MA213 Second Year Essay
Hilbert's Third Problem

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1 Introduction

At the Second International Congress of Mathematicians in 1900, David Hilbert read his report [2]"Mathematical Problems" in which he says:

> In two letters to Gerling, Gauss expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, i. e., in modern phraseology, upon the axiom of continuity (or upon the axiom of Archimedes). Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now the analogous problem in the plane has been solved. Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts. Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility.

This was justified by Max Dehn, a student of Hilbert, in the very same year it was posed. This quotation motivates the following definitions:

Definition 1.1 *If F and G are two figures then we define $F + G$ to be the* **pairwise disjoint union** *of F and G.[1]*

Definition 1.2 *Two figures F and G are* **congruent***, denoted $F \cong G$, if there exists an isometry $f : F \to G$. An isometry is a bijection that preserves distances.*

The large part of the following two chapters is from V G Boltianskii "Hilbert's Third Problem" [1].

2 Scissors Congruence of Polygons

To give a background to the problem, I first discuss the "analogous problem in the plane". The first step is to define what scissors congruence means:

Definition 2.1 *We say two polygons F and G are* **scissors congruent***, denoted $F \sim G$, if there exist polygons $F_1, ..., F_k$ and $G_1, ..., G_k$ such that $F = F_1 + ... + F_k$, $G = G_1 + ... + G_k$; $F_1 \cong G_1, ..., F_k \cong G_k$.[1] I shall call $F_1, ..., F_k$ a partition of F.*

Intuitively, the idea of scissors congruence is to "cut" one polygon into finitely many polygons, and then rearrange into another. Thus it should be obvious that they have the same area. Also directly from the definition, as $F_i \cong G_i$ then they have the same area (from the fact that an isometry preserves distances), and so disjoint unions of these must have the same area. The converse to this is a more interesting problem, i.e. if they have the same area then are they scissors congruent. For polygons this is true, so

$$Area(F) = Area(H) \iff F \sim H$$

The result of this is profound. It means that in the plane one can use a definition of area without using the methods of calculus, as Hilbert alluded to in his Mathematical problems [2]. Say, for example, that you define the area of a rectangle to be the product of its two sides. Then you can calculate the area of a pentagon, for example, by splitting it up into triangles, and thus into rectangles (lemma 2.2), of which you know the area. If this were not so, then one would have to integrate over a suitable region to find the area of a pentagon. Clearly the latter is much harder.

From defintion 2.1, one formulates some relatively simple results, and then from these prove the planar analogue to Hilbert's third problem, here called Theorem 2.4.

Proposition 2.1 *Scissors congruence defines an equivalence relation on the set of all polygons.*

Proof To be an equivalence relation it must satisfy the following three properties. Suppose here that A, B, C are three polygons.

1. $A \sim A$.
2. $A \sim B \implies B \sim A$
3. If $A \sim B$ and $B \sim C$ then $A \sim C$

Points 1 and 2 are obviously satisfied. Consider 1: Any partition (see definition 2.1) of A would satisfy the definition, since clearly $A_i \cong A_i$. Point 2: If $A \sim B$ then there are partitions $A_1, ..., A_k$ of A and $B_1, ..., B_k$ of B such that there are isometries $f_i : A_i \to B_i$ from each element in the partition of A to the partition of B. One simply now notes that every isometry has an inverse which is itself an isometry, and then one gets that $B \cong A$ using the inverses $f_i^{-1} : B_i \to A_i$.

Point 3 requires slightly more work to be proved. The idea is to partition B into polygons which are congruent to polygons in a partition of A, and partition B into polygons which are congruent to polygons in a partition of C. One then takes the intersections of one polygon from one partition with one from the other and show that this "finer" partition can be rearranged to form both A and C. You can see this for a specific example in figure 1. There we have in a) a square which is scissors congruent to a regular hexagon, and in b) a regular pentagon scissors congruent to a regular hexagon Suppose that $A \sim B$ and $B \sim C$, and further more that $A = A_1 + ... + A_n$, $B = B_1 + ... + B_n = B_1' + ... + B_m'$ and $C = C_1 + ... + C_m$ such that $A_i \cong B_i$ and $B_i' \cong C_j$, with f_i and g_j isometries such that $A_i = f_i(B_i)$ and $C_j = g_j(B_j')$. Now consider all $F_{ij} = B_i \cap B_j'$ over $1 < i < n$, $1 < j < m$. Note that no two F_{ij} have common interior points. This is because if $i \neq i^*$ then $B_i \cap B_{i^*} = \emptyset$ and so $F_{ij} \cap F_{i^*j} = \emptyset$, and similarly for j. Now summing all F_{ij} over j the following is true:

$$\bigcup_{j=1}^{m} F_{ij} = \bigcup_{j=1}^{m} (B_i \cap B_j) = B_i \cap (\cup_{j=1}^{m} B_j) = B_i$$

