f|(x) &\le ||M^\sharp|f|(x)||_\infty \\ &\le 2\sup_{Q\ni x}\inf_a \frac{1}{|Q|}\int_Q ||f|(y)-a|dy \\ &\le 2\sup_{Q\ni x}\frac{1}{|Q|}\int_Q ||f(y)|-|f_Q||dy \\ &\le 2\sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)-f_Q|dy \\ &\le 2M^\sharp f(x)\end{aligned}$$

Q.E.D.

Corollary 4.3 *If $f \in BMO$ ($\|M^\sharp f(x)\|_\infty < 2$) then $|f| \in BMO$, i.e. $\|M^\sharp |f|(x)\|_\infty \le 2c$. The converse is false however.*

Observe that $L^\infty \subset BMO$ and BMO is larger that L^∞. Consider $f(x) = \begin{cases} \log \frac{1}{|x|} & x \le 1 \\ 0 & x \ge 1 \end{cases}$ in one dimension. Then consider $\operatorname{sgn}(x) f(x)$. We claim that $f = |f|$ is in BMO but $\operatorname{sgn}(x) f(x)$ is not.

The moral of this is that for BMO, size is not the only thing that matters.

Theorem 4.4 *Consider $Tf(x) = \int K(x-y) f(y) dy$ such that K is a tempered distribution agreeing with a function on $\mathbb{R}^n \setminus \{0\}$ such that $|\hat{K}(\xi)| \le A$ and K satisfies a Hormander type condition $\int_{|x|>2|y|} |K(x-y) - K(x)| dx \le B$ for all y. Then T maps L^∞ into BMO and*

$$\|Tf\|_{BMO} \le C\|f\|_\infty$$

Note that the BMO norm measures the oscillations. Also oscillations are really important, recall Riemann-Lebesgue. $\hat{f}(\xi) \to 0$ mainly due to the oscillations of $e^{-2\pi i x\xi}$.

The BMO norm measures the oscillations of f at every scale. $|f - f_Q|$ measures the difference between f and its average f_Q. Then $M^\sharp$ measures that difference, at every scale.

Theorem 4.5 (Interpolation) *T a bounded operator on L^{p_0}, $\|Tf\|_{p_0} \le c_{p_0} \|f\|_{p_0}$ and bounded from L^∞ to BMO. Then T is bounded in L^p for $p > p_0$.*

This in some sense generalises Marcinkiewicz.

Theorem 4.6 (John-Nirenberg) *Suppose that $f \in BMO$, then there exist C_1, C_2 such that*

$$|\{x \in Q : |f - f_Q| > \lambda\}| \le C_1 e^{-C^2 \lambda / \|f\|_\star} |Q|$$

5 Weak Derivatives and Distributions.

5.1 Weak Derivatives

We look carefully at the integration by parts formula. Suppose that $\Omega \subset \mathbb{R}^n$ is open, $\phi \in C_c^\infty(\Omega)$ and $u \in C^1(\Omega)$. Then

$$\int_\Omega \partial_{x_j} u \phi dx = -\int_\Omega u \partial_{x_j} \phi dx$$

and if $u \in C^{|\alpha|}$ then

$$\int_\Omega \partial_x^\alpha u \phi dx = -\int_\Omega u \partial_x^\alpha \phi dx$$

Definition 5.1 *Let $u, v \in L^1_{loc}$, and α a multiindex. We say that v is the* ***α-weak derivative*** *of u if*

$$\int v\phi dx = (-1)^{|\alpha|} \int u \partial_x^\alpha \phi dx$$

for all $\phi \in C_c^\infty$

Observe that this is unique if it exists. If we consider $f(x) = |x|$ then $f'(x) = \operatorname{sgn}(x)$. If $g(x) = \chi_{[0,\infty)}$ then this has no weak derivative.

Consider $u_t + u_x = 0$ for $x \in \mathbb{R}$ and $t > 0$. This has a solution $f(x-t)$ for some f. If we add $u(x,0) = f(x)$ then $u(x,t) = f(x-t)$ should be the unique solution.

