# MA4C0 Differential Geometry

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These notes are based on the 2012 `MA4C0 Differential Geometry` course, taught by Peter Topping, typeset by Matthew Egginton.
No guarantee is given that they are accurate or applicable, but hopefully they will assist your study.
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# 1 Introduction

The following is a very sketchy background in certain parts of the course, included to give some context to what is done. None of the definitions and theorems are stated rigorously. We briefly look at the following

1. manifolds with a notion of length/angle
2. way of differentiating vector fields
3. shortest path between two points
4. curvature

**Definition 1.1 (Manifold)** *A manifold is a topological space such that each point has a neighbourhood that is homeomorphic to a ball in* $\mathbb{R}^n$.

Some examples of manifolds are $\mathbb{R}$, $S^1$, $S^2$, $T^2$, $\mathbb{R}P^n$, $SO(n)$. An example that isn't a manifold is where three half lines meet at the same point. The point is where it fails, as at this point it is not homeomorphic to a line.

## 1.1 Intrinsic Curvature of a Surface (Gauss Curvature)

### 1.1.1 Geodesic Flow

Curvature is very closely connected with the stability of geodesic flow. A positive curvature encourages geodesics to focus, whereas a negative curvature encourages geodesics to diverge. This is seen on the sphere, where the geodesics from a point focus on the antipodal point.

### 1.1.2 Volume Growth

Consider the function $F : r \to Vol(B(p, r))$. By comparing this function with the same function in $\mathbb{R}^n$ we can measure curvature, for example if the manifold is positively curved then the function $F$ is less than the equivalent Euclidean function.

### 1.1.3 Topology

**Theorem 1.2 (Gauss Bonnet)** *If $\mathcal{M}$ is a compact surface and $K$ is the Gauss curvature then*

$$\int_{\mathcal{M}} K dV = 4\pi(1-g)$$

Observe that there is no geometry involved in the right hand side, there is only topology

**Theorem 1.3 (Bit of Uniformisation theorem)** *Every compact surface $M$ has a Riemannian metric with constant curvature*

- *+1 if $\mathcal{M}$ is the sphere*
- *0 if $g = 1$*
- *-1 if $g \geq 2$*

**Theorem 1.4 (Cartan -Hadamard)** *Suppose $\mathcal{M}$ is a Riemannian manifold which is simply connected with curvature less than 0. Then $\mathcal{M}$ is diffeomorphic to* $\mathbb{R}^n$.

**Theorem 1.5 (Bonnet's)** *Suppose $\mathcal{M}$ is a complete Riemannian manifold , with sectional curvature at least $\lambda > 0$. Then $\mathcal{M}$ is compact.*

Note that one could define the Gauss curvature $K$ on a surface at $p$ by $length(\partial B_r) = 2\pi r(1 - \frac{K}{6}r^2 + O(r^2))$

# 2 Prerequisite Manifold Theory

**Definition 2.1** *A* **smooth manifold** $\mathcal{M}$ *of dimension $n$ is a Hausdorff topological space which is second countable, together with an open cover $\{U_\alpha\}_{\alpha\in I}$ of $\mathcal{M}$ and maps $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ homeomorphic onto their image such that whenever $U_\alpha \cap U_\beta \neq \emptyset$ then $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ is smooth*

The $\phi_\alpha$ or $(U_\alpha, \phi_\alpha)$ are called **charts**, and $\phi_\alpha \circ \phi_\beta^{-1}$ are called **transition maps**. The collection of charts is called an **atlas**.

**Definition 2.2**

1. *A function $f : \mathcal{M} \to \mathbb{R}$ is* **smooth** *if for any chart $(U_\alpha, \phi_\alpha)$ we have $f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{R}$ is smooth as a function $\mathbb{R} \to \mathbb{R}$.*
2. *A map $F : \mathcal{M} \to \widehat{\mathcal{M}}$ between manifolds is* **smooth** *if for any charts $\widehat{\phi_\beta} \circ F \circ \phi_\alpha^{-1}$ is smooth.*
3. *A map $F : \mathcal{M} \to \widehat{\mathcal{M}}$ is a* **diffeomorphism** *if it is smooth and also a bijection with smooth inverse.*
4. *A map $F : [-1,1] \to \mathcal{M}$ is* **smooth** *if its restriction to $(-1,1)$ is smooth and all (partial) derivatives are continuous up to the boundary.*

**Definition 2.3** *A* **tangent vector** *$X$ at a point $p \in \mathcal{M}$ is a linear functional on the space of smooth functions in a neighbourhood of $p$ which can be written locally as $f \to \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}(p)$*

Note that the $a_i$ in the above definition depend on the chart used to define the coordinates.

**Definition 2.4** *The* **pushforward** *$u_\star(X)$ of a tangent vector $X$ at $p \in \mathcal{M}$ under a map $u : \mathcal{M} \to \mathcal{N}$ is the vector at $u(p)$ defined, for all smooth $f$, by*

$$(u_\star(X))f = X(f \circ u)$$

*for $f \in C^\infty(\mathcal{N})$.*

We can see this operation as a linear map from vectors at $p$ to vectors at $u(p)$. The tangent space $T_pM$ is the space of all vectors at $p$. This linear map is normally written as $du : T_pM \to T_{u(p)}N$.

**Definition 2.5** *$u : M \to N$ is an* **immersion** *if for all $p \in M$ we have that $du : T_pM \to T_{u(p)}N$ is injective. It is a* **submersion** *if $du$ is surjective for all $p$ and it is an* **embedding** *if it is an immersion and a topological embedding.*

## 2.1 Tangent Bundle and Vector Bundles

The tangent bundle is the space of all points $p \in U$ and tangent vectors $X$ at $p$, i.e. it is all pairs $(p, X)$. This will be a $2n$ dimensional manifold where $n = \dim \mathcal{M}$, and will have some extra structure; that of a vector bundle.

**Definition 2.6** *A* **tangent vector field** *is an assignment to each point $p \in \mathcal{M}$ of a vector in $T_p\mathcal{M}$, which can be written $a^i(x)\frac{\partial}{\partial x_i}$ for smooth local coefficients $a^i(x)$.*

**Definition 2.7** *A* **smooth real vector bundle** *$(E, \mathcal{M}, \pi)$ of rank $k \in \mathbb{N}_0$ is a smooth manifold $E$ of dimension $m + k$, a smooth manifold $\mathcal{M}$ of dimension $m$ and a smooth surjective map $\pi : E \to \mathcal{M}$ such that*

1. *there is an open cover* $\{U_\alpha\}$ *of* $\mathcal{M}$ *and diffeomorphisms* $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$
2. *for all* $p \in \mathcal{M}$, $\psi_\alpha(\pi^{-1}(p)) = \{p\} \times \mathbb{R}^k$
3. *whenever* $U_\alpha \cap U_\beta \neq \emptyset$, *the map* $\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ *takes the form* $(x, v) \mapsto (x, A_{\alpha\beta} v)$ *where* $A_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ *and is smooth.*

**Remark** If the rank is zero then $E = \mathcal{M}$ up to diffeomorphism.

**Definition 2.8** $E_p := \pi^{-1}(p)$ *is called the* **fibre** *over* $p$*, and has a natural vector space structure.*

We often write $E \xrightarrow{\pi} \mathcal{M}$ instead of the triple.

**Definition 2.9** *A* **smooth bundle morphism** *(map)* $H$ *from* $(E, \mathcal{M}, \pi)$ *to* $(E', \mathcal{M}', \pi')$ *is a smooth map* $H : E \to E'$ *which maps* $E_p$ *to some* $E'_{p'}$ *as a linear map*

**Definition 2.10** *Two bundles* $(E, \mathcal{M}, \pi)$ *and* $(E', M', \pi')$ *are* **isomorphic** *if there is a smooth bundle map* $H : (E, \mathcal{M}, \pi) \to (E', \mathcal{M}', \pi')$ *which has an inverse that is a smooth bundle map*

**Remark** We only consider vector bundles up to isomorphism.

The tangent bundle is a vector bundle of rank $m = \dim \mathcal{M}$. Given a manifold $\mathcal{M}$ with charts $(U_\alpha, \phi_\alpha)$ then the total space $T\mathcal{M}$ is the space of pairs $(p, X)$ with $p \in \mathcal{M}$ and $X \in T_p\mathcal{M}$ and $\pi(p, X) = p$ and the local trivialisations $\psi_\alpha$ are defined by $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$ given by $(p, a^i(x)\frac{\partial}{\partial x_i}) \mapsto (p, a^1, ..., a^m)$

**Definition 2.11** *A* **smooth section** *of a vector bundle* $(E, \mathcal{M}, \pi)$ *is a smooth map* $s : \mathcal{M} \to E$ *such that* $\pi \circ s = Id_{\mathcal{M}}$. *The space of all sections is denoted* $\Gamma(\mathcal{M})$.

In other words the smooth sections are assignments of a vector in $E$ to each point $p \in \mathcal{M}$.

# 3 Prerequisite linear algebra

It is presumed that one knows dual spaces.

## 3.1 Upper and Lower indices

Let $V$ be a vector space and $V^\star$ be the dual space. We have a convention on indices. We write bases of $V$ as $\{e_i\}$, with a lower index. We write a general element as $a^i e_i$, with an upper index on the coefficients. We write bases of $V^\star$ as $\{\theta^i\}$, with upper indices, and a general element as $b_i\theta^i$. To understand why, see question 1.10 in problems.

## 3.2 Inner products and dual spaces

We can equip $V$ with an inner product $\langle , \rangle$. This then induces an isomorphism $\Theta : V \to V^\star$ by $[\Theta(v)](w) = \langle v, w \rangle$ for $v, w \in V$. By $\Theta(v)$ we mean an element in the dual space, nothing more. This allows us to extend the inner product to $V^\star$ by

$$\langle \Theta(v), \Theta(w) \rangle := \langle v, w \rangle$$

### 3.3 Tensor Products

Given two vector spaces $V$ and $W$ how can we combine them? One possibility is $V \oplus W$, and another is $V \otimes W$. We consider the latter.

Define $F(V,W)$ to be the free vector space over $\mathbb{R}$ whose generators are elements of $V \times W$. Let $R(V,W)$ be the subspace generated by elements of the form

1. $(\lambda v, w) - \lambda(v,w)$
2. $(v, \lambda w) - \lambda(v,w)$
3. $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$
4. $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$

Now define $V \otimes W = F(V,W)/R(V,W)$ and write the coset containing $(v,w)$ by $v \otimes w$. In practise this means that $V \otimes W$ can be considered as all finite linear combinations of $v \otimes w$ where we agree that $(\lambda v) \otimes w = \lambda v \otimes w = v \otimes (\lambda w)$ etc following the above. If $\{e_i\}_i$ is a basis for $V$ and $\{f_j\}_{j\in J}$ is a basis for $W$ then $\{e_i \otimes e_j\}_{I,J}$ is a basis for $V \otimes W$. However, one should note that not every element of $V \otimes W$ can be written $v \otimes w$, for example $v_1 \otimes w_1 + v_2 \otimes w_2$ for $v_1, v_2$ and $w_1, w_2$ not linearly dependent in their respective spaces.

We will often consider tensors of the type $(p,q) \in \mathbb{N}_0 \times \mathbb{N}_0$, i.e. elements of $V \otimes ... \otimes V \otimes V^\star \otimes ... \otimes V^\star$ with $p$ $V$s and $q$ $V^\star$s. We will also focus on the case $V = T_p\mathcal{M}$. Such a tensor can be written as

$$T^{i_1,\dots,i_p}_{j_1,\dots,j_q} e_{i_1} \otimes ... \otimes e_{i_p} \otimes \theta_{j_1} \otimes ... \otimes \theta_{j_q}$$

and for brevity we often only write the coefficients $T$.

### 3.4 Extending inner product to tensors

Given $\langle , \rangle$ on $V$ we can define an inner product on $V \otimes ... \otimes V \otimes V^\star \otimes ... \otimes V^\star$ by

$$\langle v_1 \otimes ... \otimes v_p \otimes L^1 \otimes ... \otimes L^q, \tilde{v_1} \otimes ... \otimes \tilde{v_p} \otimes \tilde{L^1} \otimes ... \otimes \tilde{L^q} \rangle = \langle v_1, \tilde{v_1} \rangle ... \langle v_p, \tilde{v_p} \rangle \langle L_1, \tilde{L_1} \rangle ... \langle L_q, \tilde{L_q} \rangle$$

i.e. this is the natural way to do it.

### 3.5 Contraction and Traces

Given an element $V^\star \otimes V$, a sum of terms like $L \otimes v$ we can take a **contraction** by taking each term $L \otimes v$ and replace with $L(v)$, for example $T^j_i \theta^j \otimes e_i$ then the contraction is $T^i_i$. Given an element of $V \otimes V$ (or $V \otimes V$) we can use the isomorphism $\Theta : V \to V^\star$ to turn it into an element of $V^\star \otimes V$ and then contract. This is called taking a **trace**. Given a $(p,q)$ tensor you can contract/trace 2 entries to give either $(p-2,q)$or $(p-1,q-1)$ or $(p,q-2)$ tensor.

One can only trace when there is an isomorphism $V \to V^\star$. This is apparent later.

### 3.6 Symmetric and Anti symmetric tensors

**Definition 3.1** *We define We call a tensor in $V \otimes V$* **symmetric** *if it is invariant if we switch the entries of each term. Also $T = T^{ij} e_i \otimes e_j$ is symmetric if $T^{ij}$ is a symmetric matrix. More generally a $(p,0)$ tensor is symmetric if it is invariant when we switch any two entries. The vector space of all such tensors is written $Sym^p(V)$.*

An example of this is $v_1 \otimes v_2 + v_2 \otimes v_1$.

**Definition 3.2** *A $(p,0)$ tensor is* **antisymmetric** *or alternating or skew symmetric if switching any two entries reverses the sign. The vector space of all such tensors is denoted $\Lambda^p V$*

An example of this is $v_1 \otimes v_2 - v_2 \otimes v_1$.

### 3.7 Natural Projections

The natural projection from $V \otimes V \to Sym^2(V)$, written $sym$, is defined by $v_1 \otimes v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$ and extends naturally to $(p,0)$ tensors.

The natural projection from $V \otimes V \to \Lambda^2(V)$ , written $Alt$ is defined by $v_1 \otimes v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1)$ and again extends naturally to $(p,0)$ tensors.

### 3.8 Exterior or Wedge product

Here we combine $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$ to give a tensor in $\Lambda^{k+l}(V)$ by

$$\omega \wedge \eta := Alt(\omega \otimes \eta)\frac{(k+l)!}{k!l!}$$

where the $k!l!$ is just a normalising number that the lecturer has chosen.

