# MA3G8 Functional Analysis II

June 10, 2013

## Contents



These notes are based on the 2012 `MA3F6 Functional Analysis II` course, taught by Keith Ball, typeset by Matthew Egginton.
No guarantee is given that they are accurate or applicable, but hopefully they will assist your study.
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# 0 Introduction

At the beginning of the 20th century.

# 1 Basics

## 1.1 Normed Spaces

**Definition 1.1** *A* **norm** $||\cdot||$ *on a vector space* $V$ *is a map* $x \mapsto ||x|| \in \mathbb{R}$ *satisfying*

1. $||x|| \geq 0$ *for all* $x$ *and* $||x|| = 0 \iff x = 0$.
2. $||\lambda x|| = |\lambda|||x||$ *for all* $x \in V$ *and* $\lambda \in \mathbb{R}$
3. $||x + y|| \leq ||x|| + ||y||$ *for* $x, y \in V$.

A norm gives us a way to measure distance: $d(x, y) = ||x - y||$. The unit ball $B$ plays an important role (geometry of space is invariant under translation) and it is a convex set. In other words, if $x, y \in B$ and $\lambda \in [0, 1]$ then $(1 - \lambda)x + \lambda y \in B$

**Lemma 1.2** *If* $N : V \to \mathbb{R}$ *satisfies*

1. $N(x) \geq 0$ *for all* $x$ *and* $N(x) = 0 \iff x = 0$
2. $N(\lambda x) = |\lambda|N(x)$ *for* $\lambda \in \mathbb{R}$ *and* $x \in V$
3. $\{x \in V : N(x) \leq 1\}$ *is convex*

*then* $N$ *is a norm.*

The proof essentially uses a simple trick.
**Proof** We need to check the triangle inequality. Suppose $x, y \in V$. If $N(x) = 0$ then $x = 0$ and then $x + y = y$ and so $N(x + y) = N(y) = N(x) + N(y)$. Thus suppose $a = N(x)$, $b = N(y)$ and are strictly positive. Then $N(a^{-1}x) = a^{-1}N(x) = 1$ and so $a^{-1}x \in \{x : N(x) \leq 1\}$ and the same for $b^{-1}y \in \{x : N(x) \leq 1\}$ and thus

$$\frac{a}{a+b}a^{-1}x + \frac{b}{a+b}b^{-1}y = \frac{x+y}{a+b} \in \{x : N(x) \leq 1\}$$

by convexity. Thus

$$N(x + y) = (a + b)N(\frac{x + y}{a + b}) \leq (a + b)1 = a + b = N(x) + N(y)$$

*Q.E.D.*

We shall consider a number of "standard" spaces.

1. $l_p$ the space of sequences $(x_i)$ with finite norm $||(x_i|| = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ for $1 \leq p < \infty$ and $l_\infty$ is the space of bounded sequences with the norm $||(x_i)|| = \sup_i |x_i|$.
2. Consider the finite dimensional versions $l_p^n$, the space of sequences of length $n$ with norm $||(x_i)|| = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$.
3. Also $c_0$ the space of sequences converging to 0 with norm $||(x_i)|| = \max |x_i| = \sup |x_i|$. Note $c_0 \subset l_\infty$ in the obvious way.

4. $C[0,1]$ the space of continuous functions on $[0,1]$ with norm $||f|| = \sup|f(x)| = \max|f(x)|$

5. $L_p[0,1]$ the space of measurable "functions" with $||f|| = \left(\int_0^1 |f(x)|^p dx\right)^{1/p} < \infty$. We actually mean equivalence classes of functions, i.e. $f \sim g$ if $f = g$ a.e.

6. $L_\infty[0,1]$ is the space of measurable functions such that $||f|| = \sup|f(x)| < \infty$.