Suppose $f \in L^p \setminus C^1$. Then assuming you can, for $\phi \in C_c^\infty$ we have

$$0 = \int (u_t + u_x)\phi dxdt = \int u_t\phi + u_x\phi dxdt = -\int u\phi_t + u\phi_x dxdt = -\int u(\phi_t + \phi_x)dxdt$$

and the right hand side exists if $u \in L^1_{loc}$. We say that u is a weak solution of $u_t + u_x = 0$ if it satisfies $\int u(\phi_t + \phi_x)dxdt = 0$ for all $\phi \in C_c^\infty$. One can check that if f has a weak derivative then $u(x,t) = f(x-t)$ is actually a weak solution.

An example is to solve $-\Delta u = f$ in $\mathbb{R}^n$. First look for radial solution of $-\Delta u = 0$ and for $n \geq 3$, one such is $u(x) = c_n \frac{1}{|x|^{n-2}}$ from formal calculations. Away from $x = 0$ we have $-\Delta u = 0$ and so $u(x) = c_n \int \frac{1}{|x-y|^{n-2}} f(y)dy$ solves $-\Delta u = f$. Formally $-\Delta u = c_n \int (-\Delta)\frac{1}{|x-y|^{n-2}} f(y)dy = f(x)$ and this works because $\frac{1}{|x-y|^{n-2}}$ is essentially the distribution δ_x.

5.2 Distributions

Suppose that $X, \Omega \subset \mathbb{R}^n$ are open.

Definition 5.2 *Let u be a linear form on C_c^∞. Then u is called a **distribution** if it satisfies: For all compact $K \subset X$ there exists $C = C(K)$ and $N = N(K) \in \mathbb{N}$ such that*

$$|u(\phi)| = |\langle u, \phi\rangle| \leq C \sum_{|\alpha|\leq N} \sup_x |\partial^\alpha \phi|$$

for all $\phi \in C_c^\infty(K)$.

Note that C_c^∞ is not a Banach space, so has no natural norm. However it is a Fréchet space so has a family of seminorms, and these make up the sum above.

One would like to write the following. u a distribution if it is a bounded linear functional on C_c^∞. The reason for the strange definition is that C_c^∞ has no natural norm.

We observe that $C_c^\infty \subset \mathcal{S} \subset C^\infty$. The useful one of these for us is $\mathcal{S}$. We call these tempered distributions. For C^∞ we have distributions of compact support. $\mathcal{D}'$ is the space of distributions.

Theorem 5.3

$$C^0 \hookrightarrow \mathcal{D}'$$

Proof $f \in C^0$ define $\langle f, \phi\rangle = \int f\phi dx$ for all $\phi \in C_c^\infty$. Then

$$|\langle f, \phi\rangle| \leq \|\phi\|_\infty \int_{\operatorname{spt}(\phi)} |f| dx$$

for $\phi \in C_c^\infty(K)$ and then

$$|\langle f, \phi\rangle| \leq \int_K |f| dx \|\phi\|_\infty$$

Q.E.D.

We observe that $L^1_{loc} \subset \mathcal{D}'$ and so $L^p \subset \mathcal{D}'$.

We define δ by $\langle \delta, \phi\rangle = \phi(0)$ and note that $|\langle \delta, \phi\rangle| \leq \|\phi\|_\infty$ and here C, N are independent of K. δ_y is defined by $\langle \delta_y, \phi\rangle = \phi(y)$.

5.2.1 Convergence of Distributions

We think first of convergence in C_c^∞.

Definition 5.4 *$X \subset \mathbb{R}^n$ open. Then $\phi_j \in C_c^\infty(X)$* **converges to** 0 *in $C_c^\infty(X)$ if*

1. *$\text{spt}(\phi_j) \subset K$ for some $K \subset X$ compact for all j. K is thought of as being fixed.*
2. *For all α, $\partial^\alpha \phi_j \to 0$ uniformly as $j \to \infty$.*

Theorem 5.5 *$u : C_c^\infty(X) \to \mathbb{R}$ is a distribution if and only if*

$$\lim_{j\to\infty} \langle u, \phi_j \rangle = 0$$

for all ϕ_j that converge to 0.