## 4 New Bundles from Old

### 4.1 Dual Bundles, cotangent bundle

Picture the vector bundle $(E, \mathcal{M}, \pi)$ as $\coprod_{p\in\mathcal{M}} E_p = \bigcup_{p\in\mathcal{M}} \{p\} \times E_p$. We then define the dual bundle $E^\star$ to be the bundle obtained by replacing each fibre $E_p$ by its dual $E_p^\star$. There is the obvious projection; if $v \in E_p^\star$ then $\pi(v) = p$.

The transition functions for $E^\star$, denoted $A_{\alpha\beta}^{dual}$ would be obtained from $A_{\alpha\beta}$ by $A_{\alpha\beta}^{dual} = A_{\alpha\beta}^{-T}$.

**Definition 4.1** *The* **dual bundle** *of the tangent bundle $T\mathcal{M}$ is called the cotangent bundle, and denoted by $T^\star\mathcal{M}$.*

**Definition 4.2** *A smooth section of $T^\star\mathcal{M}$ is called a* **1-form**.

Locally the tangent bundle has a local frame given by $\{\frac{\partial}{\partial x^i}\}_{i=1,\ldots,n}$ where $x^1, \ldots, x^n$ are local coordinates. The dual local frame will be denoted $\{dx^i\}$.

### 4.2 Tensor product of bundles

Suppose $(E, \mathcal{M}, \pi)$ and $(\tilde{E}, \mathcal{M}, \tilde{\pi})$ are two vector bundles of ranks $k$ and $l$ respectively over the same manifold. The tensor product $E \otimes \tilde{E}$ is a vector bundle over $\mathcal{M}$ of rank $kl$, whose fibre over $p \in \mathcal{M}$ is the tensor product of the fibres $E_p \otimes \tilde{E}_p$.
**Remark** We can also consider $Sym^p(E)$ or $\Lambda^p(E)$

### 4.3 Endomorphism Bundle

**Definition 4.3** *Given a vector bundle $E$, define $End(E) = E^\star \otimes E$.*

One example of this is one that takes a tangent vector on $S^2$ and rotates it anticlockwise by $\frac{\pi}{2}$.

### 4.4 Pull Back Bundles

Imagine vector fields on $S^1$. These are all multiples of $\frac{\partial}{\partial\theta}$. Now consider $u : S^1 \to \mathbb{R}^2$. We want to define vector fields along u., i.e. for each $p \in S^1$ we want a vector in $T_{u(p)}\mathbb{R}^2$. Such vector fields along $u$ are considered as sections of the pull back bundle $u^\star(T\mathbb{R}^2)$.

Generally suppose $u : \mathcal{M} \to \mathcal{N}$ is a smooth map between manifolds and $(E, \mathcal{N}, \pi)$ is a vector bundle. We define the pullback bundle $u^\star(E)$ to be a vector bundle over $\mathcal{M}$ whose fibre at $p \in \mathcal{M}$ is an element of $E_{u(p)}$.

The total space $u^\star(E)$ of this is the subspace of $\mathcal{M} \times E$ of all pairs $(x, v)$ such that $u(x) = \pi(v)$. The base space is $\mathcal{M}$. The projections are $\underline{\pi} : u^\star(E) \to \mathcal{M}$ defined by $(x, v) \mapsto x$. The local trivialisations are given as follows. Given $\{V_\alpha\}$ the covering of $\mathcal{N}$ in the definition of $E$, and local trivialisations $\psi_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^k$, consider the open cover of $\mathcal{M}$ by $\{u^{-1}(V_\alpha)\} =: \{\tilde{V_\alpha}\}$. Then the new local trivialisations are given by $\Psi_\alpha : \underline{\pi}^{-1}(\tilde{V_\alpha}) \to \tilde{V_\alpha} \times \mathbb{R}^k$, $\Psi(x, v) = (x, Pr_2(\psi_\alpha(v))$, where $Pr_2$ is the projection $V_\alpha \times \mathbb{R}^k \to \mathbb{R}^k$

# 5 Tensors

We used the word tensor to denote an element of $\otimes^p V \otimes \otimes^q V^\star$. The most common situation for us is when $V = T_p\mathcal{M}$

**Definition 5.1** *A* $(p, q)$ **tensor field** $T$ *is a section of the bundle* $\otimes^p T\mathcal{M} \otimes \otimes^q T^\star\mathcal{M}$. $(p, 0)$ *tensors are often called contravariant and* $(0, q)$ *tensors are often called covariant.*

We can write a tensor field in coordinates (locally) as

$$T = T^{i_1,\dots,i_p}{}_{j_1,\dots,j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

### 5.1 Pulling Back covariant tensors

We saw before how to push forward a vector under a map $u : \mathcal{M} \to \mathcal{N}$, by $u_\star : T_p\mathcal{M} \to T_{u(p)}\mathcal{N}$. The dual linear map is given by $u^\star : T^\star_{u(p)}\mathcal{N} \to T^\star_p\mathcal{M}$ and is defined by

$$[u^\star(L)]v = L(u_\star(v))$$

for $v \in \mathcal{M}$.

More generally, to pull back a $(0, q)$ tensor

$$[u^\star(L_1 \otimes \dots \otimes L_q)](v_1, \dots, v_q) = L_1(u_\star(v_1))\dots L_q(u_\star(v_q))$$

### 5.2 Differential Forms

Given a manifold $\mathcal{M}$ denote by $\Lambda^k(\mathcal{M})$ the bundle of tensors in $\otimes^k T^\star\mathcal{M}$ which are alternating.

**Definition 5.2** *A* **(differential) k-form** *is a section of* $\Lambda^k(\mathcal{M})$. *The space of all such sections is* $\Omega^k(\mathcal{M})$. *There is a convention that* $\Omega^0(\mathcal{M}) = \{$*smooth functions on* $\mathcal{M}\}$

We define the wedge product of two forms as follows. If $\omega \in \Omega^k(\mathcal{M})$ and $\eta \in \Omega^l(\mathcal{M})$ then the wedge $\omega \wedge \eta \in \Omega^{k+l}(\mathcal{M})$ is given by taking the wedge in each fibre.

Locally we can write a k-form as

$$\omega = a_{i_1,\dots,i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

We can write $n$ forms locally as

$$adx^1 \wedge \dots \wedge dx^n$$

where $a$ is a smooth function on $\mathcal{M}$

**Definition 5.3** *A manifold is* **orientable** *if there exists a nowhere vanishing* $n$*-form.*

### 5.3 Integration of Differential n-forms

On an orientable manifold we can identify any two nowhere vanishing n-forms $\omega_1, \omega_2$ as follows. If $\omega_1 = f\omega_2$ for some positive function $f : \mathcal{M} \to (0, \infty)$, rather than a negative function, then they are equivalent. The two equivalence classes are the two **orientations** of M, and an oriented manifold is an orientable manifold with a choice of one of these two equivalence classes.

Suppose $\omega = adx^1 \wedge ... \wedge dx^n$ is an n form on $\mathbb{R}^n$ with compact support. Then we define

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} adx^1 ... dx^n$$

More generally if $\omega$ is an n form on an oriented manifold $\mathcal{M}$ with compact support within a single chart $(U, \phi)$, then

$$\int_{\mathcal{M}} \omega = \pm \int_{\phi(U)} adx^1 ... dx^n$$

where $\{x^1, ..., x^n\}$ are coordinates of $(U, \phi)$ and $\omega = adx^1 \wedge ... \wedge dx^n$. Here we use the coordinate map to map it into the chart. We chose a single chart for simplicity of notation. It is possible if the support isn't in a single chart.

One takes a $+$ if $dx^1 \wedge ... \wedge dx^n$ is a positive multiple of the elements of the equivalence class, or a $-$ similarly. One can check that this is all well defined.

Note here that this only works for an $n$-form, where $n = \dim \mathcal{M}$. For a $k$-form we have no notion of integration on a general manifold.

### 5.4 Exterior Derivative

We already defined derivatives, also called pushforwards, of maps between manifolds $u : \mathcal{M} \to \mathcal{N}$. In the special case of functions $f : \mathcal{M} \to \mathbb{R}$, consider $df$ as a 1-form: $df(X) = X(f)$. In local coordinates $\{x^i\}$, in terms of $\{\frac{\partial}{\partial x^i}\}$ and $\{dx^i\}$

$$df = \frac{\partial f}{\partial x^i} dx^i$$

and one can think of this "d" as an operator $d : \Omega^0(\mathcal{M}) \to \Omega(\mathcal{M})$. This extends uniquely (it turns out), to $d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ that is linear and has the properties

1. if $\omega \in \Omega^k(\mathcal{M})$ and $\eta \in \Omega^l(\mathcal{M})$ then

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$$

2. $d \circ d \equiv 0$ or also written $d^2 = 0$.

We call this $d$ the exterior derivative. In coordinates, if

$$\omega = a_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$$

then

$$d\omega = da_{i_1,...,i_k} \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}$$

**Theorem 5.4 (Stokes' theorem)** *If $\mathcal{M}^n$ is a compact and oriented manifold, and $\omega$ is an $(n-1)$-form on $\mathcal{M}$ then*

$$\int_{\mathcal{M}} d\omega = 0$$

# 6 Connections

A connection gives a way of taking a directional derivative of a section of a vector bundle.

## 6.1 Motivation

Imagine a vector field $V$ on $\mathbb{R}^2$, a point $p \in \mathbb{R}^2$ and a vector $X \in T_p\mathbb{R}^2$. We write the directional derivative of $V$ at $p$ in the direction $X$ as $\nabla_X V$.

We would like to generalise this to, for example, differentiating vector fields $V$ on a manifold $\mathcal{M}$, in the direction $X \in T_p\mathcal{M}$, but we can't (until we have seen Riemannian metrics).

The problem is there is no good chart independent definition. To do this differentiation we need some extra structure on $\mathcal{M}$, that of a connection.

## 6.2 Definition and elementary properties

**Definition 6.1** *Let $(E, \mathcal{M}, \pi)$ be a vector bundle over $\mathcal{M}$ and $\Gamma(E)$ be the space of smooth sections. A* **connection** *on $E$ is a map $\nabla : \Gamma(T\mathcal{M}) \times \Gamma(E) \to \Gamma(E)$, $(X, v) \mapsto (\nabla_X v)$ such that*

1. *for all $v \in \Gamma(E)$, the map $X \mapsto \nabla_X v$ is " linear over $C^\infty$", namely*
$$\nabla_{fX+gY} v = f\nabla_X v + g\nabla_Y v$$
*for all $f, g \in C^\infty(\mathcal{M})$ and $X, Y \in \Gamma(T\mathcal{M})$.*
2. *for all $X \in \Gamma(T\mathcal{M})$, the map $v \mapsto \nabla_X v$ is " linear over $\mathbb{R}$", namely*
$$\nabla_X(av + bw) = a\nabla_X v + b\nabla_X w$$
*for all $a, b \in \mathbb{R}$ and $v, w \in \Gamma(E)$.*
3. $$\nabla_X(fv) = f\nabla_X v + X(f)v$$
*for all $f \in C^\infty(\mathcal{M})$ and $X \in \Gamma(T\mathcal{M})$.*

It is enough to have a single vector $X \in T_p\mathcal{M}$ and $V$ a section defined on an arbitrarily small neighbourhood of $p$ in order to make sense of $\nabla_X V$. In particular $\nabla : T_p\mathcal{M} \times \Gamma(E) \to E_p$. Alternatively we could see $\nabla$ as a map

$$\nabla : \Gamma(E) \to \Gamma(T^\star\mathcal{M} \otimes E)$$

where in coordinates this is $\nabla V = dx^i \otimes \nabla_{\frac{\partial}{\partial x^i}} V$. If $X = x^j \frac{\partial}{\partial x^i} \in \Gamma(T\mathcal{M})$ then

$$dx^i \otimes \nabla_{\frac{\partial}{\partial x^i}} V \left( x^j \frac{\partial}{\partial x^j} \right) = x^j dx^i (\frac{\partial}{\partial x^j}) \nabla_{\frac{\partial}{\partial x^i}} V = x^j \delta^i_j \nabla_{\frac{\partial}{\partial x^i}} V = x^i \nabla_{\frac{\partial}{\partial x^i}} V = \nabla_{x^i \frac{\partial}{\partial x^i}} V = \nabla_X V$$

## 6.3 Extensions of connections to Tensor bundles

Given a connection $\nabla$ on a vector bundle $(E, \mathcal{M}, \pi)$ we can extend it to $\otimes^p E \otimes \otimes^q E^\star$. We are also happy when $p = 0 = q$ as then we have $\nabla : \Gamma(T\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$

**Lemma 6.2** *There exists a unique extension of $\nabla$ such that*

1. *$\nabla_X f = X(f)$ for $f \in C^\infty(\mathcal{M})$.*

*2.* $\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2)$ *for all tensor fields* $T_1, T_2$ *of arbitrary types*

*3.* $\nabla_X(\mathrm{tr}(T)) = \mathrm{tr}(\nabla_X T)$ *where* $T$ *is of type* $(p,q)$ *with* $p,q \geq 1$ *and* $\mathrm{tr}$ *is the contraction over any pair of* $E$ *and* $E^\star$ *entries.*

If we assume that this lemma is true, what would be the extension of $\nabla$ to $E^\star$? For arbitrary $L \in \Gamma(E^\star)$ and $v \in \Gamma(E)$ and $X \in \Gamma(T\mathcal{M})$ then, for $T = L \otimes v$ we have

$$\begin{aligned} X(L(v)) &\overset{1}{=} \nabla_X(L(v)) = \nabla_X(\mathrm{tr}(L \otimes v)) \overset{3}{=} \mathrm{tr}(\nabla_x(L \otimes v)) \\ &\overset{2}{=} \mathrm{tr}((\nabla_X L) \otimes v + L \otimes (\nabla_X v)) \overset{3}{=} (\nabla_X L)(v) + L(\nabla_X v) \end{aligned}$$

and so $\nabla_X L$ would be determined by $(\nabla_X L)(v) = X(L(v)) - L(\nabla_X v)$. One could easily derive an extension to $\otimes^p E \otimes \otimes^q E^\star$ by condition 2. The existence part of the above lemma reduces to checking the above is a connection. Uniqueness is then clear by construction.