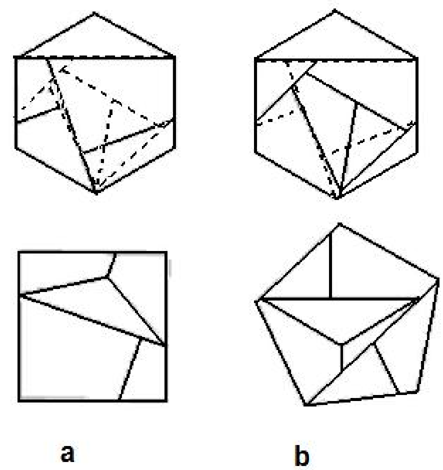

Figure 1: A hexagon congruent to a square and pentagon. Adapted from [6]

and similarly for summing the F_{ij} over the i gives

$$\bigcup_{i=1}^{n} F_{ij} = \bigcup_{i=1}^{n} (B_i \cap B_j) = (\cup_{i=1}^{n} B_i) \cap B_j = B_j$$

And using these results it follows that

$$A = \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} f_i(B_i) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} f_i(F_{ij})$$

and similarly

$$C = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} g_i(F_{ij})$$

Thus we have that $f_i(F_{ij}) \cong g_j(F_{ij})$ as they are both congruent to F_{ij}. Then as f_i and g_j are isometries their inverses exist and are also isometries, and the composition of two isometries is an isometry, we have that $g_j \circ f_i^{-1}$ is an isometry and $F_{ij} = g_j \circ f_i^{-1}(F_{ij})$, so we map an element of the partition of A to an element of the partition of C and point 3 is proven. *Q.E.D.*

Thus we are justified in using the same symbol for scissors congruence as for an equivalence relation.

Lemma 2.2 *Every triangle is scissors congruent with some rectangle*

Proof I give a proof using a specific example and method, but it can easily be formalised.

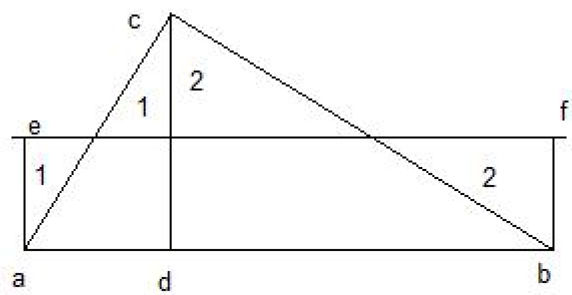

Figure 2: The construction of a rectangle scissors congruent to a triangle

Consider the largest side of the triangle abc, which here is ab. Construct the altitude through c and perpendicular to ab. Then draw the perpendicular bisector to this line, called ef in figure 2. Elevate perpendiculars to the largest side from the endpoints, here ae and bf. Then the rectangle $abfe$ constructed from these lines is scissors congruent to the triangle. This is displayed in figure 2. *Q.E.D.*

Lemma 2.3 *Any two rectangles with the same area are scissors congruent.*

The proof of this lemma in [1] is quite tedious, and so I shall omit it.

Theorem 2.4 (Bolyai-Gerwien Theorem) *Any two polygons (call them A and B) with the same area are scissors congruent.*

Idea We aim to decompose A into triangles and by lemma 2.2 also rectangles. Then we show that a sum of rectangles is scissors congruent to one large rectangle.