Proof "$\Longrightarrow$" is trivial. We have $|\langle u, \phi_j\rangle| \le C(K) \sum_{|\alpha|\le N} \|\partial^\alpha \phi\|_\infty$ by definition and the right hand side tends to zero as $\phi_j \to 0$ in $C_c^\infty(X)$.

"$\Longleftarrow$" By contradiction we have that

$$\left\{ \frac{|\langle u, \phi\rangle|}{\sum_{|\alpha|\le N} \|\partial^\alpha \phi\|_\infty}, \phi \in C_c^\infty(K) \right\}$$

is unbounded in $[0, \infty)$ for every N. Thus for every N there exists a function $\phi_N \in C_c^\infty(K)$ such that

$$\frac{|\langle u, \phi_N\rangle|}{\sum_{|\alpha|\le N} \|\partial^\alpha \phi_N\|_\infty} > N$$

Construct

$$\psi_N = \frac{\phi_N}{N \sum_{|\alpha|\le N} \|\partial^\alpha \phi_N\|_\infty}$$

and a direct calculation shows that $\psi_N \to 0$ in $C_c^\infty(K)$, and $|\langle u, \psi_N\rangle| > 1$. We have a contradiction since by hypothesis if $\psi_j \to 0$ then $|\langle u, \psi_j\rangle| \to 0$ Now

$$1 \ge \frac{|\langle u, \phi_N\rangle|}{N \sum_{|\alpha|\le N} \|\partial^\alpha \phi\|_\infty} = |\langle u, \psi_N\rangle|$$

Q.E.D.

Definition 5.6 *If the $N = N(K)$ in the definition of a distribution can be taken independent of K then we say that the lowest possible such N is the* ***order*** *of the distribution.*

It is left as an exercise to construct a distribution without finite order.

Definition 5.7 *Suppose $X \subset \mathbb{R}^n$ and $u \in \mathcal{D}'(X)$. Then the* ***support*** *of u is defined by the complement of $\{x \in X : u = 0$ on a nbhd of $x\}$*

This set is open by definition, so the support is always closed. $u = 0$ on a neighbourhood of x if and only if there exists $\Omega \ni x$ open such that $\langle u, \phi\rangle = 0$ for all $\phi \in C_c^\infty(\Omega)$.

For example the δ-distribution has support $\{0\}$ and if $f \in L^1_{loc}$ then its support as a distribution is the same as its support as a function.

The set of distributions of compact support can be identified with the dual of C^∞.

Definition 5.8 *Let $X \subset \mathbb{R}^n$ be open, and $u_j \in \mathcal{D}'(X)$. Then $u_j \to u$ in $\mathcal{D}'(X)$ if and only if*

$$\langle u_j, \phi\rangle \to \langle u, \phi\rangle \qquad\qquad \forall \phi \in C_c^\infty(X)$$

The same is true for a continuous parameter.

Riemann-Lebesgue says that $e^{ix\xi} \to 0$ in $\mathcal{D}'$. Approximations to the identity $\rho \in L^1$ then $\rho_\varepsilon(x) \to \delta$ in $\mathcal{D}'$.

We have seen $\rho_\varepsilon \star f(x) \to f(x)$ for $f \in L^\xi$ which is much stronger than the above, in other words we have proved

$$\frac{1}{\varepsilon^n}\int \rho\left(\frac{x-y}{\varepsilon}\right) f(y)dy \to f(x)$$

but here we only need

$$\frac{1}{\varepsilon^n}\int \rho\left(\frac{y}{\varepsilon}\right)\phi(y)dy \to \phi(0)$$

as $\varepsilon \to 0$.