The triangle inequality in $L_p$ is often called the Minkowski inequality

$$\left(\int |f+g|^p\right)^{1/p} \leq \left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}$$

We can see that the unit ball is convex as follows. If $\int |f|^p, \int |g|^p \leq 1$ and $\lambda \in [0,1]$ then for each $x$,

$$|(1-\lambda)f(x) + \lambda g(x)|^p \leq (1-\lambda)|f(x)|^p + \lambda|g(x)|^p$$

since the map $t \mapsto |t|^p$ is convex on $\mathbb{R}$ and so

$$\int |(1-\lambda)f(x) + \lambda g(x)|^p \leq (1-\lambda)\int |f(x)^p + \lambda \int |g(x)|^p \leq 1$$

**Lemma 1.3 (Hölder Inequality)** *If $f, g$ are measurable, and $\frac{1}{p} + \frac{1}{q} = 1$ and*

$$\int |f|^p < \infty, \int |g|^q < \infty$$

*then*

$$\int fg \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

The Cauchy-Schwartz inequality is Hölder with $p = 2 = q$.

## 1.2 Separability

**Definition 1.4** *A subset $U$ of a metric space $X$ is said to be* **dense** *in $X$ if for every $x \in X$ and $\varepsilon > 0$ there is a $u \in U$ with $d(x,u) < \varepsilon$. A normed space is called* **separable** *if it has a countable dense subset.*

**Lemma 1.5** *For a normed space $X$, the following are equivalent*

1. *$X$ is separable*
2. *$S_X = \{x \in X : ||x|| = 1\}$ is separable*
3. *$X$ contains a sequence $(x_1, x_2, ...)$ whose linear span is dense.*

We prove the following before the proof of the above

**Lemma 1.6** *A subset of a separable metric space is separable*

**Proof** Let $(x_1, x_2, ...)$ be a dense subset in $X$ and $Y \subset X$. We need to find a dense subset of $Y$.

For each $n$ and $k$, if $Y$ contains a point whose distance from $x_n$ is at most $\frac{1}{k}$ call this $u_{n,k} \in Y$. The set $\{u_{n,k}\}$ is countable. I claim that it is dense in $Y$. If $y \in Y$ and $\varepsilon > 0$ choose $x_n$ with $d(x_n, y) < \frac{1}{k}$ where $\frac{1}{k} < \frac{\varepsilon}{2}$. There is a point $u_{n,k}$ since $y$ would have been a candidate. Then

$$d(y, u_{n,k}) \leq d(y, x_n) + d(x_n, u_{n,k}) < \frac{1}{k} + \frac{1}{k} < \varepsilon$$

*Q.E.D.*

**Proof [of lemma 1.5]** 1) $\implies$ 2) is clear from the previous lemma since $S_X \subset X$.

2) $\implies$ 3) Choose $(u_i)$ dense in $S_X$. This sequence suffices for 3). Even the multiples $(\lambda u_i)$ are dense. If $x \in X$ and $x \neq 0$ then $\frac{x}{||x||} \in S_X$ so we can find a $u_i$ with $||\frac{x}{||x||} - u_i|| < \frac{\varepsilon}{||x||}$ and hence

$$||x - ||x||u_i|| < \varepsilon$$

3) $\implies$ 1). Let $(x_1, x_2, ...)$ be a sequence whose span is dense. The span is much bigger than countable and so 3) is weaker than 1). We want to build a countable dense set in $X$. We take the rational linear combination $\sum_1^n r_i x_i$ for $r_i \in \mathbb{Q}$. Given $x \in X, \varepsilon > 0$ we can find some $n \in \mathbb{N}$ and $\theta_1, ..., \theta_n \in \mathbb{R}$ with $||x - \sum \theta_i x_i|| < \frac{\varepsilon}{2}$. Now choose rationals $r_i$ with $|\theta_i - r_i| < \frac{\varepsilon}{2nM}$ where $M = 1 + \max\{||x_1||, ..., ||x_n||\}$ and then

$$\begin{aligned}\left|\left|\sum \theta_i x_i - \sum r_i x_i\right|\right| &= \left|\left|\sum (\theta_i - r_i) x_i\right|\right| \\ &\leq \sum ||(\theta_i - r_i) x_i|| \\ &= \sum |\theta_i - r_i| ||x_i|| \\ &\leq \sum \frac{\varepsilon}{2nM} M \\ &\leq \frac{\varepsilon}{2}\end{aligned}$$

and combining above inequalities gives the desired result. *Q.E.D.*

The homework includes proving $l_p$ is separable, and this uses the above.