Proof Every polygon F can be decomposed into finitely many triangles, and by lemma 2.2 these triangles are scissors congruent with rectangles. Thus let $A = P_1 + ... + P_k$ where $P_1, ..., P_k$ are rectangles. Then construct a line with perpendicular lines passing through the end points, called a_0 and b_0, as in figure 3. Then draw line segments $a_1b_1, ..., a_kb_k$ parallel to a_0b_0 such that rectangle $a_0b_0b_1a_1$ has the same area as P_1 and rectangle $a_{i-1}b_{i-1}b_ia_i$ (which I denote H_i) has the same area as P_i. Then we have $H_i \sim P_i$ by lemma 2.3. Thus $P_1 + ... + P_k \sim H_1 + ... + H_k =: H$, i.e. any polygon is scissors congruent to a rectangle. As scissors congruence preserves area and as $area(A) = area(B)$, A and B are scissors congruent to rectangles of the same area, and so by lemma 2.1 we have $F \sim H$.

Q.E.D.

On a mild digression, I give a method, without proof, of how one can split a convex polygon into triangles: Find the centroid of the polygon. Then construct

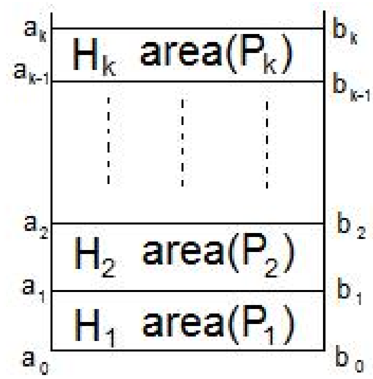

Figure 3: Rectangle H

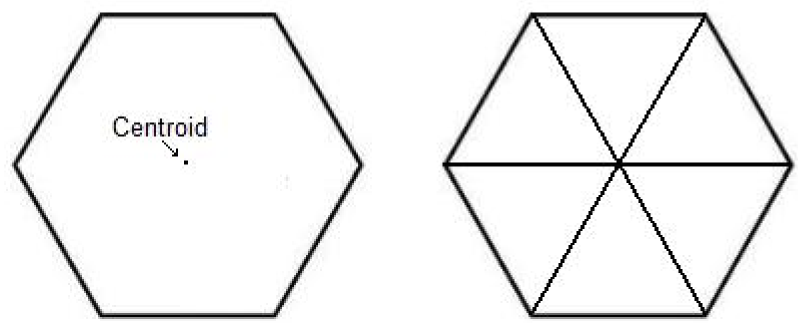

Figure 4: Splitting a hexagon into triangles

lines from the centroid to each vertex of the polygon. You have thus split the polygon into triangles. Figure 4 shows the method for a hexagon.

3 Scissors Congruence of polyhedra

In a similar way to definition 2.1, one defines scissors congruence of polyhedra by:

Definition 3.1 *We say two polyhedra F and G are* **scissors congruent***, denoted $F \sim G$, if there exist polyhedra $F_1, ..., F_k$ and $G_1, ..., G_k$ such that $F = F_1 + ... + F_k$, $G = G_1 + ... + G_k$; $F_1 \cong G_1, ..., F_k \cong G_k$.*

In the same light, with essentially the same proof, one should note that this scissors congruence again defines an equivalence relation on the set of all polyhedra. If one defines this for n-gons then one would still have an equivalence relation.

Hilbert conjectures in [2] that the equality of volume is an impossibility, in other words:

Theorem 3.1 (Hilbert's third problem) *Given any two polyhedra of equal volume, they are not in general scissors congruent.*

The following is an exposition on the proof of this statement, by exhibiting the mathematics required to give a counterexample. The counterexample is in lemma 3.4

3.1 The Solution of Hilbert's Third Problem-Dehn's proof

There are several problems which arise while attempting to use elementary methods of geometry to solve this problem. One such problem is that there exist tetrahedra with equal bases and heights which are not scissors congruent with each other. See Hill's Tetrahedron, section 3.2

One first needs the notion of an additive function in order to define a value called the Dehn invariant of a polyhedron.

Definition 3.2 *Let M be a set of real numbers. A function f defined on M is said to be* **additive** *if, for every linear dependence with integral coefficients between the elements of M, the same linear dependence holds between the corresponding values of f. In other words, if $x_1, ..., x_h \in M$ such that $n_1x_1 + .. + n_hx_h = 0$, the n_is not all zero, then*

$$n_1f(x_1) + .. + n_hf(x_h) = 0$$

One should note that this is the same as the standard definition of an additive function, where we have $f(x + y) = f(x) + f(y)$.