5.3 Derivatives of Distributions

We use integration by parts. Suppose that $u, \phi \in C_c^\infty$ and then

$$\int u_{x_i}\phi dx = -\int u\partial_{x_i}\phi dx$$

Definition 5.9 *Suppose that $u \in \mathcal{D}'(\mathbb{R}^n)$ and then define the α***th derivative of** *u by*

$$\langle \partial^\alpha u, \phi\rangle := (-1)^{|\alpha|}\langle u, \partial^\alpha \phi\rangle$$

We now check that this definition makes sense. Observe that $u \in \mathcal{D}'$ then $\partial^\alpha u \in \mathcal{D}'$. Given $K \subset X$ compact then there exists C, N such that

$$|\langle u, \phi\rangle| \le C \sum_{|\beta|\le N} ||\partial^\beta \phi||_\infty$$

and we want to have

$$|\langle \partial^\alpha u, \psi\rangle| \le \bar{C} \sum_{|\beta|\le \bar{N}} ||\partial^\beta \psi||_\infty$$

but

$$|\langle \partial^\alpha u, \psi\rangle| = |(-1)^{|\alpha|}\langle u, \partial^\alpha \psi\rangle| \le C \sum_{|\beta|\le N+|\alpha|} ||\partial^\beta \psi||_\infty$$

and so the above definition does indeed make sense, as we have $\partial^\alpha u \in \mathcal{D}'$.

Proposition 5.10 *$u_j, u \in \mathcal{D}'$ and $u_j \to u$ in $\mathcal{D}'$ then*

$$\partial^\alpha u_j \to \partial^\alpha u$$

in $\mathcal{D}'$ for all α.

Proposition 5.11 *Suppose that $f, g \in C^0$ and consider them as distributions, i.e. $\langle f, \phi\rangle = \int f\phi dx$. Assume that $\frac{\partial f}{\partial x_i}$ equals g in the sense of distributions. Then $\frac{\partial f}{\partial x_i}$ exists in the classical sense and equals g.*

The hypothesis is $\langle g, \phi\rangle = \langle \frac{\partial f}{\partial x_i}, \phi\rangle$ for all $\phi \in C_c^\infty$

Definition 5.12 *If $f \in C^\infty$ and $u \in \mathcal{D}'$ then define the* **product** fu *by*

$$\langle fu, \phi \rangle := \langle u, f\phi \rangle$$

for all $\phi \in C_c^\infty$.

Theorem 5.13 (Product rule) *Suppose that $f \in C^\infty$, and $u \in \mathcal{D}'$. Then*

$$\partial^\alpha (fu) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \partial^\beta f \partial^\gamma u$$

For example consider the δ distribution and take $f \in C^\infty$. Then

$$\langle f\delta, \phi \rangle = \langle \delta, f\phi \rangle = f(0)\phi(0) = f(0)\langle \delta, \phi \rangle$$

and so people write $f\delta = f(0)\delta$. Also be careful to note that

$$\langle f\partial_{x_i}\delta, \phi \rangle = f(0)\langle \partial_{x_i}\delta, \phi \rangle - (\partial_{x_i} f)\langle \delta, \phi \rangle$$

5.4 Distributions of compact support

We denote by $\mathcal{E}'$ the dual of C^∞, and these are the distributions of compact support.

Definition 5.14 *Suppose that we have a sequence $\phi_j \in C^\infty(X)$. We say ϕ_j converges to* 0 *if for all $K \subset X$ compact we have*

$$\partial^\alpha \phi_j \to 0$$

for all α uniformly on K.

Definition 5.15 *We say that $u \in \mathcal{E}'$ is a* **distribution of compact support** *if it is a linear map on C^∞ such that there exists a compact set K, and constants $C, N \geq 0$ such that*

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq N} ||\partial^\alpha \phi||_{L^\infty(K)}$$

for all $\phi \in C^\infty$.

The following theorem links $\mathcal{D}'$ and $\mathcal{E}'$ together.

Theorem 5.16 *Suppose that $u \in \mathcal{D}'(X)$ and X is open. If the support of u is compact then there exists a unique extension of u to C^∞ that is in $\mathcal{E}'(X)$.*

Given $v \in \mathcal{E}'$ the restriction of v to C_c^∞ is a distribution, i.e. $V|_{C_c^\infty} \in \mathcal{D}'$. Moreover it has compact support.