## 1.3 Linear Maps

A linear map $T : X \to Y$ between normed spaces may or may not be continuous. It is continuous if and only if it is bounded, i.e. there exists $K > 0$ with

$$||Tx|| \leq K||x||$$

For such a map we define its norm as $\inf\{K : ||Tx|| \leq K||x|| \forall x\} = ||T|| = \sup\{||Tx|| : ||x|| \leq 1\}$. This is in some sense the maximum scale factor.

The space of bounded linear maps form a normed vector space under this norm. This space $\mathcal{B}(X, Y)$ is complete if and only if $Y$ is complete. A complete normed space is called a Banach space. In finite dimensions, $T : U \to V$ then $\dim(\ker T) + \dim(\text{Im}T) = \dim U$.

Two Banach spaces $X$ and $Y$ are called **isomorphic** if there is an invertible linear map $T : X \to Y$ with $||T||$ and $||T^{-1}||$ bounded. They are **isometric** if $||T|| = 1 = ||T^{-1}||$.

There is a natural (algebraic) isomorphism between $U/\ker T$ and $ImT$, by " cutting the electric cable".

We can define the **quotient** in a Banach space. If $Y$ is a closed subspace of $X$ we define $X/Y$ to be the space of translates $\{x+Y : x \in X\}$ with $x+Y = z+Y$ if $x-z \in Y$. As in the finite dimensional case, addition and scalar multiplication make sense. We define the norm

$$||x+Y|| = \inf\{||x+y|| : y \in Y\}$$

and in some sense this is the closest point of the translate to 0, but the closest point may not exist.

1. The norm is well defined; i.e. $||x+Y|| = ||z+Y||$ if $x-z \in Y$. This is because we have seen a form of the norm without reference to a representative.

2. $||\lambda(x+Y)|| = |\lambda|||x+Y||$ is easy to show

3. We show the triangle inequality explicitly. Given $\varepsilon > 0$ choose $y_x \in Y$ and $y_z \in Y$ with $||x+y_x|| < ||x+Y|| + \varepsilon$ and $||x+y_z|| < ||z+Y|| + \varepsilon$ and then

   $$||x+z+Y|| \leq ||x+z+y_x+y_z|| \leq ||x+y_x|| + ||z+y_z|| \leq ||x+Y|| + ||z+Y|| + 2\varepsilon$$

   and taking the infimum over $\varepsilon$ gives us our result.

4. Show $||x+Y|| = 0$ if and only if $x+Y = 0$, i.e. $x \in Y$.

   If $x \in Y$ then choose $y = -x$ and so $x+y = 0$ and so $||x+Y|| = 0$. Conversely suppose $||x+Y|| = 0$. Then we can find a sequence $(y_k)$ in $Y$ with $||x+y_k|| \to 0$, i.e. $x+y_k \to 0$ or $y_k \to -x$, and since $Y$ is closed, $-x \in Y$ and so $x \in Y$.

5. $X/Y$ is complete. We take a Cauchy sequence and then construct a subsequence and use the completeness of $X$. We might not find a convergent sequence of representatives. It suffices to find a convergent subsequence. Choose $n_k$ so that $||x_{n_k} + Y - x_m + Y|| \leq \frac{1}{2^k}$ if $m \geq n_k$. Hence $||x_{n_k} + Y - x_{n_{k+1}} + Y|| \leq \frac{1}{2^k}$. Choose $y_{n_k}$ one by one so that

   $$||x_{n_k} + y_{n_k} - (x_{n_{k+1}} + y_{n_{k+1}})|| \leq \frac{2}{2^k}$$

   and so $x_{n_k} + y_{n_k}$ is Cauchy and has a limit $u$. Then $x_{n_k} + Y \to u + Y$ as required. We have used:

**Lemma 1.7** *If $(x_k)$ is a sequence in a complete metric space with $d(x_k, x_{k+1}) \leq \varepsilon_k$ and $\sum_1^\infty \varepsilon_k < \infty$ then $(x_k)$ is Cauchy.*

## 2 Duality and the separation theorems

If $X$ is a normed space then a linear functional on $X$ is a linear map $X \to \mathbb{R}$ (or the underlying field). The space of bounded linear functionals is denoted $X^\star$; it is a Banach space with the operator norm $||\phi|| = \sup\{|\phi(x)| : ||x|| \leq 1\}$. If $X$ is a Hilbert space then every bounded linear functional is of the form

$$x \mapsto \langle x, y \rangle$$

for some $y \in X$ and these maps are all in $X^\star$. Thus we can identify $X^\star$ with $X$. What about $l_p$ and $L_p$?