Another way of writing the extended connection is by seeing a tensor as a multilinear map , eg if $T$ is of type $(1,1)$, a section of $E^\star \otimes E$ we can see it at each point $p \in M$ as a multilinear map on $E_p \otimes E_p^\star$. If $L \in \Gamma(E^\star)$ and $v \in \Gamma(E)$ then

$$(\nabla_X T)(v, L) = X\left[T(v,L)\right] - T(\nabla_X v, L) - T(v, \nabla_X L) \tag{6.1}$$

## 6.4 Pullback connection

Recall that given $u : \mathcal{M} \to \mathcal{N}$ and a vector bundle $(E, \mathcal{N}, \pi)$ we have the pull back bundle $u^\star(E)$, a vector bundle over $\mathcal{M}$ of vectors along $u$.

If $\nabla$ is a connection on $E$ then a natural pullback connection $\tilde{\nabla}$ is induced on $u^\star(E)$ as we now define. Consider $v \in \Gamma(E)$. It induces a section $v \circ u$ for $u^\star(E)$. Suppose $X \in T_p\mathcal{M}$, then we wish to write down a formula for $\tilde{\nabla}_X(v \circ u)$, as follows.

$$[\tilde{\nabla}_X(v \circ u)](p) = [\nabla_{u_\star(X)} v](u(p))$$

But not all sections of $u^\star(E)$ are of the form $v \circ u$, i.e. consider the case that $u$ is the constant map.

However there is a natural way of extending what we have just defined. Given $p \in \mathcal{M}$, choose a local frame $\{e_i\}$ for $E$ near $u(p)$. Write a general section $w \in \Gamma(u^\star(E))$ as $w(x) = a^i(x)[e_i(x) \circ u(x)]$ as $e_i \circ u$ is a local frame for $u^\star(E)$. By the axioms of connections

$$\tilde{\nabla}_X w = \tilde{\nabla}_X(a^i e_i \circ u) = X(a^i) e_i \circ u + a^i \tilde{\nabla}_X(e_i \circ u)$$

and the last thing we just made sense of.

For example consider $\gamma : (-1,1) \to \mathbb{R}^2$. Then $\tilde{\nabla}_{\frac{\partial}{\partial s}} w$ is the directional derivative along the curve.

## 6.5 Parallel sections

**Definition 6.3** *A section* $v \in \Gamma(E)$ *is* **parallel** *if* $\nabla v \equiv 0$. *We are here seeing* $\nabla$ *as a map* $\Gamma(E) \to \Gamma(T^\star\mathcal{M} \otimes E)$*, so equivalently for all* $X \in T\mathcal{M}$ *we have* $\nabla_X v = 0$.

For example if $E = T\mathbb{R}^2$ then $v$ is parallel if $v$ is parallel. Suppose $\gamma : (-1,1) \to \mathcal{M}$ and consider $w \in \Gamma(\gamma^\star(E))$. According to the definition, $w$ is parallel if $\tilde{\nabla}_{\frac{\partial}{\partial s}} w = 0$.

### 6.6 Parallel Translation

Informally, one would like to push a vector along a curve, keeping it parallel. Formally, this is defined as:

Suppose $\gamma : (-1, 1) \to \mathcal{M}$ and $E$ is a vector bundle over $\mathcal{M}$, with connection $\nabla$. If we are given a vector $w_0 \in T_{\gamma(0)}\mathcal{M}$, can we find $w \in \Gamma(\gamma^\star(E))$ such that $w(0) = w_0$ and $\tilde{\nabla} w \equiv 0$?

The answer is yes, and we will call $w$ the parallel translate of $w_0$. This is constructed as follows.

Observe that $\gamma^\star(E)$ is trivial, i.e. we can pick a global frame $e_\alpha \in \Gamma(\gamma^\star(E))$. Consider a general section $w \in \Gamma(\gamma^\star(E))$ as $w(s) = a^\alpha(s)e_\alpha(s)$ and then by definition we have

$$\begin{aligned}
\tilde{\nabla}_{\frac{\partial}{\partial s}} w &= \tilde{\nabla}_{\frac{\partial}{\partial s}} (a^\alpha(s)e_\alpha(s)) \\
&= \frac{\partial a^\alpha}{\partial s} e_\alpha + a^\beta \tilde{\nabla}_{\frac{\partial}{\partial s}} (e_\beta) \\
&= \frac{\partial a^\alpha}{\partial s} e_\alpha + a^\beta \Gamma^\alpha_{1,\beta}(e_\alpha) \\
&= \left[ \frac{\partial a^\alpha}{\partial s} + a^\beta \Gamma^\alpha_{1,\beta} \right] e_\alpha
\end{aligned}$$

where $\Gamma^\alpha_{1,\beta} = \theta^\alpha(\tilde{\nabla}_{\frac{\partial}{\partial s}} e_\beta)$ is called a Christoffel symbol.

Thus $w$ parallel means the above equation must be zero. This is a system of $k$ first order ODEs and the basic theory gives a solution for given initial conditions $w_0$. Beware, that given $w_0 \in E_p$ for $p \in M$, in general you cannot extend to a parallel section $w$ in a neighbourhood of $p$.

### 6.7 Holonomy

A slight variation of the parallel translation construction would involve a map $\gamma : [-1, 1] \to \mathcal{M}$ and with specification of $w_0 \in E_{\gamma(-1)}$. We would then find the parallel $w \in \Gamma(\gamma^\star(E))$ as before.

Let us consider the special case $\gamma(-1) = \gamma(1) = p$ in the case of $E = T\mathbb{R}^2$ with $\nabla$ being the usual directional derivative, and then parallel translating $w_0$ around $\gamma$ would give $w_0$ at the end. However in general we would get back something else.

Consider for example $\mathcal{M} = S^2$ and $\nabla$ the Levi-Civita connection (to be seen soon). Here on lines of longitude the vector points in a tangential direction and keeps the same modulus but changes direction to remain tangential. On equatorial lines it keeps the same direction and modulus. Thus if we prescribe a path from the north pole to the equator, then go round a quarter of the equator and then back to the north pole on a line of longitude, we have rotated the vector by $\pi/2$ In this case, the amount the vector is rotated equals the integral of the Gauss curvature $K$ in $\Omega$, the enclosed region. However $K \equiv 1$ on $S^2$ and so the amount of rotation is Area($\Omega$)

The phenomenon that $w_0$ is parallel translated to a different vector is called **holonomy**. This process induces a linear map from $E_p \to E_p$. We will see later that measuring holonomy is measuring curvature.

## 7 Riemannian Metrics

### 7.1 Bundle metrics

From Q1.13 in the exercises, on a vector space $V$, bilinear forms $B : V \times V \to \mathbb{R}$ correspond to elements of $B \in V^\star \otimes V^\star$.

A bundle metric on a vector bundle $E$ is an assignment to each fibre $E_p$ of an inner product, that is smoothly varying in the following sense.

**Definition 7.1** *A* **bundle metric** *$h$ on a vector bundle $E$ is a positive definite section of $Sym^2(E^\star) \subset E^\star \otimes E^\star$*

Given $v, w \in E_p$ we write their inner product as either $\langle v, w\rangle_h$ or $h(v, w)$.
**Remark** All the linear algebra we did with inner products now carries over to this setting, for example we can take traces.

If the vector bundle also has a connection $\nabla$ then we can extend it to a connection on $E^\star \otimes E^\star$ and define

**Definition 7.2** *$\nabla$ is said to be* **compatible** *with a bundle metric $h$ if $h$ is parallel, i.e. $\nabla h \equiv 0$.*

Equivalently we could have said $X(h(v, w)) = h(\nabla_X v, w) + h(v, \nabla_X w)$ for all $X \in \Gamma(T\mathcal{M})$ and $v, w \in \Gamma(E)$ which comes from an equation similar to 6.1.

## 7.2 A Riemannian metric definition

We here introduce a way of measuring angles and distances on a manifold that will induce a special connection; the Levi-Civita connection and curvature.

**Definition 7.3** *A* **Riemannian metric** *$g$ on a manifold $\mathcal{M}$ is a bundle metric on $T\mathcal{M}$.*

A Riemannian manifold $(\mathcal{M}, g)$ is a manifold with Riemannian metric. For example the standard flat Riemannian metric $g$ on $\mathbb{R}^n$ that is given by

$$g = dx^1 \otimes dx^1 + ... + dx^n \otimes dx^n$$

and note that this is the same as the standard inner product. In other words what this does is eat two vectors by summing component wise.

Riemannian metrics will allow us to define

1. distance between two points, and length of curves
2. notion of volume
3. a special connection
4. curvature

## 7.3 Isometries

**Definition 7.4** *Two Riemannian manifolds $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ are* **isometric** *if there is a diffeomorphism $u : \mathcal{M}_1 \to \mathcal{M}_2$ such that $u^\star(g_2) = g_1$.*

Riemannian geometry is invariant under isometries

## 7.4 Induced Metrics

**Definition 7.5** *Suppose that $u : \mathcal{M} \to \mathcal{N}$ is an immersion, and $g$ is a Riemannian metric on $\mathcal{N}$. Then the* **induced metric** *on $\mathcal{M}$ is $u^\star(g)$*

Observe that the immersion property is to make $u^\star(g)$ positive definite.

**Definition 7.6** *An immersion $u : (\mathcal{M}, g) \to (\mathcal{N}, h)$ is* **isometric** *if $g = u^\star h$*

For example, an isometric immersion $\gamma : (-1, 1) \to \mathbb{R}^2$ with the standard metric from last time would be a unit speed parameterisation of a curve in $\mathbb{R}^2$.

### 7.5 Volume forms for Oriented manifolds

On an oriented Riemannian manifold $\mathcal{M}$, we can define a " Hodge star" operator as follows:

$$\bigstar : \Lambda^k(\mathcal{M}) \to \Lambda^{n-k}(\mathcal{M})$$

for $k = 0, 1, ..., n$. If $\theta^1, ..., \theta^n$ is a local dual orthonormal basis for which $\theta^1 \wedge ... \wedge \theta^n$ corresponds to the orientation, then $\bigstar(\theta^1 \wedge ... \wedge \theta^k) = \theta^{k+1} \wedge ... \wedge \theta^n$

**Definition 7.7** *Given an oriented Riemannian manifold, we define the* **(Riemannian) volume form** $dV \in \Gamma(\Lambda^n(\mathcal{M}))$ *by* $dV = \bigstar 1$

in other words $dV = \theta^1 \wedge ... \wedge \theta^n$, with the $\theta^i$ as above.

### 7.6 Integration on manifolds

We know how to integrate $n$ forms from before. Now we can define integration of a function.

**Definition 7.8** *Given an oriented Riemannian manifold* $\mathcal{M}$ *and* $f : \mathcal{M} \to \mathbb{R}$ *a smooth map with compact support, the* **integral** *of* $f$ *is defined to be* $\int_{\mathcal{M}} f dV$.

This definition generalised a lot.

## 8 Levi-Civita connection

### 8.1 Definition and Motivation

The Levi-Civita connection is going to be a canonical metric on the tangent bundle of a Riemannian manifold. It is the closest we can get to "directional derivative".

Suppose we have a connection $\nabla$ on $TM$. This extends to a connection $\nabla$ on $T^\star M$. given a 1 form $\omega \in \Gamma(T^\star M)$ there are two ways of differentiating it to get a 2 form:

1. Apply the exterior derivative $d$
2. Apply $\nabla$ to get a section of $T^\star M \otimes T^\star M$ and project onto the alternating tensors and so in all is $\mathrm{Alt}(\nabla\omega)$

The first requirement of the Levi-Civita connection is "torsion free" which means 1 and 2 agree (up to normalisation). Here $d\omega = 2\mathrm{Alt}(\nabla\omega)$

The second requirement of the Levi-Civita connection is that it is compatible with the Riemannian metric.

**Theorem 8.1 (Fundamental Theorem of Riemannian Geometry)** *Given a Riemannian manifold* $(\mathcal{M}, g)$ *there exists a unique connection* $\nabla$ *on* $T\mathcal{M}$ *(the Levi Civita connection) which is both torsion free and compatible with the metric. It is determined by the Koszal formula:*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}[X\langle Y, Z\rangle - Z\langle X, Y\rangle + Y\langle Z, X\rangle + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle + \langle X, [Z, Y]\rangle]$$

We first introduce torsion properly. Take $\omega \in \Gamma(T^\star\mathcal{M})$ and $X, Y \in \Gamma(T\mathcal{M})$. Then

$$\begin{aligned}
2\mathrm{Alt}(\nabla\omega)(X, Y) &= \nabla\omega(X, Y) - \nabla\omega(Y, X) \\
&= \nabla_X\omega(Y) - \nabla_Y\omega(X) \\
&= X\omega(Y) - \omega(\nabla_X Y) - Y\omega(X) + \omega(\nabla_Y X) \\
&= X\omega(Y) - Y\omega(X) - \omega([X, Y]) - \omega(\nabla_X Y - \nabla_Y X - [X, Y])
\end{aligned}$$

From exercise sheet 3, $d\omega(X, Y) = X\omega(Y) - Y(\omega(X)) - \omega([X, Y])$, and comparing these motivates us to define the torsion as follows:

**Definition 8.2** *We define the* **torsion** *as*

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

*and then if* $\tau \equiv 0$ *we say this is* **torsion free**.

Note that this is $C^\infty(\mathcal{M})$ linear and so $\tau$ is a tensor field. Indeed $\tau \in \Gamma(\bigotimes^2 T^\star\mathcal{M} \otimes T\mathcal{M})$ and is even in $\Gamma(\Lambda^2 T^\star\mathcal{M} \otimes T\mathcal{M})$.

There is also an alternative viewpoint. We could instead define

$$\tau := d - \mathrm{Alt} \circ \nabla : \Gamma(T^\star\mathcal{M}) \to \Gamma(\Lambda^2 T^\star\mathcal{M})$$

but this is equivalent to before because it is also $C^\infty(\mathcal{M})$ linear and is an element of $\Gamma(T\mathcal{M} \otimes \Lambda^2 T^\star\mathcal{M})$.

**Proof (of theorem 8.1)** First, assume $\nabla$ has the required properties. Then we get from compatibility that

$$\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle \tag{8.1}$$

and then we use the torsion free condition to write

$$\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle - \langle Y, [X,Z]\rangle - \langle Y, \nabla_Z X\rangle \tag{8.2}$$

and if we cycle $X, Y, Z$ we get the following two equations:

$$\langle \nabla_Z X, Y\rangle = Z\langle X, Y\rangle - \langle X, [Z,Y]\rangle - \langle X, \nabla_Y Z\rangle \tag{8.3}$$

$$\langle \nabla_Y Z, X\rangle = Y\langle Z, X\rangle - \langle Z, [Y,X]\rangle - \langle Z, \nabla_X Y\rangle \tag{8.4}$$

Now using (8.4) in (8.3) and the resulting expression in (8.2) we get

$$\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle - \langle Y, [X,Z]\rangle - [Z\langle X, Y\rangle - \langle X, [Z,Y]\rangle - [Y\langle Z, X\rangle - \langle Z, [Y,X]\rangle - \langle Z, \nabla_X Y\rangle]]$$

and simplifying gives the required equation. Thus if $\nabla$ is a connection with the required properties, then it must be determined by this formula, and so is unique.