Now suppose A is a polyhedron with edge lengths $l_1, ..., l_p$ dihedral angles $\alpha_1, ..., \alpha_p$ with corresponding dihedral angles $\alpha_1, ..., \alpha_p$ and that $M = \{\alpha_1, ..., \alpha_p\}$. A dihedral angle is in general the angle between two planes. In this case, I take it to be the angle between two intersecting faces of the polyhedron.

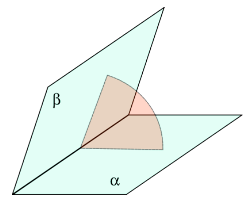

Figure 5: Dihedral angle between two planes[3]

Definition 3.3 *We then define the* **Dehn invariant** *to be the value* $\Delta(A)$ *such that*

$$\Delta(A) = l_1\delta(\alpha_1) + ... + l_p\delta(\alpha_p)$$

where $\delta : M \to \mathbb{R}$ *is an additive function.*

While I have called Δ the Dehn invariant, the definition gives it in terms of a general additive function. In my opinion, for this to be a valid definition I would expect it to be the same regardless of the function used to define it. However this is not the case! Take for example the function used in lemma 3.4. This has $\delta(\varphi) = 1$, but any non zero real number would suffice instead of 1, and still is equally valid and works just as well. If it was zero, then the function δ would still be additive with integral coefficients, but wouldn't give the counterexample.

What is also remarkable is that it is not at all obvious that this number is invariant under scissors congruence, and yet this is the key reason why this value is so important in solving the third problem. This statement is rephrased in the following theorem.

Theorem 3.2 *Given a polyhedron* A *with dihedral angles* $\alpha_1, ..., \alpha_p$ *and a polyhedron* B *with dihedral angles* $\beta_1, ..., \beta_q$*, let* G *be a set of real numbers containing the numbers* $\pi, \alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q$*. Suppose there exists an additive function* f *defined on* G *such that* $f(\pi) = 0$*, while the corresponding Dehn invariants of the polyhedra* A *and* B *are unequal, i.e.* $f(A) \neq f(B)$*. Then the polyhedra* A *and* B *are not scissors congruent.*

For the sake of completeness I include a proof of theorem 3.2, as it is the key result for the third problem. The proof is largely from [1], although I have somewhat abridged it.

Proof We prove the contrapositive of the theorem. Let A and B be scissors congruent, so that

$$A = P_1 + ... + P_k,\ B = Q_1 + ... + Q_k,\ P_i \cong Q_i\ \forall i = 1, 2, ..., k$$

and define an additive function f on the set G, which contains $\pi, \alpha_1, ..., \alpha_p$, $\beta_1, ..., \beta_q$. We want to be able to add all the dihedral angles of the P_i and Q_j to this set G. Suppose we add another value, γ, say, to this set G to get $G' = G \cup \{\gamma\}$. Then we can define f on this new set in such a way that we keep it additive. To do so, consider the following. We have two cases, either there is no linear dependence with the coefficient of γ non zero as in definition 3.2. Here we can define $f(\gamma)$ as we wish. If there is a linear dependence, we set $f(\gamma) = -\sum_{i=1}^{p} \frac{n_i}{n} f(x_i)$, where each $x_i \in G$. Thus one can add all the dihedral angles of the P_is to this set G. Note that the Q_is have the same dihedral angles as the P_is, and so we only need to add the angles from one of the two polyhedra. I claim that one can write the additive function of the polyhedra as follows:

$$f(A) = f(P_1) + ... + f(P_k)$$

$$f(B) = f(Q_1) + ... + f(Q_k)$$

To see this consider the following. There are four cases:

1. The edges of polyhedra meet entirely in the interior of A
2. The edges of polyhedra meet on a face of another polyhedra in A
3. The edges of polyhedra meet on a face of A
4. The edges of polyhedra meet at an edge of A

In 1. the values of the angles can be summed to 2π and so we have a linear dependence on G, and as $f(\pi) = 0$ we have that the sum of f on all these angles is zero. In 2. the sum of the angles equals π and so by definition of f this is zero, and this is the same as in 3. In 4. the sum of the angles equals the dihedral angle α_i for that edge, and so this is $f(\alpha_i)$. Thus we have the above equations.

Now one notes that, as the corresponding dihedral angles of P_i and Q_i are the same, $f(P_i) = f(Q_i)$ and so $f(A) = f(B)$ and we have proved the theorem.

Q.E.D.