**Theorem 2.1 (Duality of $l_p$)** *If* $X = \begin{cases} c_0 \\ l_p & 1 < p < \infty \\ l_1 \end{cases}$*, then* $X^\star$ *is isometric to* $\begin{cases} l_1 \\ l_q \\ l_\infty \end{cases}$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

**Remark** If $p = 2$ then $q = 2$ and $l_p^\star = l_p$.
**Proof** For each $y$ of the supposed dual (in each case) we can define $\phi_y : X \to \mathbb{R}$ by

$$\phi_y(x) = \sum_1^\infty y_i x_i$$

where $x = (x_1, x_2, ...)$ and this makes sense by Holder:

$$\sum_1^\infty |y_i x_i| \leq \begin{cases} ||x||_\infty ||y||_1 \\ ||x||_p ||y||_q \\ ||x||_1 ||y||_\infty \end{cases}$$

and so the series converges absolutely.

$\phi_y$ is clearly linear and the map $y \mapsto \phi_y$ $(l_q \to X^\star)$ is linear

$$\phi_{y+z}(x) = \sum (y_i + z_i) x_i = \sum y_i x_i + \sum z_i x_i = \phi_y(x) + \phi_z(x)$$

We also have $|\phi_y(x)| \leq ||x||_p ||y||_q$ and so on so the map has $||\phi_y|| \leq ||y||$. Note that the map $y \mapsto \phi_y$ is obviously 1-1 but we shall get this automatically when we prove $||y|| \leq ||\phi_y||$

It remains to show that each $\phi \in X^\star$ is of the form $\phi_y$ for some $y$ (in the supposed dual) with $||y|| \leq ||\phi_y||$.

Let $(e_i)$ be the standard unit vector basis of $X$. For each $i$ set $y_i = \phi(e_i)$. Then $\phi$ and $\phi_y$ agree on the $(e_i)$. The problem is the norm. We show that $||y|| \leq ||\phi||$ as then $\phi$ is of the form $\phi_y$ for that $y$. If $1 < p < \infty$ then

$$\begin{aligned} \sum_1^m |y_i|^q &= \sum_1^m y_i |y_i|^{q-1} \text{sgn}(y_i) \\ &= \sum_1^m \phi(e_i) |y_i|^{q-1} \text{sgn}(y_i) \\ &= \phi(\sum_1^m |y_i|^{q-1} \text{sgn}(y_i) e_i) \\ &\leq ||\phi|| || \sum_1^m |y_i|^{q-1} \text{sgn}(y_i) e_i ||_p \\ &= ||\phi|| \left( \sum_1^m |y_i|^{(q-1)p} \right)^{\frac{1}{p}} \\ &= ||\phi|| \left( \sum_1^m |y_i|^q \right)^{\frac{1}{p}} \end{aligned}$$

We have shown that

$$\sum_1^m |y_i|^q \leq ||\phi|| \left( \sum_1^m |y_i|^q \right)^{\frac{1}{p}}$$

and since $\frac{1}{p} < 1$ the estimate has some content. If $y = 0$ then $\phi = 0$ and there was nothing to prove. Thus we may assume that $\sum |y_i| > 0$ and then

$$\left(\sum_1^m |y_i|^q\right)^{\frac{1}{q}} \leq ||\phi||$$

and so taking limits as $m \to \infty$ we have

$$\left(\sum_1^m |y_i|^q\right)^{\frac{1}{q}} = ||y|| \leq ||\phi||$$

we know that $\phi$ and $\phi_y$ agree on $(e_i)$ and are continuous linear functionals. In $l_p$ for $1 < p < \infty$ or $c_0$ or $l_1$ the closed span of the $(e_i)$ is $X$. Thus the functionals agree on $X$.