Now it is enough to show that $\nabla$ defined by this formula satisfies the connection properties. *Q.E.D.*

**Remark** Torsion only makes sense on $T\mathcal{M}$ (or $T^\star\mathcal{M}$) but compatibility of a connection with a metric makes sense on any vector bundle.

**Remark** The Levi-Civita connection corresponds to directional derivatives on $\mathbb{R}^n$. Take $\{e_i\}$ as the standard basis. Then the connection would be determined by $\nabla_{e_i} e_j$ or by $\langle \nabla_{e_i} e_j, e_k\rangle$ so just need to check that $\langle D_{e_i} e_j, e_k\rangle = 0$. Remember that $[e_i, e_j] = 0$ for all $i$ and $j$.

## 8.2 Levi-Civita connection and Traces

Just like with a general connection, the L-C connection can be extended to $\bigotimes^p T\mathcal{M} \otimes \bigotimes^q T^\star\mathcal{M}$. One of the properties was

$$\nabla_X(\mathrm{tr} T) = \mathrm{tr}(\nabla_X T)$$

where $T$ is a $(p,q)$ tensor field and tr is a contraction over any pair of $T\mathcal{M}$ and $T^\star\mathcal{M}$ entries. But now , with the metric, we can contract over any pair of indices. Thus if $T = T^{ij} e_i \otimes e_j$ then $\mathrm{tr} T = g_{ij} T^{ij} = \mathrm{tr}_{13}\mathrm{tr}_{24} g \otimes T$ where $\mathrm{tr}_{13}$ means the contraction of the first and third entries of the $(2,2)$ tensor $g \otimes T$. Thus if $\nabla$ is the L-C connection then

$$\begin{aligned}
\nabla_X(\mathrm{tr} T) &= \nabla_X(\mathrm{tr}_{13}\mathrm{tr}_{24} g \otimes T) \\
&= \mathrm{tr}_{13}\mathrm{tr}_{24} \nabla_X (g \otimes T) \\
&= \mathrm{tr}_{13}\mathrm{tr}_{24}[(\nabla_X g) \otimes T + g \otimes (\nabla_X T)] \\
&= \mathrm{tr}_{13}\mathrm{tr}_{24}(g \otimes \nabla_X T) \\
&= \mathrm{tr}(\nabla_X T)
\end{aligned}$$

and so the L-C connection commutes with any traces

Let $u : \mathcal{M} \to \mathbb{R}^n$ be an immersion. Define $g = u^\star(g_{\mathbb{R}^N})$ and so $u$ is an isometric immersion. We write $\nabla$ for the L-C connection of $\mathcal{M}$. We want to understand $\nabla_X Y$ in terms of $\nabla^{\mathbb{R}^n}$. First, consider $u_\star(Y)$, the section of $u^\star(T\mathbb{R}^n)$. Let $\tilde{\nabla}$ be the pullback connection of $\nabla^{\mathbb{R}^n}$ under $u$. Then we claim that

$$u_\star(\nabla_X Y) = [\tilde{\nabla}_X u_\star(Y)]^T$$

where $^T$ is projection onto the tangent space. Equivalently

$$g(\nabla_X Y, Z) = \langle \tilde{\nabla}_X u_\star(Y), u_\star(Z) \rangle_{\mathbb{R}^n}$$

It is quite common though to confuse $X$ and $u_\star(X)$ and to write both as $X$ and also to write $\tilde{\nabla}$ as $\nabla^{\mathbb{R}^n}$. Thus the above becomes

$$\nabla_X Y = [\nabla_X^{\mathbb{R}^n} Y]^T$$

in other words the L-C connection is the derivative in $\mathbb{R}^n$ projected onto the tangent space.

## 8.3 Taking more than one derivative; Rough Laplacian

Suppose that $(E, \mathcal{M}, \pi)$ is a vector bundle with connection $\nabla$ and $\mathcal{M}$ also has a Riemannian metric $g$ (which induces the L-C connection). Then we differentiate sections of any tensor product of $T\mathcal{M}, T^\star\mathcal{M}, E, E^\star$ via the product rule. For example it extends to $T\mathcal{M} \otimes E$ by

$$\nabla_X(Y \otimes V) = (\nabla_X^{LC} Y) \otimes V + Y \otimes (\nabla_X^E V)$$

To take two derivatives, if $V \in \Gamma(E)$ then $\nabla V \in \Gamma(T^\star\mathcal{M} \otimes E)$ and then we can now differentiate again

$$\nabla(\nabla V) =: \nabla^2 V \in \Gamma(T^\star\mathcal{M} \otimes T^\star\mathcal{M} \otimes E)$$

and we call this the **Hessian** of $V$.

We can now use the Riemannian metric $g$ to trace over the two $T^\star\mathcal{M}$ entries in $\nabla^2 V$ giving us a section of $E$.

**Definition 8.3** *The* **connection Laplacian (or rough Laplacian)** *of* $V \in \Gamma(E)$ *is defined to be*

$$\Delta V := \mathrm{tr}_{12} \nabla^2 V$$

Observe that some people take minus this as their Laplacian.

**Remark** If $E$ is of rank zero, sections of $E$ are functions on $\mathcal{M}$ and then this Laplacian is the **Laplace-Beltrami** operator. With respect to local coordinates $x^i$ (this gives a frame $\{\frac{\partial}{\partial x^i}\}$), and writing $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$, and $g^{ij} = (dx^i, dx^j)$ then

$$\Delta_{LB} f := g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - g^{ij} \Gamma_{ij}^k \frac{\partial f}{\partial x^k}$$

## 8.4 Index notation

This is good for computations where we have lots of derivatives or lots of traces.

Suppose $\{e_i\}$ is a local frame for $\mathcal{M}$ and $\{\theta^i\}$ is the dual frame and that we have a Riemannian metric $g$ on $\mathcal{M}$ and that $\{e_i\}$ is orthogonal. Normally upper and lower indices are used to keep track of how objects transform as we change basis. If $A$ transforms objects with upper indices then $A^{-T}$ transforms lower indices. However if we restrict to an orthonormal basis then we can restrict to $A \in SO(n)$, i.e. $A = A^{-T}$. The upshot is that we only need consider lower indices.

If $\{e_i\}$ is an ONB then the dual is also an ONB and then there is an isomorphism $V \to V^\star$ given by $e_i \mapsto \theta^i$, and so we can write tensors with only lower indices.

### 8.4.1 Notation for taking derivatives

If we take one derivative (using the connection $\nabla$) of , say $T \in \Gamma(T\mathcal{M} \otimes T^\star\mathcal{M})$ then we get $\nabla T \in \Gamma(T^\star\mathcal{M} \otimes T\mathcal{M} \otimes T^\star\mathcal{M})$ and we could write this as

$$\nabla_i T_{jk} \theta_i \otimes e_j \otimes \theta_k$$

This defines $\nabla_i T_{jk}$ as coefficients. Here we are taking the tensor, applying $\nabla$ and then expressing it in terms of the basis, not the other way around.

If we are taking a second derivative, then we often just write

$$\nabla_i \nabla_j T_{kl}$$

we could write the rough Laplacian of $T$ as $\nabla_i \nabla_i T_{jk}$. However, be careful as writing $\nabla_i \nabla_j$ is not the same as writing $\nabla_{e_i} \nabla_{e_j}$. Think about the following:

$$\nabla_i \nabla_j f := \nabla^2 f(e_i, e_j)$$

$$\nabla_{e_i}(\nabla_{e_j} f) = \nabla_{e_i}(e_j(f)) = \nabla_{e_i}(df(e_j)) = (\nabla_{e_i} df)(e_j) + df(\nabla_{e_i} e_j) = \nabla^2 f(e_i, e_j) + df(\nabla_{e_i} e_j)$$

and so in the latter we have an extra term.

## 8.5 Divergence

**Definition 8.4** *The* **divergence operator** $\delta : \Gamma(\bigotimes^{q+1} T^\star\mathcal{M}) \to \Gamma(\bigotimes^q T^\star\mathcal{M})$ *is defined by*

$$\delta T = -\mathrm{tr}_{12} \nabla T$$

For example, if $q = 1$ then $\delta T = -\nabla_i T_{ij}$.

In the special case that $\delta$ is applied to differential forms, we can show that, for $n = \dim \mathcal{M}$,

$$\delta = (-1)^{nq+1} \bigstar d \bigstar \tag{8.5}$$

This gives a nice expression for the Laplace-Beltrami operator:

$$\Delta_{LB} f = \mathrm{tr} \nabla^2 f = \mathrm{tr} \nabla df = -\delta df = \bigstar d \bigstar df \tag{8.6}$$

A consequence of this is the Divergence theorem.

**Theorem 8.5 (Divergence)** *Suppose $\mathcal{M}$ is a compact oriented manifold, and $\omega \in \Gamma(T^\star\mathcal{M})$. Then*

$$\int_{\mathcal{M}} (\delta\omega) dV = 0$$

**Proof**

$$\begin{aligned}
\int_{\mathcal{M}} (\delta\omega) dV &= \int_{\mathcal{M}} [(-1)^{nq+1} \bigstar d \bigstar \omega] \bigstar 1 \\
&= (-1)^{nq+1} \int_{\mathcal{M}} \bigstar[\bigstar d \bigstar \omega] \\
&= \pm \int_{\mathcal{M}} d(\bigstar\omega) \overset{\text{Stokes}}{=} 0
\end{aligned}$$

*Q.E.D.*

### 8.6 Integration by Parts

**Lemma 8.6** *On an oriented compact manifold* $(\mathcal{M}, g)$*, suppose* $S \in \Gamma(\bigotimes^q T^\star\mathcal{M})$ *and* $T \in \Gamma(\bigotimes^{q+1} T^\star\mathcal{M})$*. Then*

$$\int_{\mathcal{M}} \langle \delta T, S\rangle = \int_{\mathcal{M}} \langle T, \nabla S\rangle$$

**Proof** Define the 1-form $\omega_k := T_{k,i_1,\ldots,i_q} S_{i_1,\ldots,i_q}$ and then $\delta\omega = \langle \delta T, S\rangle - \langle T, \nabla S\rangle$ and then integrate using the divergence theorem. *Q.E.D.*

It is best to do the calculation in the proof in index notation. We restrict to the case $q = 1$ for simplicity. Then $\omega_k = T_{ki}S_i$ and then

$$\begin{aligned}\delta\omega :&= -\nabla_k\omega_k \\ &= -\nabla_k(T_{ki}S_i) \\ &= -(\nabla_k T_{ki})S_i - T_{ki}\nabla_k S_i \\ &= \langle \delta T, S\rangle - \langle T, \nabla S\rangle\end{aligned}$$

Here we use the product rule, the fact that $\nabla$ commutes with traces etc. Without index notation, this is a nightmare:

$$\nabla_X\omega = \nabla_X(\mathrm{tr}_{23} T \otimes S) = \mathrm{tr}_{23}\nabla_X(T \otimes S) = \mathrm{tr}_{23}\left[\nabla_X T \otimes S + T \otimes \nabla_X S\right]$$

and so

$$\nabla\omega = \mathrm{tr}_{34}[\nabla T \otimes S] + \mathrm{tr}_{24}[T \otimes \nabla S]$$

and so

$$\delta\omega = -\mathrm{tr}\nabla\omega = \mathrm{tr}_{12}\mathrm{tr}_{34}[\nabla T \otimes S] + \mathrm{tr}_{13}\mathrm{tr}_{24}[T \otimes \nabla S]$$

which is what we want.

Clearly index notation is better. You could also write the integration by parts formula in index notation:

$$-\int_{\mathcal{M}} (\nabla_k T_{ki}) S_i dV = \int_{\mathcal{M}} T_{ki}\nabla_k S_i dV$$

This formula is maybe more explanative and looks similar to the IBP we knew from second year.

## 9 Geodesics, lengths of paths, Riemannian Distance

### 9.1 Definition of Geodesic

Consider a map $\gamma$ from an interval $I \subset \mathbb{R}$ with standard coordinate $s$, to a Riemannian manifold $(\mathcal{M}, g)$. Informally $\gamma$ will be a geodesic if it has zero acceleration.

**Definition 9.1** *We define the velocity as* $\gamma'(s) := \gamma_\star\left(\frac{\partial}{\partial s}\right)$

We write $\tilde{\nabla}$ for the pull-back of the L-C connection $\nabla$ under $\gamma$. We adopt the shorthand $D_s = \tilde{\nabla}_{\frac{\partial}{\partial s}}$

**Definition 9.2** $\gamma$ *is called a* **geodesic** *if* $D_s(\gamma'(s)) \equiv 0$*, or also written* $\tilde{\nabla}_{\frac{\partial}{\partial s}}\left(\gamma_\star\left(\frac{\partial}{\partial s}\right)\right) = 0$

The intuition, as I see it here, is that the velocity is the speed in $\mathbb{R}$ pushed forward to lie on the manifold, and so for each point in $\mathbb{R}$ we have speed as a vector in the tangent space of the manifold.

In other words, $\gamma'(s)$ is a parallel section of $\gamma^\star(T\mathcal{M})$. In the literature it is common to fudge notation and write $X = \gamma'(s)$ and the equation of the geodesic as $\nabla_X X = 0$.

**Theorem 9.3** *Given $p \in \mathcal{M}$ and $X \in T_p\mathcal{M}$ then there exists an open interval $I \subset \mathbb{R}$ containing 0 and a geodesic $\gamma : I \to \mathcal{M}$ with $\gamma(0) = p$ and $\gamma'(0) = X$. Any other geodesic $\tilde{\gamma} : \tilde{I} \to \mathcal{M}$ with $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}'(0) = X$ agrees with $\gamma$ on the connected intersection $I \cap \tilde{I}$.*

The proof of this is essentially writing down the condition to be a geodesic into an ODE and then using existence of ODEs.