The following lemma is necessary in order for the function δ which I define in lemma 3.4 to be additive, and therefore a Dehn invariant, and is also a nice use of trigonometric identities.

Lemma 3.3 $\frac{1}{\pi} \arccos \frac{1}{n}$ *is irrational for every natural* $n \geq 3$.

Proof We prove by contradiction. Suppose that $\frac{\arccos \frac{1}{n}}{\pi} = \frac{p}{q}$, for some $p, q \in \mathbb{Z}^+$. Let $\varphi = \arccos \frac{1}{n}$. Then we have that $q\varphi = p\pi$, giving $\cos q\varphi = \pm 1$, so $\cos q\varphi$ is an integer. We now find a contradiction, namely that $\cos q\varphi$ is not an integer for $p = 1, 2, ...$ We now use the formula

$$\cos (q+1)\varphi + \cos (q-1)\varphi = 2 \cos q\varphi \cos \varphi \tag{1}$$

which gives us

$$\cos (q+1)\varphi = \frac{2}{n} \cos q\varphi - \cos (q-1)\varphi$$

We then use case analysis on whether n is even or odd. Suppose firstly that n is odd. We perform induction on q, by showing that $\cos q\varphi$ can be expressed as a fraction with denominator n^k and numerator coprime to this. Suppose $q = 1$.

We the have that $\cos\varphi = \cos(\arccos\frac{1}{n}) = \frac{1}{n}$. Now this is not an integer, as $n \geq 3$. Then for $q = 2$ we have that $\cos 2\varphi = 2\cos^2\varphi - 1 = \frac{2-n^2}{n^2}$ and since n is odd, $2 - n^2$ and n^2 are coprime. Observe that if $2 - n^2$ and n^2 are coprime then $2 - n^2$ and n are coprime. Now for this to be an integer we would have $2 - n^2$ and n not coprime, or $n^2 = 1$. As the latter cannot be the case, and the former is not true, $\cos 2\varphi$ is not an integer. Thus we have established the result for $n = 1$ and $n = 2$.

Suppose that our assumption holds for $q = 1, 2, ..., k$. Then we have that:

$$\cos q\varphi = \frac{a}{n^k} \;\; \cos(q-1)\varphi = \frac{b}{n^{k-1}}$$

where a and b are coprime with n. then by (1) we get that

$$\cos(k+1)\varphi = \frac{2}{n}\cos q\varphi - \cos(q-1)\varphi = \frac{2a - bn^2}{n^{k+1}}$$

and since 2, a and b are coprime with n, we have that $2a - bn^2$ is coprime with n. Thus by induction we have that $\cos q\varphi$ is not an integer for any q. A similar inductive idea, where we write $n = 2m$ and show that $\cos q\varphi$ can be written as a fraction with denominator $2m^k$ and coprime numerator, can be used for n even. I omit the details for brevity. *Q.E.D.*

The following result is the solution to the third problem. I call it merely a lemma as it is only relevant to two specific polyhedra.

Lemma 3.4 *A regular tetrahedron and cube with the same volume have different Dehn invariants (and hence are not scissors congruent).*

Proof

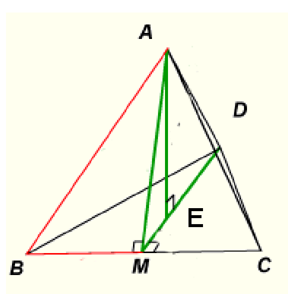

Figure 6: Regular Tetrahedron showing Dihedral angle. Adapted from [5]

Let T be the regular tetrahedron and C the cube with same volume, and let φ be the dihedral angle of T. One can calculate that $\varphi = \arccos\frac{1}{3}$, as from figure 6 you can see that $|EM| = \frac{1}{3}|DM| = \frac{1}{3}|AM|$ where AE is the perpendicular to the plane BCD passing through A. We also have that the dihedral angle of C is $\frac{\pi}{2}$. Thus using theorem 3.2 with $G = \{\pi, \frac{\pi}{2}, \varphi\}$ we define the function δ by:

$$\delta(x) = \begin{cases} 0 & x = \pi, \frac{\pi}{2} \\ 1 & x = \varphi \end{cases}$$

I now show that this function is additive. Suppose we have the relation $n_1\pi + n_2\frac{\pi}{2} + n_3\varphi = 0$, where $n_i \in \mathbb{N}$. If we suppose that $n_3 \neq 0$ we obtain that $\frac{\varphi}{\pi} = -\frac{n_1 + \frac{n_2}{2}}{n_3}$ contradicting lemma 3.3, and so we get that $n_3 = 0$, and so for any choice of n_1 and n_2 we have $n_1 f(\pi) + n_2 f\left(\frac{\pi}{2}\right) + n_3 f(\varphi) = 0$ and so the function f is additive.