If $X = c_0$ then

$$\begin{aligned}\sum_1^m |y_i| &= \sum_1^m y_i \text{sgn}(y_i) \\ &= \sum_1^m \phi(e_i)\text{sgn}(y_i) \\ &= \phi\left(\sum_1^m \text{sgn}(y_i)e_i\right) \\ &\leq ||\phi|| || \sum_1^m \text{sgn}(y_i)e_i|| \\ &= ||\phi||\end{aligned}$$

and if $X = l_1$ then

$$|y_i| = |\phi(e_i)| \leq ||\phi|| ||e_i|| = ||\phi||$$

*Q.E.D.*

1. In each case we show that if $y$ is in the suggested dual then it gives a $\phi_y \in X^\star$, by using Holder. In particular each $y \in l_\infty$ acts on $l_1$ and each $y \in l_1$ acts on ($l_\infty$ and hence) $c_0$.

2. For each case, given $\phi \in X^\star$ there is a $y$ in the supposed dual with $\phi(e_i) = \phi_y(e_i)$ for each $i$. Since the closed span of the $e_i$ is the space required this gives $\phi = \phi_y$ on $X$.

However, since $l_\infty$ is not separable, we cannot conclude that $l_\infty^\star = l_1$. Moreover $l_\infty^\star$ is not isomorphic to $l_1$. Note that the above proof uses separability in the construction ofthe basis.

The word duality suggests some kind of symmetry between $X$ and $X^\star$. In finite dimensions, indeed $(X^\star)^\star \cong X$ in a canonical way: if $x \in X$ and $\phi \in X^\star$ then the action of $x$ on $\phi$ is the action of $\phi$ on $x$.

In Banach spaces, $X$ still acts on $X^\star$, so $X \subset X^{\star\star}$ as for example $c_0$ acts on $l_1$ and $c_0 \subset l_\infty$ but $X$ may not fill up $X \subset X^{\star\star}$, eg $c_0 \neq l_\infty$ (they are massively different).

Things in $L_p$ are somewhat similar. There is no analogue of $c_0$ and $L_\infty^\star$ is every bit as nasty as $l_\infty^\star$. $C[0, 1]$ is a little bit like $c_0$. We have $L_1^\star \cong L_\infty$ and $L_p^\star \cong L_q$ if $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p < \infty$.

The proof of $L_p^\star \cong L_q$ is very similar to the proof that $l_p^\star \cong l_q$. If $g \in L_q$ we can define $\phi_g$ on $L_p$ by $\phi_g(f) = \int_0^1 fg$ and Hölder still works. On the other hand, if we have a functional $\phi$ on $L_p$ we have no "basis" vectors to apply $\phi$ to.

We can apply $\phi$ instead to $1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ so we get numbers $\phi(1_A)$ and the map $A \mapsto \phi(1_A)$ is a signed measure. So by the Radon-Nikodým theorem there is a function $g$ with $\phi(1_A) = \int g 1_A$ for all $A$. So $\phi$ is given by $f \mapsto \int fg$ for $f = 1_A$ and by the standard machine $\phi = \phi_g$ on $L_p$.

We motivate the next steps we take. Suppose $X$ is a Banach space, how might we define a functional on $X$? Choose a basis $\{e_\alpha\}_{\alpha \in A}$ and then $\phi(\sum \lambda_\alpha e_\alpha) = \lambda_1$ is a linear functional. This though will not in general be continuous, or how do we show so? We need, ironically, the axiom of choice to prove that a continuous functional exists.

First though we consider $C[0,1]$ and its dual space. If $\mu$ is a finite Borel measure on $[0,1]$ then the map $f \mapsto \int_0^1 f d\mu$ is a bounded linear functional on $C[0,1]$. It turns out that $C[0,1]^\star \cong \{\text{finite Borel signed measures}\}$ but this is a subtle theorem, the Riesz representation theorem. Newton showed that the map $f \mapsto \int_0^1 f dx$ is a bounded linear functional. Thus Riesz says that the ordinary integral is given by a measure, so Riesz implies the existence of the Lebesgue measure. Every linear functional on $C[0,1]$ is given by a signed measure.