Theorem 5.17 *Suppose that $u \in \mathcal{D}'(X)$ and $\phi \in C_c^\infty(X \times Y)$. Then*

$$\langle u(x), \phi(x,y) \rangle \in C_c^\infty(Y)$$

Suppose that $u \in \mathcal{E}'(X)$ and $\psi \in C^\infty(X \times Y)$. Then

$$\langle u(x), \psi(x,y) \rangle \in C^\infty(Y)$$

Suppose that $f(x) \in L^1(\mathbb{R})$ and $\phi(x,\xi) = e^{-2\pi i x\xi}$ then

$$\langle f(x), \phi(x,\xi) \rangle = \int f(x) e^{-2\pi i x\xi} dx$$

which is the Fourier transform, so the weirdness in the theorem above is not that weird.

5.5 Convolutions

Suppose that f, g are functions. Then we had before that $f \star g(x) = \int f(x-y)g(y)dy$ and we use this to define the convolution with distributions.

$$\begin{aligned}\langle f \star g(x), \phi(x)\rangle &= \iint f(x-y)g(y)dy\phi(x)dx \\ &= \iint f(z)g(y)dy\phi(z+y)dz\end{aligned}$$

where we set $x = y + z$ in the x integration. Then the natural definition for distributions would be the same:

$$\langle u \star v, \phi\rangle = \langle u(x), \langle v(y), \phi(x,y)\rangle\rangle$$

If $v \in \mathcal{D}'$ then $\langle v(y), \phi(x,y)\rangle$ is not necessarily in C_c^∞ and so this doesnt work. It is impossible to define $u \star v$ for $u, v \in \mathcal{D}'$ even if v is a function. The way to get around it is to demand that either u or v is in $\mathcal{E}'$, say. Then

$$\langle u \star v, \phi\rangle = \langle u(x), \langle v(y), \phi(x,y)\rangle\rangle$$

works because $\langle v(y), \phi(x,y)\rangle$ is in C^∞.

We have various properties: If $u \in \mathcal{E}'$ and $v \in \mathcal{D}'$ then

$$\partial_j(u \star v) = \partial_j(u) \star v = u \star (\partial_j v)$$

Also

$$\delta \star u = u$$

for all $u \in \mathcal{D}'$.

5.6 Tempered Distributions

Recall that $\mathcal{S}$ is the space of C^∞ functions ϕ such that

$$||\phi||_{\alpha,\beta} = \sup_x |x^\alpha \partial^\beta \phi| < C$$

for all α, β. Then these define a family of seminorms. Convergence in $\mathcal{S}$ is given by $\phi_j \to 0$ if and only if

$$||\phi||_{\alpha,\beta} \to 0$$

for all α, β.

Definition 5.18 *Let u be a linear functional on $\mathcal{S}$. We say it is a* **tempered distribution** *if it satisfies: there exists N such that*

$$|\langle u, \phi\rangle| \le \sum_{|\alpha|,|\beta|\le N} ||x^\alpha \partial^\beta \phi(x)||_\infty$$

Observe that $C_c^\infty \subset \mathcal{S}$ and so every tempered distribution is a distribution.

Theorem 5.19 *The space of tempered distributions equals the space of distributions that have an extension to $\mathcal{S}$.*

We now introduce the Fourier transform, using the Plancherel formula

$$\int \hat{f} g = \int f \hat{g}$$

for $f, g \in \mathcal{S}$.

Definition 5.20 *Given $u \in \mathcal{S}'$ we define the* **Fourier transform** $\hat{u}$ *by*

$$\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle$$

Definition 5.21 *$u_j \in \mathcal{S}'$ converges to $u \in \mathcal{S}'$ if and only if*

$$\langle u_j, \phi \rangle \to \langle u, \phi \rangle$$

for all $\phi \in \mathcal{S}$.

Theorem 5.22 (Structural theorem) *Every tempered distribution is the derivative of a function with polynomial growth, i.e. $u \in \mathcal{S}'$ then $u = \partial^\alpha f$ where f satisfies*

$$|f(x)| \leq (1 + |x|)^k$$

for some k

We hope that if we Fourier transform as a function and as a tempered distribution, then the two methods should agree.