**Proof** Pick local coordinates $\{x^i\}$ near $p \in \mathcal{M}$ and without loss of generality $p$ corresponds to $x^i = 0$ for all $i$. Write $\gamma(s) = (x^1(s), ..., x^n(s))$. We will write the equation for $\gamma$ being a geodesic as an ODE for $x^1, ..., x^n$. We can write $\gamma'(s) = (\dot{x}^1(s), ..., \dot{x}^n(s))$ or more precisely as $\gamma'(s) = \dot{x}^i(s)\frac{\partial}{\partial x^i}(\gamma(s))$. Beware, because $\frac{\partial}{\partial x^i}(\gamma(s))$ is not differentiating $\gamma(s)$. It is the tangent vector $\frac{\partial}{\partial x^i}$ at $\gamma(s) \in \mathcal{M}$. By definition of the pull-back connection we have

$$
\begin{aligned}
D_s\gamma'(s) &= \tilde{\nabla}_{\frac{\partial}{\partial s}}(\dot{x}^i(s)\frac{\partial}{\partial x^i}) \\
&= \frac{\partial}{\partial s}\dot{x}^i(s)\frac{\partial}{\partial x^i} + \dot{x}^i(s)\tilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial}{\partial x^i} \\
&= \ddot{x}^i(s)\frac{\partial}{\partial x^i} + \dot{x}^i(s)\nabla_{\gamma'(s)}\frac{\partial}{\partial x^i} \\
&= \ddot{x}^k(s)\frac{\partial}{\partial x^k} + \dot{x}^i(s)\dot{x}^j(s)\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} \\
&= (\ddot{x}^k(s) + \dot{x}^i(s)\dot{x}^j(s)\Gamma^k_{ij})\frac{\partial}{\partial x^k} \\
&= 0
\end{aligned}
$$

and so we have a nice ODE, and so by ODE theory we have a local solution if we specify $x^i(0)$ and $\dot{x}^i(0)$, as we have.

We now prove uniqueness. Suppose the two geodesics are not equal. They must be equal on a neighbourhood of $s = 0$ by ODE theory. Thus there must be a value $s_0 > 0$ where the geodesics diverge. At this point, both geodesics have the same velocity, and so by applying the ODE argument they agree locally near $s_0$. This is a contradiction. *Q.E.D.*

## 9.2 Maximal Geodesics; geodesic completeness

By exploiting the existence and uniqueness of theorem 9.3 we can deduce that there exists a well defined **maximal geodesic** associated with an arbitrary $p \in \mathcal{M}$ and $X \in T_p\mathcal{M}$ (associated with an arbitrary $X \in T\mathcal{M}$ for which we define $\pi(X) = p$) i.e. it cannot be extended to any larger interval.

**Definition 9.4** *The Riemannian manifold $(\mathcal{M}, g)$ is* **geodesically complete** *if the maximal geodesic associate with each $X$ is defined for all $x \in \mathbb{R}$.*

As an example, the flat disc is not geodesically complete. Soon (Hopf-Rinow theorem) we will see that geodesic completeness is equivalent to completeness with respect to a natural metric space structure that we will define on a Riemannian manifold.

## 9.3 Exponential Map

Using the previous section we will define the exponential map as a map from part of $T\mathcal{M}$ to $\mathcal{M}$ which takes $X \in T\mathcal{M}$ and maps it to $\gamma(1)$ where $\gamma$ is the unique geodesic associated with $X$.

**Definition 9.5** *We define $D\mathcal{M}$ to be the subset of $T\mathcal{M}$ consisting of all $X \in T\mathcal{M}$ for which there is a geodesic associated with $X$ which is defined on an interval $I$ containing $[0,1]$.*

*The* **exponential map** $\exp : D\mathcal{M} \to M$ *is defined by* $\exp(X) = \gamma(1)$ *where $\gamma$ is the unique geodesic associated with $X$.*

We define $D\mathcal{M}$ in order to make the exponential map make sense. For example, if $\mathcal{M} = S^2$ then $D\mathcal{M} = T\mathcal{M}$.

**Remark** If we restrict to one fibre $T_p\mathcal{M} \cap D\mathcal{M}$ then we write the exponential map as $\exp_p : T_p\mathcal{M} \cap D\mathcal{M} \to \mathcal{M}$

On the sphere $\exp_p$ maps $D(0,\pi)$, the ball of radius $\pi > 0$ in the tangent space in $T_p\mathcal{M}$, onto $S^2 \setminus \{\text{south pole}\}$.

**Lemma 9.6 (Properties of the exponential map.)** *The following properties hold.*

1. *$D\mathcal{M}$ is an open subset of $T\mathcal{M}$ containing all zero length vectors, and each fibre is "star shaped", i.e. each point can be connected to $0 \in T_p\mathcal{M}$ by a straight line.*
2. *Given $X \in D\mathcal{M}$, the curve $s \mapsto \exp(sX)$ is a geodesic defined at least for $s \in [0,1]$ and is the same geodesic $\gamma$ that defined $\exp(X) = \gamma(1)$ (see Q5.3 Rescalling lemma).*
3. $\exp$ *is a smooth map*

We omit the proof. One can either see the Rescaling lemma, or can consider the so called "geodesic flow".

## 9.4 Normal Coordinates

Throughout this section, $\mathcal{M}$ is a Riemannian manifold.

**Lemma 9.7 (Normal Neighbourhood lemma)** *Given $p \in \mathcal{M}$ there exists $\epsilon(p) > 0$ such that $\exp_p$ is a diffeomorphism from $D(0,\epsilon) \subset T_p\mathcal{M}$ to its image in $\mathcal{M}$.*

We will call the supremum of all such $\epsilon$ the **injectivity radius** at $p$. Note that this can be infinity. For example, on the sphere this is $\pi$. To prove this lemma we will need the inverse function theorem for manifolds.

**Proof (sketch)** We want to take the derivative of $\exp_p : T_p\mathcal{M} \cap D\mathcal{M} \to \mathcal{M}$ at $0 \in T_p\mathcal{M}$. We get that this is a map

$$d(\exp_p) : T_0(T_p\mathcal{M} \cap D\mathcal{M}) \to T_p\mathcal{M}$$

but there is a natural identification of $T_0(T_p\mathcal{M})$ with $T_p\mathcal{M}$ and so we can write

$$d(\exp_p) : T_p\mathcal{M} \to T_p\mathcal{M}$$

and by unravelling the definitions we get that

$$d(\exp_p) = Id$$

We can show this last line as follows. If we take $\tau(t) = tX$ then

$$d(\exp_p) = \left.\frac{d}{dt}\right|_{t=0} \exp_p \circ \tau(t) = \left.\frac{d}{dt}\right|_{t=0} \exp_p tX = \gamma_X(t)|_{t=0} = X$$

where $\gamma_X$ is the geodesic at $p$ initially in direction $X$.

In particular it is invertible and so we can apply the inverse function theorem. This then gives the statement of the lemma. *Q.E.D.*

We introduce some terminology. For $r \in (0,$ injectivity radius $)$ we call $\exp_p(D(0,r))$ a **geodesic ball** of radius $r$ about $p$. We call $\exp_p(\partial D(0,r))$ the **geodesic sphere** about $p$.

To define normal coordinates pick $p \in \mathcal{M}$, and choose an orthonormal basis $\{e_i\}$ of $T_p\mathcal{M}$. Define a chart $\phi$ by

$$\phi^{-1}(x) = \exp_p(x^i e_i)$$

where $x = (x^1, ..., x^n)$ are the coordinates of $x^i e_i$.

**Definition 9.8** *The coordinates associated with this chart are called* **normal coordinates**

Normal coordinates $\{x^i\}$ are very useful, e.g. for making computations because $\nabla_{\frac{\partial}{\partial x^i}} = 0$ at $p$. Also, $g_{ij} = \delta_{ij}$ and so in this case $\{\frac{\partial}{\partial x^i}\}$ are orthogonal.

### 9.4.1 Uniform Normal Neighbourhoods

It is a problem in lemma 9.7 that the injectivity radius could depend horribly on $p$; it could be discontinuous with respect to $p$.

**Lemma 9.9 (Uniform normal neighbourhood lemma)** *For all $p \in \mathcal{M}$ and any neighbourhood $U$ of $p$ there exists a neighbourhood $W \subset U$ of $p$ such that there exists a $\delta > 0$ such that for all $q \in W$, the restriction of $\exp_q$ to the ball $D(0,\delta) \subset T_q\mathcal{M}$ is a diffeomorphism and $\exp_q(D(0,\delta)) \supset W$.*

**Proof** [Non-examinable?] The key idea here is instead of linearising $\exp_p$ at $0 \in T_p\mathcal{M}$ we linearise $X \mapsto (\pi(X), \exp(X))$ at $0 \in T_p\mathcal{M}$ within $T\mathcal{M}$. For a full proof see Lee p78. *Q.E.D.*

## 9.5 Adapted Local Frames

We said normal coordinates are good for doing computations. Also useful are adapted local frames. From 3.7 near a point $p \in \mathcal{M}$ there exists a local orthonormal frame $\{e_i\}$.

**Lemma 9.10** *Near any $p \in \mathcal{M}$ there exists a local orthonormal frame $\{e_i\}$ such that at $p$, $\nabla^{LC} e_i = 0$ for all $i$.*

**Proof (sketch)** Pick any ONB $\{e_i\}$ of $T_p\mathcal{M}$. Extend $\{e_i\}$ to a neighbourhood of $p$ (any geodesic ball) by parallel translating each $e_i$ radially along geodesics. By Q3.10, this gives an orthonormal frame. By ODE theory, $\{e_i\}$ is smooth. *Q.E.D.*

Note we don't have $\nabla e_i = 0$ in a whole neighbourhood of $p$, in general.

## 9.6 Lengths of paths/curves

**Definition 9.11** *The* **length** *of a smooth path $\gamma : [a,b] \to (\mathcal{M}, g)$ is defined to be*

$$L(\gamma) = \int_a^b |\gamma'(s)| ds$$

*where* $|X| = g(X,X)^{\frac{1}{2}}$

Observe that $L$ is independent of the parameterisation, see Q5.6.
**Remark** We can extend this to " piecewise smooth" curves, i.e. continuous $\gamma : [a,b] \to \mathcal{M}$ such that there exist finite subdivision of $[a,b]$ where $a < a_0 < a_1 < ... < a_k < b$ such that $\gamma|_{[a_{i-1},a_i]}$ is smooth for all $i \in \{1,...,k\}$. Also allow $\gamma$ to be constant in which case $L(\gamma) = 0$. We call all such curves "**admissible**".

## 9.7 Riemannian distance

Assume that $(\mathcal{M}, g)$ is connected. Given $p, q \in \mathcal{M}$, define $\Sigma_{p,q}$ to be the set of all admissible curves $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

**Definition 9.12** *The* **Riemannian distance** *is a function $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ defined by*

$$d(p,q) = \inf_{\gamma \in \Sigma_{p,q}} (L(\gamma))$$

Observe that this definition implicitly contains the Riemannian metric $g$. We use Q5.7 to check that $\Sigma_{p,q}$ is non empty for arbitrary $p, q$, if $\mathcal{M}$ is connected.

**Lemma 9.13** *With $d$ defined as above, any connected Riemannian manifold is a metric space, whose topology agrees with that of the manifold.*

## 9.8 Minimising curve and geodesics

**Definition 9.14** *An admissible curve $\gamma : [a,b] \to \mathcal{M}$ is* **minimising** *if $L(\gamma) \leq L(\tilde{\gamma})$ where $\tilde{\gamma}$ is any other admissible curve from $\gamma(a)$ to $\gamma(b)$.*

Equivalently we could define $L(\gamma) = d(\gamma(a), \gamma(b))$. Q3.10 gives the fact that geodesics have constant speed. Note that we can always parametrise a curve to have constant speed without changing length.

**Theorem 9.15** *If an admissible curve is minimising and it has constant speed then it is a geodesic. In particular it is smooth, and not just piecewise smooth.*

We prove this later.

### 9.8.1 Variations of Curves

Given $\gamma : [a,b] \to \mathcal{M}$ that is smooth we want to consider a variation of $\gamma$, more precisely a smooth map $F : (-\delta, \delta) \times [a,b] \to \mathcal{M}$ such that $F(0, \cdot) = \gamma$. We will consider for $\eta \in (-\delta, \delta)$ the curves $F(\eta, \cdot)$. By smooth here we mean it is smooth on the restriction to $(-\delta, \delta) \times (a,b)$ and the derivatives extend continuously to $(-\delta, \delta) \times [a,b]$ with respect to charts.

### 9.8.2 First Variation Formula

The goal is to find a formula for $\frac{d}{d\eta} L(F(\eta, \cdot))|_{\eta=0}$. We will assume that $\gamma = F(0, \cdot)$ has unit speed parametrisation, i.e. $|\gamma'(s)| = 1$ for all $s \in [a,b]$. We will use the notation

$$X = \frac{\partial F}{\partial s} = \frac{\partial F}{\partial s}(\eta, s) = F_\star \left( \frac{\partial}{\partial s} \right)$$

and that

$$Y = \frac{\partial F}{\partial \eta} = \frac{\partial F}{\partial \eta}(\eta, s) = F_\star \left( \frac{\partial}{\partial \eta} \right)$$

Observe that $|X| = 1$ when $\eta = 0$ and $X$ is the tangent vector along the curves and $Y$ is the variation vector field.

**Theorem 9.16** *Given a smooth $F$ as above, for which $F(0, \cdot)$ has unit speed parametrisation,*

$$\frac{d}{d\eta} L(F(\eta, \cdot))|_{\eta=0} = [\langle X, Y \rangle]_{s=a}^{s=b} - \int_a^b \langle Y, D_s X \rangle ds$$

Here the first term on the right hand side describes the change due to the endpoints changing, and the second term the change in shape of the curve.

In the proof of this, we will need the "Symmetry lemma" in Q5.9, which tells us that

$$D_s Y = D_\eta X$$

or equivalently that $D_s \frac{\partial F}{\partial \eta} = D_\eta \frac{\partial F}{\partial s}$, i.e. in this case the derivatives commute.