We now calculate the Dehn invariants of T and C using this function δ. For the cube C, which we suppose to have side length l, we have

$$\Delta(C) = 12l\delta(\frac{\pi}{2}) = 0$$

Now, supposing we have side length m for T, we have

$$\Delta(T) = 6m\delta(\varphi) = 6m \neq 0$$

Thus the Dehn invariants of T and C are different and so by theorem 3.2 we have that T and C are not scissors congruent. This shows that volume and scissors congruence are not equivalent for polyhedra, and so solves the third problem with Hilbert's expected answer. *Q.E.D.*

Observe that as here we are forced to have $\delta(\pi) = 0$ we then neccessarily must have $\delta(\frac{\pi}{2}) = 0$ by the additivity constraint. However there are no such constraints upon the value of $\delta(\varphi)$ and so we set it to be non zero to show they are not scissors congruent. The fact that we could have set it to zero means nothing, as for two polyhedra to be scissors congruent you would expect theorem 3.2 to hold for any dehn invariant.

As Hilbert alludes to in [2], this result means that any proof of a formula for area in three dimensional space requires a form of limiting process or calculus.

3.2 Hill's Tetrahedron

Hill's tetrahedron of the first type is the polyhedron defined as follows. Let $\alpha \in (0, 2\pi/3)$ and let $v_1, v_2, v_3 \in \mathbb{R}^3$ be three unit vectors with the angle α between every two of them. Then Hill's tetrahedron (see figure 7)is $\mathbb{H} = \{c_1v_1 + c_2v_2 + c_3v_3 : 0 \leq c_1 \leq c_2 \leq c_3 \leq 1\}$. A more understandable explanation for this definition is the following. Suppose we have a parallelepiped P constructed by three vectors c_1v_1, c_2v_2, c_3v_3 of the same length and which pairwise make the same angle. Draw three planes passing through a diagonal and two opposite vertices. This divides the parallelepiped into 6 tetrahedra, and these are called Hill's tetrahedra of the first type.

This tetrahedron is scissors congruent to a right triangular prism, hence to a parallelepiped and so to a cube. As such, it is not scissors congruent to the regular tetrahedron as in lemma 3.4. It follows from the scissors congruence to a cube that $\Delta(\mathbb{H}) = 0$. To verify this I calculate the Dehn invariant separately.

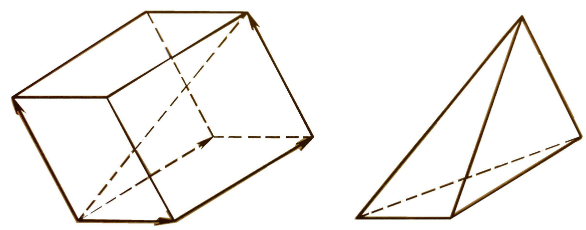

Figure 7: Hill's tetrahedron of the first type [1]

Now the lengths and corresponding dihedral angles of this tetrahedron are as follows:

Edges	Lengths	Dih. angles
ab	$\sin\alpha$	α
ac	$\sqrt{3}\cos\alpha$	$\pi/3$
ad	1	$\pi/2$
bc	1	$\pi/2$
bd	$\sin\alpha$	$\pi - 2\alpha$
cd	$\sin\alpha$	α

where α is the angle between the vectors constructing the parallelepiped. I now construct the following function on the set of dihedral angles of $\mathbb{H}$ and on π. One defines $\delta : \mathbb{R} \to \mathbb{R}$ by:

$$\delta(\beta) = \begin{cases} 1 & \beta = \alpha \\ 0 & \beta = \pi/3 \\ 0 & \beta = \pi/2 \\ 0 & \beta = \pi \\ -2 & \beta = \pi - 2\alpha \end{cases}$$