Theorem 5.23 *If $u \in L^1$ defines a tempered distribution then its Fourier transform as a tempered distribution agrees with the distribution generated by $\hat{u}$, where $\wedge$ is the Fourier transform of a function.*

Theorem 5.24 *$\wedge$ in $\mathcal{S}'$ is an isometry.*

Proof $\wedge$ is clearly linear. Now

$$\langle \hat{\hat{\hat{u}}}, \phi \rangle = \langle \hat{\hat{u}}, \hat{\phi} \rangle = \langle u, \hat{\hat{\hat{\phi}}} \rangle = \langle u, \check{\phi} \rangle$$

where $\check{f}(x) = \int f(y) e^{2\pi i x y} dy$ and we had $\hat{\hat{\phi}} = \check{\phi}$ for $\phi \in \mathcal{S}$. also note that $\hat{\hat{\hat{\hat{\phi}}}} = \phi$.

Since $\wedge$ is linear, proving it is injective reduces to proving that $\hat{u} = 0 \implies u = 0$ in the sense of tempered distributions. If we assume that

$$\langle \hat{u}, \hat{\hat{\phi}} \rangle = 0$$

for all $\phi \in \mathcal{S}$ then this immediately implies that

$$\langle u, \check{\phi} \rangle = 0$$

for all $\phi \in \mathcal{S}$. Since the inverse Fourier transform is an isometry, we have

$$\langle u, \psi \rangle = 0$$

for all $\psi \in \mathcal{S}$, as required.

To prove that $\wedge$ is surjective, note that

$$\hat{\hat{\hat{\hat{\hat{u}}}}} = \hat{u}$$

Q.E.D.

Theorem 5.25 *If $u \in \mathcal{S}'$ then*

1. $\partial^\alpha u \in \mathcal{S}'$ *and* $\widehat{\partial^\alpha u} = (2\pi i \xi)^\alpha \hat{u}$ *in the sense of distributions*

2. $x^\alpha u \in \mathcal{S}'$ *and* $\widehat{x^\alpha u} = ()\partial^\alpha \hat{u}$ *in the sense of distributions*

3. *Every formula we know for the Fourier transform translates to* $\mathcal{S}'$.

and

Theorem 5.26 *Every distribution with compact support is a tempered distribution.*

Theorem 5.27 *The Fourier transform of every distribution with compact support is a function. Moreover*

$$\hat{u} = \langle u(x), e^{-2\pi i x\cdot\xi}\rangle$$

The last part of this means that the map given by $\hat{u}$, i.e. $\langle \hat{u}, \phi\rangle$ is the same as the map $\langle\langle u(x), e^{-2\pi i x\cdot\xi}\rangle, \phi\rangle$

Example 5.1 *The* δ *distribution. Clearly this is a distribution with compact support. Then*

$$\langle \hat{\delta}, \phi\rangle := \langle \delta, \hat{\phi}\rangle := \hat{\phi}(0)$$

and also $\hat{\phi}(\xi) = \int \phi(x) e^{-2\pi i x\cdot\xi} dx$ *and so* $\hat{\phi}(0) = \int \phi(x) dx$ *and so*

$$\hat{\phi}(0) = \int \phi(x) dx = \langle 1, \phi\rangle$$

and so $\langle \hat{\delta}, \phi\rangle = \langle 1, \phi\rangle$

We saw that if $u \in \mathcal{E}'$ and $v \in \mathcal{D}'$ then $u \star v$ makes sense. Now we have

Lemma 5.28 *If* $u, v \in \mathcal{E}'$ *then* $u \star v$ *exists and is in* $\mathcal{E}'$ *and moreover*

$$\widehat{(u \star v)} = \hat{u}\hat{v}$$

6 Sobolev Spaces

Suppose $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ with Ω open. We have a notion of a weak derivative as we saw before, using integration by parts.