**Proof** By the definition of $L$, we have

$$L(F(\eta,\cdot)) = \int_a^b |X| ds = \int_a^b g(X,X)^{\frac{1}{2}} ds$$

and so using the fact that $|X| = 1$ we get that

$$\begin{aligned}
\frac{d}{d\eta} L(F(\eta,\cdot))|_{\eta=0} &= \int_a^b \frac{1}{2} g(X,X)^{-\frac{1}{2}} \frac{\partial}{\partial \eta} g(X,X) ds \bigg|_{\eta=0} \\
&= \int_a^b g(D_\eta X, X) ds \bigg|_{\eta=0} \\
&= \int_a^b g(D_s Y, X) ds \bigg|_{\eta=0} \\
&= \int_a^b \left[ \frac{\partial}{\partial s} [g(X,Y)] - g(Y, D_s X) \right] ds \\
&= [g(X,Y)]_{s=a}^{s=b} - \int_a^b g(Y, D_s X) ds \\
&= [\langle X, Y \rangle]_{s=a}^{s=b} - \int_a^b \langle Y, D_s X \rangle ds
\end{aligned}$$

*Q.E.D.*

### 9.8.3 Proof of Minimising curves are geodesics (Theorem 9.15)

We want to show that if $\gamma : [a,b] \to \mathcal{M}$ is a minimising curve with constant speed then $D_s \gamma'(s) \equiv 0$. Let us suppose that $Y \in \Gamma(\gamma^\star(T\mathcal{M}))$ with $Y(a) = 0 = Y(b)$. Construct, for sufficiently small $\delta > 0$ a function $F : (-\delta,\delta) \times [a,b] \to \mathcal{M}$ such that $F(0,\cdot) = \gamma$, $F(\eta,a) = \gamma(a)$, $F(\eta,b) = \gamma(b)$ and $\frac{\partial F}{\partial \eta}(0,\cdot) = Y$, and as usual we set $X = \frac{\partial F}{\partial s}$.

By minimising hypothesis we have

$$\frac{d}{d\eta} L(F(\eta,\cdot))|_{\eta=0} = 0$$

and the first variation formula gives $\frac{d}{d\eta} L(F(\eta,\cdot))|_{\eta=0} = -\int_a^b \langle Y, D_s X \rangle ds$ due to the end points not changing. Therefore $\int_a^b \langle Y, D_s X \rangle ds = 0$ for arbitrary $Y$. Pick a "cut-off" function $\phi : \mathbb{R} \to [0,\infty)$ such that $\phi(x) > 0$ precisely when $x \in (a,b)$, and so it is zero otherwise. Set $Y(s) = \phi(s) D_s X$ and then

$$\int_a^b \phi(s) \langle D_s X, D_s X \rangle ds = \int_a^b \phi(s) |D_s X|^2 ds = 0$$

and so $D_s X \equiv 0$ i.e. $\gamma$ is a geodesic since $D_s \gamma'(s) = 0$ as $X(0,s) = \frac{\partial F}{\partial s}(0,s) = \gamma'(s)$.

## 9.9 Geodesics are locally minimising

**Definition 9.17** *A curve $\gamma : I \to \mathcal{M}$ from some interval $I \subset \mathbb{R}$ is called* **locally minimising** *if for all $c \in I$ there exists a neighbourhood $U \subset I$ of $c$ such that for all $a', b' \in U$ then the restriction of $\gamma|_{[a',b']}$ is minimising.*

**Remark** If $\gamma$ is minimising then it is locally minimising.

This says that it is locally minimising if it is minimising on a small neighbourhood of the domain.

**Theorem 9.18** *Every geodesic is locally minimising.*

We proceed to develop the tools to prove this statement

### 9.9.1 Gauss Lemma

Given $p \in \mathcal{M}$ define the **radial distance function** on a geodesic ball centred at $p$ by picking normal coordinates $\{x^i\}$ centred at $p$ and setting

$$r(x) = \left( \sum_{i=1}^{n} (x^i)^2 \right)^{\frac{1}{2}}$$

This is well defined because it is independent of the choice of orthonormal basis $\{e_i\}$ from the construction of normal coordinates.

Consider the radial geodesic $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = p, \gamma(1) = x$ i.e. $\gamma(s) = \exp_p(sx^i e_i)$. Then $\gamma'(0) = x^i e_i$ and therefore $|\gamma'(s)| = |\gamma'(0)| = |x^i e_i| = r(x)$ and therefore $L(\gamma) = r$. We will shortly see that $\gamma$ is minimising, i.e. $d(x,p) = L(\gamma) = r(x)$.

Given such an $r$, it doesn't immediately make sense to write $\frac{\partial}{\partial r}$. Normally this would mean differentiating with respect to $r$ while keeping other variables fixed. However these other variables aren't defined yet.

To define $\frac{\partial}{\partial r}$ at $x$ in the geodesic ball (not at $p$), take radial geodesic $\gamma$ discussed above. In coordinates $\gamma(s) = (x^1, ..., x^n)$ so $\gamma'(s) = x^i \frac{\partial}{\partial x^i}|_{\gamma(s)}$ which has norm $r$. We thus define a **unit radial vector field** to be

$$\frac{\partial}{\partial r} = \frac{x^i}{r} \frac{\partial}{\partial x^i}$$

We could have tried to define this as $\nabla r = (dr)^{\#}$, where sharp is the natural isomorphism from $T^\star\mathcal{M}$ to $T\mathcal{M}$. We now want to show that $\frac{\partial}{\partial r} = \nabla r$. This will need the Gauss lemma.

What we are trying to show is the same as, for all $X \in T_q\mathcal{M}$ $q \neq p$ we need to show $dr(X) = \langle \nabla r, X \rangle = \langle \frac{\partial}{\partial r}, X \rangle$. If we set $X = \frac{\partial}{\partial r}$ then we have the right hand side equal to

$$RHS = \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = \left| \frac{\partial}{\partial r} \right|^2 = 1$$

and

$$LHS = \frac{x^i}{r} dx^i \left( \frac{x^j}{r} \frac{\partial}{\partial x^j} \right) = \frac{x^i x^j}{r} \delta_{ij} = 1$$

and so they are equal. We can decompose a general $X$ as a radial vector and a vector tangent to the geodesic sphere at $p$ through $q$. Therefore it remains to prove the above for $X$ a tangent vector to the geodesic sphere through $q$, i.e. for $X$ such that $\langle \nabla r, X \rangle = dr(X) = X(r) = 0$. Thus we must prove that

$$\langle \frac{\partial}{\partial r}, X \rangle = 0$$

**Lemma 9.19 (Gauss)** *The unit radial vector field $\frac{\partial}{\partial r}$ is orthogonal to the geodesic spheres.*

As a consequence $\nabla r = \frac{\partial}{\partial r}$ and in particular $|\nabla r| = 1$.
**Proof** Let $q$ lie in a geodesic ball centred at $p$, say on the sphere $\exp_p(\partial D(0,R))$ and write $q = \exp_p(V)$ for $V \in \partial D(0,R)$.

Let $X \in T_q\mathcal{M}$ be tangent to the geodesic sphere $\partial D(0,R)$ i.e. $X(R) = 0$. $\exp_p$ is a diffeomorphism on geodesic balls and therefore there exits a $W \in T_V(T_p\mathcal{M})$ such that $d\exp_p(W) = X$. Because $X$ is tangent to geodesic spheres we know that $W$ is tangent to $\partial D(0,R)$ at $V$.

In particular, we can choose a curve $\sigma : (-\delta,\delta) \to \partial D(0,R)$ such that $\sigma(0) = V$ and $\sigma'(0) = W$. Consider $F : (-\delta,\delta) \times [0,1] \to \mathcal{M}$ defined by $F(\eta,s) = \exp_p(s\sigma(\eta))$. Therefore for all $\eta \in (-\delta,\delta)$ then $F(\eta,\cdot)$ is a radial geodesic with speed $R$. The objective is to show that $\frac{1}{r}\frac{\partial F}{\partial s}(0,1)$ is orthogonal to $X$.

We proceed with a few computations:

$$\frac{\partial F}{\partial \eta}(0,0) = \frac{d}{d\eta}|_{\eta=0} \exp_p(0) = 0$$

$$\frac{\partial F}{\partial \eta}(0,1) = \frac{d}{d\eta}|_{\eta=0} \exp_p(\sigma(\eta)) = X$$

by construction. Then

$$\frac{\partial}{\partial s}\left(\left\langle \frac{\partial F}{\partial \eta}, \frac{\partial F}{\partial s}\right\rangle\right) = \left\langle D_s\frac{\partial F}{\partial \eta}, \frac{\partial F}{\partial s}\right\rangle + \left\langle \frac{\partial F}{\partial \eta}, D_s\frac{\partial F}{\partial s}\right\rangle$$

but since $F(\eta,\cdot)$ is a geodesic, we have $D_s\frac{\partial F}{\partial s} = 0$ and the symmetry lemma (Q5.9) gives $D_s\frac{\partial F}{\partial \eta} = D_\eta\frac{\partial F}{\partial s}$ and thus we have

$$\frac{\partial}{\partial s}\left(\left\langle \frac{\partial F}{\partial \eta}, \frac{\partial F}{\partial s}\right\rangle\right) = \left\langle D_\eta\frac{\partial F}{\partial s}, \frac{\partial F}{\partial s}\right\rangle = \frac{1}{2}\frac{\partial}{\partial \eta}\left\langle \frac{\partial F}{\partial s}, \frac{\partial F}{\partial s}\right\rangle = \frac{1}{2}\frac{\partial}{\partial \eta}\left|\frac{\partial F}{\partial s}\right|^2 = 0$$

since $F(\eta,\cdot)$ is a geodesic so $\frac{\partial F}{\partial s}$ is constant, so its rate of change is zero.

At $s = 0$ we have $\frac{\partial F}{\partial \eta}(0,0) = 0$ and therefore

$$\left.\left\langle \frac{\partial F}{\partial \eta}, \frac{\partial F}{\partial s}\right\rangle\right|_{s=0,\eta=0} = 0$$

and therefore if we integrate from $s = 0$ to $s = 1$ to get, for $\eta = 0$,

$$\int_0^1 \frac{\partial}{\partial s}\left(\left\langle \frac{\partial F}{\partial \eta}, \frac{\partial F}{\partial s}\right\rangle\right) ds = \left.\left\langle \frac{\partial F}{\partial \eta}, \frac{\partial F}{\partial s}\right\rangle\right|_{s=1,\eta=0} - \left.\left\langle \frac{\partial F}{\partial \eta}, \frac{\partial F}{\partial s}\right\rangle\right|_{s=0,\eta=0} = 0$$

and so $\langle X, \frac{\partial F}{\partial s}\rangle = 0$. *Q.E.D.*

### 9.9.2 Radial Geodesics are Initially minimising

**Lemma 9.20** *Suppose $p \in \mathcal{M}$ and $\epsilon \in (0,\infty)$ is no more than the injectivity radius. Then for all $q \in \exp_p(D(0,\epsilon))$ (i.e. $q = \exp_p(X)$ for $X \in D(0,\epsilon)$), the radial geodesic $s \mapsto \exp_p(sX)$ for $s \in [0,1]$ is a minimising curve from $p$ to $q$.*

*Moreover, this is the unique minimising curve from $p$ to $q$ within the class of piecewise smooth curves from $p$ to $q$ modulo reparametrisation.*

**Proof** Define $R := |X|$ and consider the radial geodesic with arclength parametrisation $\gamma : s \mapsto \exp_p(\frac{sX}{R})$. for $s \in [0, R]$. Observe that $|\gamma'(s)| = 1$.

First we need to prove that any other curve from $p$ to $q$ has length at least $R$, $R$ being the length of $\gamma$.

We can assume the other curve is smooth (easy to extend to piecewise smooth). Also, by chopping off the start of the curve we may assume it never returns to $p$ (this only decreases the length). By chopping off the end of the curve we may assume we have a curve $\sigma : [0, 1] \to \exp_p(D(0, R))$ such that $\sigma(0) = p$ and $\sigma(1) \in \exp_p(\partial D(0, R))$. We will show even $\sigma$ has length at least $R$.

We decompose orthogonally to get $\sigma'(s) = a(s)\frac{\partial}{\partial r} + V(s)$ for $a(s) \in \mathbb{R}$ and $V(s) \perp \nabla r$ and this is for $s > 0$. Therefore

$$|\sigma'(s)|^2 = |a(s)|^2|\nabla r|^2 + |V(s)|^2 \geq |a(s)|^2$$

Also the inner product of the orthogonal decomposition with $\nabla r$ gives

$$a(s) = \langle \sigma'(s), \nabla r \rangle = dr(\sigma'(s))$$

and so

$$\begin{aligned}
L(\sigma) &= \lim_{\epsilon \to 0} L(\sigma|_{[\epsilon,1]}) \\
&= \lim_{\epsilon \to 0} \int_\epsilon^1 |\sigma'(s)| ds \\
&\geq \lim_{\epsilon \to 0} \int_\epsilon^1 dr(\sigma'(s)) ds \\
&= \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{d}{ds}[r(\sigma(s))] ds \\
&= \lim_{\epsilon \to 0} [r(\sigma(1)) - r(\sigma(\epsilon))] \\
&= R - r(\sigma(0)) \\
&= R
\end{aligned}$$

It thus remains to show uniqueness. Suppose we have some other curve from $p$ to $q$ with the least possible length. By the beginning of the proof, our new "competitor" curve has that it doesnt return to $p$, and never leaves $\exp_p(D(0, \epsilon))$ and that $\sigma'(s) = \frac{\partial}{\partial r}$ (if we reparameterise so that $|\sigma'(s)| = 1$).

Thus $\sigma$ and the radial geodesic are integral curves of $\frac{\partial}{\partial r}$ and they "end" at the $q$. We cannot use the beginning as $\frac{\partial}{\partial r}$ is not defined there. Therefore they are the same, using ODE theory. *Q.E.D.*

**Corollary 9.21** *Within any geodesic ball, the radial distance function $r(x)$ equals the Riemannian distance function $d(x, p)$.*

Thus geodesic balls $\exp_p(D(0, r))$ can be written $\mathcal{B}(p, r)$ where more generally we define $\mathcal{B}(p, r) = \{x \in \mathcal{M} : d(x, p) < r\}$

### 9.9.3 Proof that Geodesics are locally minimising

**Proof (of theorem 9.18)** Given a geodesic curve $\gamma : I \to \mathcal{M}$ pick $c \in I$. We need to find a neighbourhood about $c$ where $\gamma$ is minimising. Without loss of generality we can assume that $I$ is an open interval.

We then appeal to lemma 9.9 to find a uniformly normal neighbourhood $W$ about $\gamma(c)$, i.e. there exists a $\delta > 0$ such that each point of $W$ has a geodesic ball of radius $\delta$ that contains the whole of $W$.

Let $(a, b) \subset I$ be the connected component of the preimage of $W$, i.e $(\gamma^{-1}(W))$ containing $c$. We claim that if $a < a' < b' < b$ then $\gamma|_{[a',b']}$ is minimising, with $c \in (a', b')$.