## 2.1 1-Codimensional Subspaces

**Definition 2.2** *A subspace $Y$ of a Banach space $X$ is called* **1-codimensional** *if*

1. $Y \neq X$
2. *If $Z$ is a subspace such that $Y \subset Z \subset X$ then $Z = Y$ or $Z = X$*

**Theorem 2.3** *If $Y$ is a subspace of a Banach space $X$ then the following are equivalent.*

1. *$Y$ is 1-codimensional*
2. *$Y \neq X$ but if $x \in X \setminus Y$ then* $\operatorname{span}\{Y \cup \{x\}\} = X$
3. *$Y = \ker \phi$ for some non-zero linear functional $\phi$ on $X$. (note $\phi$ is not necessarily bounded)*

**Proof** $1 \iff 2$ is easy. If $Y$ is 1-codimensional and $x \in X \setminus Y$ then $Y \subset \operatorname{span}\{Y \cup \{x\}\}$ but $Y \neq \operatorname{span}\{Y \cup \{x\}\}$ and so $\operatorname{span}\{Y \cup \{x\}\} = X$. On the other hand if $Y$ satisfies 2 and $Y \subset Z \subset X$ and $Z \neq Y$ then choose $z \in Z \setminus Y$. Then $Z \supset \operatorname{span}\{Y \cup \{z\}\} = X$.

$1, 2 \implies 3$. Assume $Y$ is 1-codimensional and choose $x \in X \setminus Y$. Each element $z \in X$ can be written as $y + \lambda x$ for some $y \in Y$ and $\lambda \in \mathbb{R}$. The representation is unique because if $y_1 + \lambda_1 x = y_2 + \lambda_2 x$ then $(\lambda_1 - \lambda_2)x = y_2 - y_1 \in Y$ and so $\lambda_1 = \lambda_2$ and hence $y_1 = y_2$. Now define $\phi(y + \lambda x) = \lambda$. Clearly $Y \subset \ker \phi$ and $\phi \neq 0$. $\ker \phi = Y$ because $Y \subset \ker \phi \subsetneq X$ and so must be $Y$.

$3 \implies 2$. Suppose $Y = \ker \phi$ and $\phi \neq 0$. Take any $x$ such that $\phi(x) \neq 0$. We want to show that $Y \cup \{x\}$ spans $X$. Given $z \in X$ we want to choose $y, \lambda$ such that $z = y + \lambda x$. We do this as follows. Set $\lambda = \frac{\phi(z)}{\phi(x)}$ and $y = z - \lambda x$. Then $\phi(y) = \phi(z - \lambda x) = 0$ and so $y \in Y$ as required. *Q.E.D.*

**Lemma 2.4** *If $Y = \ker \phi$ is 1-codimensional in $X$ then*

1. *$Y$ is closed if and only if $\phi$ is continuous*

2. *$Y$ is dense if and only if $\phi$ is not continuous.*

**Proof** Since $Y \neq X$ it cannot be closed and dense. So it suffices to prove that $Y$ is closed if $\phi$ is continous and $Y$ is dense if not.

If $\phi$ is continuous, then $\ker\phi = \phi^{-1}(\{0\})$ and so is closed. We need to show if $\phi$ is not continuous then $\ker\phi$ is dense.

$\phi$ is unbounded (i.e. not continuous) so choose a sequence $(x_n)$ in $X$ with $||x_n|| \leq 1$ with $|\phi(x_n)| \to \infty$. If $x \in X$ choose $y_n = x - \frac{\phi(x)}{\phi(x_n)}x_n$ and then $\phi(y_n) = 0$ so $y_n \in \ker\phi$. On the other hand

$$||y_n - x|| = || - \frac{\phi(x)}{\phi(x_n)}x_n|| = \left|\frac{\phi(x)}{\phi(x_n)}\right| ||x_n|| \to 0$$

and so $x \in \text{closure}(Y)$. *Q.E.D.*

## 2.2 Separation Theorems

We state the important theorem, give some applications and then prove it, as this way you are motivated why the theorem is so important.