Definition 6.1

$$H^k = \{f : \mathbb{R}^n \to \mathbb{R} : f \in L^2, \frac{\partial^\alpha f}{\partial x^\alpha} \text{ exists weakly and } \int \left|\frac{\partial^\alpha f}{\partial x^\alpha}\right|^2 dx < \infty \text{ for } |\alpha| \le k\}$$

H^k is a Banach space with the norm

$$||f||_{H^k} = \sum_{|\alpha|\le k} ||\partial^\alpha f||_2$$

or equivalently

$$||f||_{H^k} = \left(\sum_{|\alpha|\le k} ||\partial^\alpha f||_2^2\right)^{\frac{1}{2}}$$

We define

$$W^{k,p} = \{f : \mathbb{R}^n \to \mathbb{R} : f \in L^p, \frac{\partial^\alpha f}{\partial x^\alpha} \text{ exists weakly and } \int \left|\frac{\partial^\alpha f}{\partial x^\alpha}\right|^p dx < \infty \text{ for } |\alpha| \leq k\}$$

This is a Banach space with the norm

$$||f||_{W^{k,p}} = \sum_{|\alpha|\leq k} ||\partial^\alpha f||_p$$

In H^k we only know how to differentiate for orders in $\mathbb{N}$, and we could use the Fourier transform to compute the norm. We use Plancherel:

$$||\partial^\alpha f||_2 = ||\widehat{\partial^\alpha f}||_2 = C||\xi^\alpha \hat{f}(\xi)||_2$$

and for $f \in H^k$ we need

$$||\xi^\alpha \hat{f}(\xi)||_2 \leq |||\xi|^k \hat{f}(\xi)||_2 < C$$

for $|\alpha| \leq k$.

Definition 6.2 *We define the Sobolev space H^s, $s \in \mathbb{R}$, using distributions as follows*

$$H^s = \{u \in \mathcal{S}' : \hat{u} \text{ is a function for which} \int \left((1+|\xi|^2)^{s/2}\hat{u}(\xi)|\right)^2 d\xi < \infty\}$$

We claim that if $s \in \mathbb{N}$ then this definition is the same as the one before. We also observe that if $s \in \mathbb{R}$ then we are allowed fractional orders of differentiation, and negative orders as well.

We could equivalently have said

$$\int ((1+|\xi|)^s \hat{u}(\xi)|)^2 d\xi < \infty$$

because there exist c, C such that

$$c(1+|\xi|^2)^{s/2} \leq (1+|\xi|)^s \leq C(1+|\xi|^2)^{s/2}$$

6.1 Sobolev Embeddings

Theorem 6.3

$$H^s \hookrightarrow C^0$$

provided $s > n/2$ where n is the dimension of the space.

Proof We are going to show $H^s \subset (C^0 \cap L^\infty)$. If $u \in H^s$ then $u \in \mathcal{S}'$ and so $\hat{u}$ is a function. We show $\hat{u} \in L^1$ as then we have

$$u(x) = \int \hat{u}(\xi)e^{2\pi i x\xi} d\xi$$

and this immediately gives $u \in L^\infty$ and u is continuous by properties of the Fourier transform.

We know that

$$\int \left((1+|\xi|^2)^{s/2}\hat{u}(\xi)|\right)^2 d\xi < \infty$$

and so

$$\left(\int |\hat{u}(\xi)|^2 d\xi\right)^2 = \left(\int \frac{1}{(1+|\xi|^2)^{s/2}}(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|^2 d\xi\right)^2$$
$$\overset{\text{Hölder}}{\leq} \int \left((1+|\xi|^2)^{s/2}|\hat{u}(\xi)|^2\right)^2 d\xi \int \left(\frac{1}{(1+|\xi|^2)^{s/2}}\right)^2 d\xi$$

and so we need

$$I = \int \frac{1}{(1+|\xi|^2)^s} d\xi < \infty$$

and so $\frac{1}{(1+|\xi|^2)^s}$ needs to decrease faster than $\frac{1}{|\xi|^n}$ and so we need $2s > n$ to make it finite.

Q.E.D.

Corollary 6.4 *If $u \in H^s$ and $s > n/2 + k$ for $k \in \mathbb{N}$ then $u \in C^k$.*

Observe that $H^s \subset H^t$ if $s \geq t$ and so in a sense s measures the regularity of the functions. There is a pairing structure between H^s and H^{-s} as follows. If $u \in H^s$ and $v \in H^{-s}$ it is possible to define $\langle u, v\rangle$ that satisfies

$$|\langle u, v\rangle| \leq ||u||_{H^s}||v||_{H^{-s}}$$