To show this note that $\gamma([a', b'])$ is contained in $W$ by construction and therefore within a geodesic ball of radius $\delta$ centred at $\gamma(a')$. By construction $\gamma|_{[a',b']}$ is a radial geodesic starting at $\gamma(a')$, remaining within the geodesic ball centred at $\gamma(a')$. By lemma 9.20, radial geodesics are minimising while they stay within a geodesic ball. *Q.E.D.*

### 9.10 Completeness, Geodesic completeness and the Hopf-Rinow theorem.

**Theorem 9.22 (Hopf-Rinow)** *A connected Riemannian manifold is complete as a metric space if and only if it is geodesically complete.*

*In this case, any two points can be connected by a minimising geodesic.*

For example, $\mathbb{R}^2 \setminus \{0\}$ is not complete and you cannot connect $x$ and $-x$ by a minimising geodesic. We will skip the proof but it can be found in Lee p108.

## 10 Curvature

We are aiming to talk about the curvature of a Riemannian manifold. We will do this by defining the curvature of a connection and then we will be able to take the curvature of the Levi-Civita connection.

### 10.1 Curvature of a Connection

Temporarily we stop assuming that we have a Riemannian metric, at least for this subsection.

**Definition 10.1** *Given a vector bundle $(E, \mathcal{M}, \pi)$ with connection $\nabla$, the* **curvature** *of $\nabla$ is the map $\Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(E) \to \Gamma(E)$ written $(X, Y, v) \mapsto R(X, Y)v$ defined by*

$$R(X, Y)v := -\nabla_X \nabla_Y v + \nabla_Y \nabla_X v + \nabla_{[X,Y]} v$$

It should be noted that half of everybody takes the opposite sign convention to that used above. Q6.1 says curvature is not just a map of this form. It is also a tensor. In fact $R(X, Y)$ at $p \in \mathcal{M}$ only depends on $X, Y, v$ at $p$ and is not dependent on values in some neighbourhood of $p$. Thus it is in fact a section of $T^\star\mathcal{M} \otimes T^\star\mathcal{M} \otimes E^\star \otimes E$. Also notice the symmetry

$$R(X, Y)v = -R(Y, X)v$$

and so $R$ is in fact a section of $\Lambda^2 T^\star\mathcal{M} \otimes \mathrm{End}(E)$. We can also view $R$ as the endomorphism of $E$ given by $v \mapsto R(X, Y)v$.

$R(X, Y)v$ can be seen as the direction on which $v$ is perturbed if we parallel translate it around an "infinitesimal loop" in the plane $X \wedge Y$, see section 6.7, on holonomy.

### 10.2 Curvature of a Riemannian manifold

Take $(\mathcal{M}, g)$ a Riemannian manifold with $\nabla$ the Levi-Civita connection. Then the curvature $R$ is a section of $T^\star\mathcal{M} \otimes T^\star\mathcal{M} \otimes T^\star\mathcal{M} \otimes T\mathcal{M}$ i.e. a $(1, 3)$ tensor field. In this context we can also use the metric (via musical isomorphism ) to define the $(0, 4)$ tensor $R_m(X, Y, W, Z) = \langle R(X, Y)W, Z\rangle$. Sometimes we write $R_m(g)$ if we want to emphasise which metric we are taking the curvature of.

**Remark** $R_m$ is invariant under isometries, i.e. if $\phi$ is an isometry then $R_m(\phi^\star g) = \phi^\star(R_m(g))$.

All the Riemannian metrics we are looking at are invariant under isometries (by construction).

The curvature can be extended to higher order tensors, i.e. $R(X,Y)T$ makes sense for $T$ any $(p,q)$ tensor field (Q6.2).

**Lemma 10.2** *$R_m$ has the following symmetries:*

1. $R_m(X,Y,W,Z) = -R_m(Y,X,W,Z)$
2. $R_m(X,Y,W,Z) = -R_m(X,Y,Z,W)$
3. *First Bianchi Identity* $R_m(X,Y,W,Z) + R_m(Y,W,X,Z) + R_m(W,X,Y,Z) = 0$
4. $R_m(X,Y,W,Z) = R_m(W,Z,X,Y)$

**Proof**

1. seen before
2. $$\begin{aligned} 0 &= R(X,Y)\langle W,Z\rangle \\ &= R(X,Y)(\mathrm{tr} W\otimes Z) \\ &= \mathrm{tr} R(X,Y)(W\otimes Z) \\ &= \mathrm{tr}[(R(X,Y)W)\otimes Z + W\otimes(R(X,Y)Z)] \\ &= \langle R(X,Y)W,Z\rangle + \langle W,R(X,Y)Z\rangle \end{aligned}$$
   and so $R_m(X,Y,W,Z) + R_m(X,Y,Z,W) = 0$
3. Equivalent to $R(X,Y)W + R(Y,W)X + R(W,X)Y = 0$ which is Q6.3.
4. If we take 3 and permute the entries we get
   $$R_m(X,Y,W,Z) + R_m(Y,W,X,Z) + R_m(W,X,Y,Z) = 0$$
   $$R_m(Y,W,Z,X) + R_m(W,Z,Y,X) + R_m(Z,Y,W,X) = 0$$
   $$R_m(W,Z,X,Y) + R_m(Z,X,W,Y) + R_m(X,W,Z,Y) = 0$$
   $$R_m(Z,X,Y,W) + R_m(X,Y,Z,W) + R_m(Y,Z,X,W) = 0$$
   Add to get cancellation of the first 2 columns and then by 1 and 2 on the final 2 columns we get
   $$R_m(W,X,Y,Z) - R_m(Y,Z,W,X) + R_m(W,X,Y,Z) - R_m(Y,Z,W,X) = 0$$
   and the result follows.

*Q.E.D.*

We also have a differential Bianchi identity.

**Lemma 10.3** $\nabla_V R_m(W,X,Y,Z) + \nabla_V R_m(V,W,Y,Z) + \nabla_W R_m(X,V,Y,Z) = 0$

The proof of this is Q6.4

## 10.3 Sectional Curvature

**Definition 10.4** *Given $p \in \mathcal{M}$ take a 2 dimensional subspace $\pi \subset T_p\mathcal{M}$. Take an orthonormal frame $X, Y$ for $\pi$. Then the* **sectional curvature** *of $\mathcal{M}$ at $p$ with respect to $\pi$ is*

$$K(\pi) = R_m(X, Y, X, Y)$$

Note that there is no confusion over the sign of $K$. It is always the case that spheres have positive sectional curvature.

If $\mathcal{M}$ is a two dimensional manifold then we have to take $\pi = T_p\mathcal{M}$ and $K$ is then the Gauss curvature at $p$.

Given $K(\pi)$ for all $p \in \mathcal{M}$ and $\pi \subset T_p\mathcal{M}$ we can reconstruct the full curvature $R_m$.

## 10.4 Ricci and Scalar curvatures

Sometimes $R_m$ has too much information for our purposes and is hard to compute with, and so we want nicer measures of curvature which still contain enough information.

**Definition 10.5** *The* **Ricci curvature** $\mathrm{Ric} : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M})$ *is the section of* $\mathrm{sym}^2(T^\star\mathcal{M})$ *defined by*

$$\mathrm{Ric}(X, Y) = \mathrm{tr}(R_m(X, \cdot, Y, \cdot))$$

In other words $\mathrm{Ric}(X, Y) = \mathrm{tr}\{Z \mapsto R(X, Z)Y\} = \sum_i R_m(X, e_i, Y, e_i)$ for $\{e_i\}$ an orthonormal frame field.

**Definition 10.6** *The* **scalar curvature** *is defined to be $R \in C^\infty(\mathcal{M})$ by*

$$R = \mathrm{tr}_g(\mathrm{Ric})$$

In other words $R = g^{ij}R_{ij}$ where $R_{ij}dx^i \otimes dx^j = \mathrm{Ric}$

**Remark** Ric contains just enough information to be useful:

- It contains the volume growth
- If $\Delta$ represents the Laplace -Beltrami operator, then in normal coordinates $\Delta g_{ij} = -\frac{2}{3}R_{ij}$

In two dimensions, the scalar curvature $R$ is twice the Gauss curvature.

We consider the second Bianchi identity again:

$$\nabla_V R_m(W, X, Y, Z) + \nabla_V R_m(V, W, Y, Z) + \nabla_W R_m(X, V, Y, Z) = 0$$

If we trace over $W$ and $Y$ and then over $X$ and $Z$ we get

$$\nabla_V R + \delta\mathrm{Ric}(V) + \delta\mathrm{Ric}(V) = 0$$

we summarise this as follows.

**Lemma 10.7 (Contracted second Bianchi identity)**

$$dR + 2\delta\mathrm{Ric} = 0$$

Identities such as these reflect the fact that the whole subject is invariant under pullback.

### 10.5 Einstein Metrics

An Einstein metric is a metric $g$ such that

$$\mathrm{Ric} = \lambda g$$

for some $\lambda \in \mathbb{R}$. Some people would define Einstein metrics to be those satisfying this for $\lambda : \mathcal{M} \to \mathbb{R}$, i.e. by taking trace of this giving $R = \lambda n$. Then we have

$$\mathrm{Ric} = \frac{R}{n} g$$

**Lemma 10.8 (Schur)** *The two above definitions of an Einstein metric are the same, so long as* $\dim \mathcal{M} \neq 2$

**Proof** We take the divergence to get

$$\delta \mathrm{Ric} = \delta \left( \frac{R}{n} g \right) = \frac{1}{n} \delta(Rg) = -\frac{1}{n} \mathrm{tr}(\nabla R g) = -\frac{dR}{n}$$

Then using lemma 10.7

$$dR = -n \delta \mathrm{Ric} = \frac{n}{2} dR$$

and if $n = 2$ this is vacuous. If $n \neq 2$ then $dR = 0$ and so $R$ is constant *Q.E.D.*

### 10.6 Curvature Operator

Given the symmetries of $R_m$ (1 and 2 in lemma 10.2) we can define a linear map

$$\mathcal{R} : \Lambda^2(T\mathcal{M}) \to \Lambda^2(T\mathcal{M})$$

called the curvature operator, defined by

$$\langle \mathcal{R}(X \wedge Y), W \wedge Z \rangle = R_m(X, Y, W, Z)$$

By symmetry 4 of lemma 10.2 we have that $\mathcal{R}$ is symmetric. Therefore we can diagonalise $\mathcal{R}$. If all the eigenvalues are positive we say the manifold has positive curvature operator.
**Remark** (non exam) A common theme in Riemannian geometry is that the curvature condition on a manifold implies a particular topological condition on the manifold. For example, by Gauss Bonnet , a compact surface with positive Gauss curvature must be $S^2$ or $\mathbb{R}P^2$.

**Theorem 10.9 (Böhn-Wilking 2006)** *(non exam) A simply connected compact manifold with positive curvature operator must be diffeomorphic to $S^n$.*

### 10.7 Curvature in Index notation

We introduce several conventions for the coefficients of curvature with respect to coordinates:

$$R_m = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

$$\mathrm{Ric} = R_{ij} dx^i \otimes dx^j$$

and therefore $R_{ij} = g^{kl} R_{ikjl}$ and $R = g^{ij} R_{ij} = g^{ij} g^{kl} R_{ikjl}$.

Working with respect to an orthonormal basis, or covector field, we have $R_{ij} = R_{ikjk}$ and $R = R_{ii} = R_{ikik}$ and we often use this notation in calculations. As a first example lets redo the contracted second Bianchi. This is

$$\nabla_i R_{jklm} + \nabla_k R_{ijlm} + \nabla_j R_{kilm} = 0$$

and tracing over $j$ and $l$ and $k$ and $m$ gives

$$\nabla_i R - \nabla_k R_{ik} - \nabla_j R_{ij} = 0$$

which is what we want in index notation.

# 11 Conquering Calculation

## 11.1 Switching Derivations

Given some general tensor $T$, if we apply $\nabla$ twice to get $\nabla^2 T \in \Gamma(T^\star\mathcal{M}\otimes T^\star\mathcal{M}\otimes ...)$ and then apply first $T^\star\mathcal{M}\otimes T^\star\mathcal{M}$ part to $X, Y \in \Gamma(T\mathcal{M})$ we write the result $\nabla^2_{X,Y}T$. For example if $T$ is a 1-form then $\nabla^2_{X,Y}T := \nabla^2 T(X,Y)$

$$\nabla^2_{X,Y}T = (\nabla_X \nabla T)(Y) - \nabla_X(\nabla T(Y)) - \nabla T(\nabla_X Y) = \nabla_X\nabla_Y T - \nabla_{\nabla_X Y}T$$

and as a consequence

$$-\nabla^2_{X,Y} + \nabla^2_{Y,X} = -(\nabla_X\nabla_Y - \nabla_{\nabla_X Y}) + (\nabla_Y\nabla_X - \nabla_{\nabla_Y X})$$

If it is torsion free then this is equal to

$$-\nabla_X\nabla_Y + \nabla_Y\nabla_X + \nabla_{[X,Y]} = R(X,Y)$$

so if we commute two derivatives during a computation, we pick up an extra "curvature" term.

In practise we often want to write the curvature tensor in terms of $R(X,Y)$ acting on a vector field.

For example suppose $w$ is a 1-form, using $R(X,Y)f = 0$ and the Leibniz rule we have

$$0 = R(X,Y)(w(Z)) = [R(X,Y)w](Z) + w[R(X,Y)Z]$$

and this leads to the Ricci identity

$$(-\nabla^2_{X,Y}w + \nabla^2_{Y,X}w)(Z) = -w[R(X,Y)(Z)]$$

i.e. the way we switch derivatives of a 1-form. More generally, for any $A \in \Gamma(\otimes^k T^\star\mathcal{M})$ we have

$$\begin{aligned}(-\nabla^2_{X,Y}A + \nabla^2_{Y,X}A)(w,z,...) &= [R(X,Y)A](w,z,...) = \\ &= -A(R(X,Y)w,z,...) - A(w,R(X,Y)z,...) - ...\end{aligned}$$

but this is more useful in index notation. This is with respect to an orthonormal frame $\{e_i\}$. Since $\nabla^2_{X,Y}w = \nabla^2 w(X,Y)$ we have this equal to

$$\nabla^2_{X,Y}w = (\nabla_i\nabla_j w_k)X^iY^je_k$$

and also

$$R(X,Y)w = -\nabla_i\nabla_j w_k X^iY^je_k + \nabla_j\nabla_i w_k X^iY^je_k$$

Also $R(X,Y)w = R_{ijkl}X^iY^jw_ke_l$ and we would normally then write that

$$R_{ijkl}w_k = -\nabla_i\nabla_j w_k + \nabla_j\nabla_i w_k$$

or

$$\nabla_i\nabla_j w_k = \nabla_j\nabla_i w_k + R_{ijkl}w_l$$

More generally we have

$$\nabla_i\nabla_j A_{k_1...k_m} = \nabla_j\nabla_i A_{k_1...k_m} + R_{ijkl}A_{lk_2...k_m} + R_{ijkl}A_{k_1lk_3...k_m} + ...$$

### 11.2 Bochnaw/Weitzenbock formula

The spirit of this application is that a manifold with positive Ricci curvature tells us stuff about the topology.