One must now show that this function is additive with integral coefficients, as in definition 3.2. Suppose we have a relationship of the form

$$n_1\alpha + n_2\pi/2 + n_3\pi/3 + n_4(\pi - 2\alpha) + n_5\pi = 0$$

We then want to show that

$$n_1\delta(\alpha)+n_2\delta(\pi/2)+n_3\delta(\pi/3)+n_4\delta(\pi-2\alpha)+n_5\delta(\pi) = n_1\delta(\alpha)+n_4\delta(\pi-2\alpha) = 0$$

From the first equation we get that

$$(n_1 - 2n_4)\alpha + (n_2/2 + n_3/3 + n_4 + n_5)\pi = 0$$

and so we must have that $n_1 - 2n_4 = 0$ and that $n_2/3 + n_3/2 + n_4 + n_5 = 0$. This is because $\alpha \neq 0$ else there would be no tetrahedron. Thus δ defines an additive function.

Hence, one gets that:

$$\Delta(\mathbb{H}) = \sin\alpha + 0 + 0 + 0 + (-2)\sin\alpha + \sin\alpha = 0$$

4 Algebra of Polyhedra

We will now consider the same problem but using the methods of algebra, in particular those of a group G acting on a set X. Suppose in this case that we take the group G to be the set of all isometries $\mathbb{R}^3 \to \mathbb{R}^3$, and X to be the set of all polyhedra in $\mathbb{R}^3$.

There are two sets associated to each polyhedron. Firstly the set of all lengths of the edges, $\{l_1, l_2, ..., l_n\} \subset \mathbb{R}$, and secondly the set of all dihedral angles of the polyhedron, $\{\alpha_1, \alpha_2 ..., \alpha_n\} \subset \mathbb{R}_\pi$, where by $\mathbb{R}_\pi$ I mean the quotient group $\mathbb{R}/\mathbb{Z}\pi$, i.e. the real numbers modulo π. I now define the tensor product, give some properties, and then explain why it is so useful here.

Definition 4.1 *We define the* **tensor product space** *of two vector spaces V and W over F to be $V \otimes W = (V \times W)/R$, where $R \subseteq V \times W$ is the set generated by the equivalence relations*

$$(v_1 + v_2, w) \sim (v_1, w) + (v_2, w)$$

$$(v, w_1 + w_2) \sim (v, w_1) + (v, w_2)$$

$$c(v, w) \sim (cv, w) \sim (v, cw)$$

where $c \in F$, $v_1, v_2, v \in V$ and $w_1, w_2, w \in W$.

We have a typical element of the tensor product of the form $a \otimes b = (a, b) + R$. One can then observe several general properties of a tensor product, 1 and 2 in the proposition below, and in light of the two sets of values associated to each polyhedron, namely the lengths and dihedral angles, one can show point 3 for a specific case.

Proposition 4.1 (Properties of the tensor product) *Suppose that $a \in V$, $b \in W$, $c \in F$. Then:*

1. $a \otimes 0 = 0 = 0 \otimes a$
2. $a(b \otimes c) = ab \otimes c = b \otimes ac$
3. *Suppose $V = \mathbb{R}$ and $W = \mathbb{R}_\pi$. Then $a \otimes q\pi \neq 0$ if $a \neq 0$ and $q \notin \mathbb{Q}$*

Proof

1. The first point is true as follows: $a \otimes 0 = a \otimes (b - b) = a \otimes b - a \otimes b = 0$ and similarly for the other equality.
2. The second point is obvious by the third equivalence relation in definition 4.1.
3. I prove the contrapositive of the statement. Clearly if $a = 0$ then it is zero by the first point. Suppose not, then if $q \in \mathbb{Z}$ clearly the tensor product is zero. If not, then let $q = \frac{c}{b}$ where $c \in \mathbb{Z}$ and $b \in \mathbb{N}$. Then $a \otimes \frac{c\pi}{b} = \frac{a}{b} \otimes c\pi = 0$ as $c \in \mathbb{Z}$.

Q.E.D.

In light of definition 3.3 one can specify the Dehn invariant as follows:

Definition 4.2 *[4] One defines a* **general Dehn Invariant** *to be any function $f : X \to G$, G an abelian group, with the following conditions:*

A) *$f(S) = f(T)$ if S and T are congruent polyhedra*

B) *$f(R) = f(S) + f(T)$ if S and T and polyhedra whose pairwise disjoint union is R*

Note that the function $v : X \to \mathbb{R}$ defined by $v(X) = vol(x)$ is a Dehn invariant. This is because certainly $\mathbb{R}$ is an abelian group and, if two polyhedra A and B are congruent, then by definition 1.2 they must have the same area, $v(A) = v(B)$. Similarly the volume adds as in B).