**Theorem 2.5 (Hahn-Banach)** *If $Y$ is a subspace of a real normed linear space $X$ and $\phi : Y \to \mathbb{R}$ is a continuous linear functional, then there is a continuous linear functional $\tilde{\phi} : X \to \mathbb{R}$ which extends $\phi$, i.e. $\tilde{\phi}(y) = \phi(y)$ if $y \in Y$ and $||\tilde{\phi}|| = ||\phi||$.*

**Corollary 2.6** *If $x \in X$ is a unit vector then there is a functional $\tilde{\phi} \in X^\star$ with $||\tilde{\phi}|| \leq 1$ and $\tilde{\phi}(x) = 1$.*

The corollary thus says that there is a supporting hyperplane to the unit ball $B$ at each point on the boundary.
**Proof** Let $Y = \text{span}\{x\} \subset X$ and define $\phi : Y \to \mathbb{R}$ by $\phi(\lambda x) = \lambda$. Then $\phi(x) = 1$ and $|\phi(\lambda x)| = |\lambda| ||x|| = ||\lambda x||$ and so $||\phi|| = 1$. Then by Hahn-Banach we can extend to $\tilde{\phi}$ with $||\tilde{\phi}|| = 1$ and $\tilde{\phi}(x) = 1$. *Q.E.D.*

**Corollary 2.7 (Closest point witnesses)** *Let $Y$ be a closed subspace of $X$ and $x \in X$. Let $d = \inf\{||x - y|| : y \in Y\}$. Then there is a linear functional $\phi$ on $X$ with*

1. *$\phi(y) = 0$ for all $y \in Y$*

2. *$||\phi|| = 1$*

3. *$\phi(x) = d$*

Thus $\phi$ witnesses the fact that $x$ is at least distance $d$ from $Y$, since if $y \in Y$ then

$$||x - y|| = ||\phi|| ||x - y|| \geq \phi(x - y) = \phi(x) - \phi(y) = d$$

**Proof** Let $Z = \text{span}\{Y \cup \{x\}\}$ and define $\phi : Z \to \mathbb{R}$ by $\phi(y + \lambda x) = d\lambda$. It satisfies 1 and 3 and we want to show that $|\phi(y + \lambda x)| \leq ||y + \lambda x||$. But how do we understand the right hand side of this. We know that $||x - y|| \geq d$ for all $y \in Y$ and so $||\lambda x - \lambda(\frac{-y}{\lambda})|| \geq |\lambda| d$ and so we have this bounded. Then we have $|\phi(y + \lambda x)| = |\lambda d| = |\lambda| d \leq |\lambda| ||x - \frac{-y}{\lambda}|| = ||y + \lambda x||$.By the Hahn-Banach theorem there is a functional on $X$ with the same properties. *Q.E.D.*

### 2.2.1 Reflexivity

We saw that if $x \in X$ we can define an element of $X^{\star\star}$, $\hat{x}$ by $\hat{x}(\phi) = \phi(x)$ for $\phi \in X^\star$. The map $x \mapsto \hat{x}$ is linear and $||\hat{x}|| \leq ||x||$ since $|\hat{x}(\phi)| = |\phi(x)| \leq ||\phi||||x||$. In fact $||\hat{x}|| = ||x||$ and so the map $x \mapsto \hat{x}$ is an isometric embedding of $X$ into $X^{\star\star}$, because, if $||x|| = 1$ then by the Hahn-Banach there is a $\phi \in X^\star$ with $||\phi|| = 1$ and $\phi(x) = 1$ and so $\hat{x}(\phi) = \phi(x) = 1$ and $||\phi|| = 1$ and so $||\hat{x}|| \leq 1$. If $||x|| \neq 1$ we scale.

**Definition 2.8** *If the map $x \mapsto \hat{x}$ is surjective then we say that $X$ is* **reflexive**.

### 2.2.2 Adjoints

**Definition 2.9** *If $T : X \to Y$ is linear and bounded then the* **adjoint** *$T^\star : Y^\star \to X^\star$ is defined by*

$$T^\star(\psi)(x) = \psi(Tx)$$

*for $\psi \in Y^\star$ and $x \in X$.*

**Lemma 2.10** *The following are properties of the adjoint*

1. *for each $\psi$, $T^\star\psi$ is linear*
2. *$T^\star$ is linear*
3. $||T^\star|| = ||T||$