**Proposition 11.1** *Suppose $f : \mathcal{M} \to \mathbb{R}$ with $\mathcal{M}$ a Riemannian manifold. Then*

$$\frac{1}{2}\Delta|df|^2 = |\mathrm{Hess}(f)|^2 + \langle df, d|\Delta_{LB} f|\rangle + \mathrm{Ric}(\nabla f, \nabla f)$$

We make some clarification

1. $\mathrm{Hess}(f) = \nabla(df) \in \Gamma(T^\star\mathcal{M} \otimes T^\star\mathcal{M})$ so we can talk about its norm
2. another way of writing $\langle df, dh\rangle$ would be $\langle \nabla f, \nabla h\rangle$
3. In index notation $\langle df, dh\rangle = (\nabla_i f)(\nabla_i h)$
4. In index notation $\Delta_{LB} f = \nabla_i \nabla_i f$
5. In index notation $|\mathrm{Hess}(f)|^2 = (\nabla_i \nabla_j f)(\nabla_i \nabla_j f)$
6. Hess is symmetric i.e. $\nabla_i \nabla_j f = \nabla_j \nabla_i f$
7. Ricci Identity $\nabla_i \nabla_i w_k = \nabla_j \nabla_i w_k + R_{ijkl} w_l$

**Proof**

$$\begin{aligned}
\frac{1}{2}\Delta|df|^2 &= \frac{1}{2}\nabla_i \nabla_j ((\nabla_j f)(\nabla_j f)) \\
&= \nabla_i[(\nabla_i \nabla_j f)(\nabla_j f)] \\
&= (\nabla_i(\nabla_i \nabla_j f))(\nabla_j f) + (\nabla_i \nabla_j f)(\nabla_i \nabla_j f) \\
&= (\nabla_i \nabla_j \nabla_i f)(\nabla_j f) + |\mathrm{Hess}(f)|^2 \\
&= (\nabla_j(\Delta_{LB} f))(\nabla_j f) + R_{jl}(\nabla_l f)(\nabla_j f) + |\mathrm{Hess}(f)|^2 \\
&= \langle d|\Delta_{LB} f|, df\rangle + \mathrm{Ric}(\nabla f, \nabla f) + |\mathrm{Hess}(f)|^2
\end{aligned}$$

*Q.E.D.*

## 12 Second Variation Formula

In section 8.8.2 we computed the first variation formula for the length of a path. We have the following basic set up. Let $\gamma : [a, b] \to \mathcal{M}$ be a smooth curve, and $F : (-\delta, \delta) \times [a, b] \to \mathcal{M}$ is the variation and consider the length $L(F(\eta, \cdot))$. Previously we assumed that $\gamma$ had unit speed. Now we will assume that it is a geodesic. Also assume that the end points are kept fixed i.e.

$$F(\eta, a) = \gamma(a), \qquad F(\eta, b) = \gamma(b)$$

for all $\eta \in (-\delta, \delta)$.

**Definition 12.1** *Define the* **normal projection** *of $Y$ to be*

$$Y^\perp = Y - \langle Y, \gamma'(s)\rangle\gamma'(s)$$

in other words this is $Y$ minus the part parallel to $\gamma'$.

Suppose that $X = \frac{\partial F}{\partial s}$ and $Y = \frac{\partial F}{\partial \eta}$, and note that $|X| = 1$ for $\eta = 0$.

**Theorem 12.2** *Given $F$ a smooth curve as above with $F(0,\cdot)$ a unit speed geodesic and with fixed endpoints, we have*

$$\frac{d^2}{d\eta^2}\bigg|_{\eta=0} L(F(\eta,\cdot)) = \int_a^b \Big[|D_sY^\perp|^2 - R_m(Y^\perp,\gamma'(s),Y^\perp,\gamma'(s))\Big]\,ds$$

Note that the second derivative of length on a negatively curved manifold is positive, and so all geodesics are stable, i.e. if one starts on one, then one stays on one.
**Proof** Start by taking a first derivative of $L(F(\eta,\cdot))$. This is a bit like the computation for first variation formula except we cannot yet set $\eta = 0$. Recall that, from the definition,

$$L(F(\eta,\cdot)) = \int_a^b \left|\frac{\partial F}{\partial s}\right| ds = \int_a^b g(X,X)^{\frac{1}{2}} ds$$

and also observe that $\frac{d}{d\eta}[g(X,X)] = 2g(D_\eta X, X) = 2g(D_sY, X)$ by the symmetry lemma. Thus

$$\frac{d}{d\eta}L(F(\eta,\cdot)) = \int_a^b \frac{1}{2}g(X,X)^{-\frac{1}{2}}\frac{d}{d\eta}(g(X,X))ds = \int_a^b g(X,X)^{-\frac{1}{2}}g(D_sY,X)ds$$

Differentiate with respect to $\eta$ again and now set $\eta = 0$ to get

$$\begin{aligned}
\frac{d^2}{d\eta^2}\bigg|_{\eta=0} L(F(\eta,\cdot)) &= \int_a^b -\frac{1}{2}g(X,X)^{-\frac{3}{2}}\left[\frac{d}{d\eta}g(X,X)\right]g(D_sY,X) + g(X,X)^{-\frac{1}{2}}\times \\
&\qquad\qquad \times\left(g(D_\eta D_sY,X) + g(D_sY, D_\eta X)\right)ds \\
&= \int_a^b -\frac{1}{2}\left[\frac{d}{d\eta}g(X,X)\right]g(D_sY,X) + g(D_\eta D_sY,X) + g(D_sY,D_\eta X)ds \\
&= \int_a^b -[g(D_sY,X)]^2 + g(D_\eta D_sY,X) + g(D_sY,D_sY)ds \\
&= \int_a^b -[g(D_sY,X)]^2 + |D_sY|^2 + g(D_sD_\eta Y,X) + g(\tilde{R}(\frac{\partial}{\partial s},\frac{\partial}{\partial\eta})Y,X)ds \\
&= \int_a^b -[g(D_sY,X)]^2 + |D_sY|^2 + \frac{\partial}{\partial s}[g(D_\eta Y,X)] \\
&\qquad\qquad - g(D_\eta Y, D_sX) + g(\tilde{R}(X,Y)Y,X)ds \\
&= \int_a^b -[g(D_sY,X)]^2 + |D_sY|^2 + R_m(X,Y,Y,X)ds
\end{aligned}$$

since $D_sX = 0$ for $\eta = 0$ as we are on a geodesic, and the other term is zero as it depends on the change of the endpoints, which is zero.

Now $Y = Y^\perp + Y^T$ with $Y^T = \langle Y,X\rangle X$ and so $D_s(Y^T) = \langle D_sY,X\rangle X$ as $D_sX = 0$ for $\eta = 0$. Thus

$$|(D_sY)^T|^2 = |\langle D_sY,X\rangle X|^2 = \langle D_sY,X\rangle^2$$

and also

$$(D_sY)^\perp = D_sY - \langle D_sY,X\rangle X = D_s(Y^\perp) + D_s(Y^T) - \langle D_sY,X\rangle X = D_s(Y^\perp)$$

and thus

$$|D_sY|^2 = |(D_sY)^\perp|^2 + |(D_sY)^T|^2 = |D_sY^\perp|^2 + g(D_sY,X)^2$$

We then conclude that

$$\frac{d^2}{d\eta^2}|_{\eta=0}L(F(\eta,\cdot)) = \int_a^b |D_sY^\perp|^2 + R_m(X,YY,X)ds = \int_a^b |D_sY^\perp|^2 + R_m(Y,\gamma'(s),Y,\gamma'(s))ds$$

Note that $R_m(X,Y,X,Y) = R_m(X,Y^\perp,X,Y^\perp)$ because by decomposing, we have

$$R_m(X,X,\cdot,\cdot) = 0 \qquad R_m(\cdot,\cdot,X,X) = 0$$

because of the symmetries of $R_m$. *Q.E.D.*

**Remark** This formula works because $\gamma$ is assumed to be a geodesic. This is why it only depends on $Y$ at $\eta = 0$. Analogous fact in finite dimensions is this. If $\sigma(\eta)$ is a smooth curve in $\mathbb{R}^2$ with $\sigma(0) = 0$ and $L : \mathbb{R}^2 \to \mathbb{R}$ a smooth function then

$$\left.\frac{d^2}{d\eta^2}\right|_{\eta=0} L(\sigma(\eta)) = (\text{Hess}L)(\sigma'(0),\sigma'(0))$$

provided $\nabla L = 0$ at 0. Otherwise we would get a $\sigma''$ term.

A consequence of this is given a minimising geodesic $\gamma : [a,b] \to \mathcal{M}$ then for any $Y \in \Gamma(\gamma^\star(T\mathcal{M}))$ with $Y(a) = 0 = Y(b)$ and $Y(s) \perp \gamma'(s)$ for all $s \in [a,b]$. Define $F : (-\delta,\delta) \times [a,b] \to \mathcal{M}$ by $F(\eta,s) = \exp_{\gamma(s)}(\eta Y)$ and apply the second variation formula. Then $\gamma$ minimising implies $\frac{d^2}{d\eta^2}|_{\eta=0} L(F(\eta,\cdot)) \geq 0$ and therefore we have the following.

**Corollary 12.3** *Under the circumstances above*

$$Q(Y) := \int_a^b |D_sY|^2 - R_m(Y,\gamma'(s),Y,\gamma'(s))ds \geq 0$$

# 13 Classical Riemannian Geometry Results: Bonnet-Myers

**Definition 13.1** *The* **diameter** *of a Riemannian connected manifold is defined to be*

$$\text{diam}(\mathcal{M},g) = \sup_{p,q\in\mathcal{M}} d_g(p,q) \in (0,\infty]$$

**Theorem 13.2 (Myers)** *Suppose $r > 0$ and $(\mathcal{M},g)$ is a complete connected Riemannian manifold of dimension $n$ such that $\text{Ric} \geq \frac{n-1}{r^2}g$ i.e. if $X \in T\mathcal{M}$ then $\text{Ric}(X,X) \geq \frac{n-1}{r^2}g(X,X)$. Then $\mathcal{M}$ is compact and*

$$\text{diam}(\mathcal{M},g) \leq \pi r$$

Thus positive curvature forces the manifold to close in on itself.
**Remark** This is sharp for the sphere, e.g. a sphere of radius $r$ has $\text{Ric} = \frac{n-1}{r^2}g$.
**Proof** Start by proving the diameter estimate. Pick arbitrary points $p,q \in \mathcal{M}$ and define $L = d_g(p,q)$. We want to prove that $L \leq \pi r$.

The Hopf-Rinow theorem tells us that there exists a minimising geodesic $\gamma : [0,L] \to \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma(L) = q$, and with unit speed. Pick an orthonormal basis $\{e_i\}$ of $T_p\mathcal{M}$ with $e_n = \gamma'(0)$. Parallel translate this frame $\{e_i\}$ along $\gamma$ to give sections $e_i \in \Gamma(\gamma^\star(T\mathcal{M}))$ and note that for all $s \in [0,L]$ we have $\{e_i(s)\}$ is an orthonormal basis of $T_{\gamma(s)}\mathcal{M}$. Note for all $s$ that $e_n(s) = \gamma'(s)$ because $e_n(0) = \gamma'(0)$ by construction and both $e_n(s)$ and $\gamma'(s)$ are parallel (satisfy same ODEs).

Define for all $i \in \{1,...,n-1\}$

$$Y_i(s) = \sin\left(\frac{\pi s}{L}\right) e_i(s)$$

and observe that $Y_i(0) = 0 = Y_i(L)$ and that $Y_i(s) \perp \gamma'(s)$.

Apply corollary 12.3 to get that

$$Q(Y_i) = \int_0^L |D_sY_i|^2 - R_m(Y_i,\gamma'(s),Y_i,\gamma'(s))ds \geq 0$$

but $D_s Y_i = \frac{\pi}{L}\cos\left(\frac{\pi s}{L}\right) e_i(s) + \sin\left(\frac{\pi s}{L}\right) \times 0 = \frac{\pi}{L}\cos\left(\frac{\pi s}{L}\right) e_i(s)$ and therefore

$$|D_s Y_i|^2 = \frac{\pi^2}{L^2}\cos^2\left(\frac{\pi s}{L}\right)$$

and also

$$R_m(Y_i, \gamma'(s), Y_i, \gamma'(s)) = \sin^2\left(\frac{\pi s}{L}\right) R_m(e_i, \gamma'(s), e_i, \gamma'(s))$$

therefore

$$0 \leq \int_0^L \frac{\pi^2}{L^2}\cos^2\left(\frac{\pi s}{L}\right) - \sin^2\left(\frac{\pi s}{L}\right) R_m(e_i, \gamma'(s), e_i, \gamma'(s)) ds$$

and then sum over $i = 1, ..., n-1$ and note that $R_m(e_n, \gamma'(s), e_n, \gamma'(s)) = 0$ and so

$$0 \leq \int_0^L \frac{\pi^2(n-1)}{L^2}\cos^2\left(\frac{\pi s}{L}\right) - \sin^2\left(\frac{\pi s}{L}\right) \mathrm{Ric}(\gamma'(s), \gamma'(s)) ds$$

but $\mathrm{Ric}(\gamma'(s), \gamma'(s)) \geq \frac{n-1}{r^2} g(\gamma'(s), \gamma'(s)) = \frac{n-1}{r^2}$ and therefore

$$0 \leq \int_0^L \frac{\pi^2(n-1)}{L^2}\cos^2\left(\frac{\pi s}{L}\right) - \frac{n-1}{r^2}\sin^2\left(\frac{\pi s}{L}\right) ds$$

and so

$$0 \leq \frac{\pi^2(n-1)}{L^2} - \frac{n-1}{r^2}$$

i.e. $L^2 \leq \pi^2 r^2$ and so $L \leq \pi r$ as $L > 0$.

To show that $\mathcal{M}$ is compact, note that by the diameter estimate, $\mathcal{M}$ is the image of $\overline{D(0, \pi r)}$ under $\exp_p$. The image of a compact set under a continuous map is compact, so we have the result. *Q.E.D.*