Definition 4.3 *We define the* **Dehn invariant** *of a polyhedron A to be the function $\Delta : X \to \mathbb{R} \otimes \mathbb{R}_\pi$ by $\Delta(A) = \sum_1^k l_i \otimes \alpha_i$, where the l_i, $i = 1, ..., k$ are the lengths of A and the α_i the corresponding dihedral angles.*

One should observe that this is a special case of definition 4.2, where one uses the abelian group $\mathbb{R} \otimes \mathbb{R}_\pi$. This is the modern definition of the Dehn invariant.

From an alternative and equivalent definition of the tensor product (see [7]) of two spaces, one obtains that for every bilinear mapping $\psi : U \times V \to F$ one obtains a unique linear mapping $g : U \otimes V \to F$ such that $\psi = g \circ \phi$, where ϕ is the embedding map $U \times V \to U \otimes V$ defined by $(u, v) \mapsto u \otimes v$. Thus one can see that this definition of the Dehn invariant contains all possible invariants by the previous definition 3.3, and so all the definitions given in this essay are essentially equivalent.

4.1 Hilbert's Third Problem using tensors

The general structure of the following was shown to me by Dmitriy Rumynin. Using definition 3.1, which as said before is an equivalence relation on the set X of all polyhedra, one can define the quotient group $X/\sim$, which is the set of all equivalence classes of scissors congruence. In this way, two polyhedra are in the same equivalence class if and only if they are scissors congruent. This idea is simply a special case of a quotient G-set, where we have $A \sim B \iff \exists g \in G$ such that $A = gB$.

If one defines $v_3 : X/\sim \to \mathbb{R}$, where $v_3([M]) = vol(M)$, one can see that the coset of M contains all polyhedra with the same volume as M.

Now Hilbert's problem is the same as asking whether v_3 is an isomorphism. This is because if we have an isomorphism, then we must also have a bijection between v_3 and the volume of the polyhedron, thus making every two polyhedra of the same volume scissors congruent. Note that the proof that this is not an isomorphism again uses theorem 3.2 in the solution, with the Dehn invariant defined in 4.3 as the function f. We now consider the same shapes as we did in the proof of lemma 3.4: A the unit cube and B the tetrahedron with volume 1. We now calculate the Dehn invariants of these two shapes. Then $\Delta(A) = 12(1 \otimes \frac{\pi}{2}) = 1 \otimes \frac{12\pi}{2} = 1 \otimes 6\pi = 0$.

We now consider $\Delta(B) = 6(\frac{\sqrt{2}}{\sqrt[3]{3}} \otimes \arccos\frac{1}{3} = a \otimes \arccos\frac{1}{3}$. We aim to show that $d_3(B) \neq 0$. This is non zero if $a \neq 0$ or $\arccos\frac{1}{3} \notin \mathbb{Q}$. We know $\arccos\frac{1}{3}$ is an irrational multiple of π by lemma 3.3, and we have a non zero side length, thus $d_3(B) \neq 0$. Thus we have two polyhedra with the same volume but different Dehn invariants. Thus $vol(M)$ can take the same value for different cosets, and so v_3 is not an isomorphism. We have again proved Hilbert's third problem.

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In a similar way, one could define a Dehn invariant for $\mathbb{R}^n$ by exchanging the lengths l_i into volumes of the $n-2$ dimensional "edges" of the n dimensional polygon. One could then consider the same problem in higher dimensions.

Throughout this essay, I have only considered the problem in Euclidean geometry, but one could equally validly consider it in other geometries, such as Spherical or hyperbolic. Reference [4] provides some information on this at a high level if the reader is interested.

In the definition of scissors congruent (2.1 and 3.1) we state that the polygons are decomposed into other polygons. This decompostion into polygons is crucial. If we hadn't specified polygons, then one would have to bear in mind the Banach-Tarski paradox, which would ruin the idea of shapes of the same area being scissors congruent. Without this, one would have to conclude that any two bounded subsets of 3-dimensional space with non-empty interiors are scissors congruent.

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