**Proof**

1. clear since $T$ is linear.
2. $$T^\star(\psi + \phi)(x) = (\phi + \psi)(Tx) = \phi(Tx) + \psi(Tx) = T^\star(\phi)(x) + T^\star(\psi)(x)$$
   and so $T^\star(\psi + \phi) = T^\star(\phi) + T^\star(\psi)$.
3. We have that
   $$|T^\star(\psi)(x)| = |\psi(Tx)| \leq ||\psi||||Tx|| \leq ||\psi||||T||||x||$$
   and so $||T^\star(\psi)|| \leq ||T||||\psi||$ and thus $||T^\star|| \leq ||T||$.
   We claim that $||T^\star|| \geq ||T||$. We can choose $x \in X$ with $||x|| = 1$ and $||Tx|| \geq ||T|| - \varepsilon$ and then by Hahn-Banach there is a $\psi \in Y^\star$ with $||\psi|| = 1$ and $|\psi(Tx)| \geq ||T|| - \varepsilon$. Then $|T^\star(\psi)(x)| = |\psi(Tx)| \geq ||T|| - \varepsilon$ but $||x|| = 1 = ||\psi||$ and so $||T^\star|| \geq ||T|| - \varepsilon$ and let $\varepsilon \to 0$.

*Q.E.D.*

In a Hilbert space, if we identify $H^\star$ with $H$ via the inner product then this adjoint is the usual one in the real case.

In finite dimensions we can represent $T$ as a matrix with respect to bases in $X$ and $Y$. Then the matrix of $T^\star$ is the transpose of that of $T$, if we use the dual bases for $Y^\star$ and $X^\star$.

**Theorem 2.11 ( 1 -step extension)** *If $Y$ is a 1-codimensional subspace of a normed space $X$ and $\phi : Y \to \mathbb{R}$ is a bounded linear functional then there is an extension $\tilde{\phi} : X \to \mathbb{R}$ of $\phi$ with $||\tilde{\phi}|| \leq ||\phi||$*

The inequality of the norms is crucial if we want to apply this infinitely often.

To extend into 1 dimension higher we will have a single degree of freedom.

**Proof**

By scaling we may assume that $||\phi|| = 1$. Choose $x \in X \setminus Y$. We can write every point of $X$ uniquely as $y + \lambda x$ for $y \in Y$ and $\lambda \in \mathbb{R}$. For each $\alpha$ define $\phi_\alpha : X \to \mathbb{R}$ by

$$\phi_\alpha(y + \lambda x) = \phi(y) + \lambda\alpha$$

Our aim is to show that for an appropriate $\alpha$, $||\phi_\alpha|| \leq 1$, since each $\phi_\alpha$ is clearly linear. We need to check that

$$|\phi(y) + \lambda\alpha| \leq ||y + \lambda x||$$

for all $y \in Y$ and $\lambda \in \mathbb{R}$. This is automatic if $\lambda = 0$ so by considering $\lambda^{-1}y$ instead of $y$ it suffices to establish this for $\lambda = 1$, and all $y \in Y$.

We thus want

$$|\phi(y) + \alpha| \leq ||y + x||$$

for all $y \in Y$. This is the same as

$$\phi(y) + \alpha \leq ||y + x|| \qquad \forall y \in Y \qquad \text{and} \qquad \phi(z) + \alpha \geq -||z + x|| \qquad \forall z \in Y$$

This means we need to choose $\alpha$ such that

$$-||z + x|| - \phi(z) \leq \alpha \leq ||y + x|| - \phi(y) \qquad \forall y, z \in Y$$

such an $\alpha$ exists provided

$$-||z + x|| - \phi(z) \leq ||y + x|| - \phi(y) \qquad \forall y, z \in Y$$

in other words we need

$$\phi(y) - \phi(z) \leq ||y + x|| + ||z + x||$$

but

$$\phi(y) - \phi(z) = \phi(y - z) \leq ||y - z|| = ||y + x - (z + x)|| \leq ||y + x|| + ||z + x||$$

as required.

*Q.E.D.*

**Definition 2.12** *A* **partially ordered set** $(\Omega, \leq)$ *is a set* $\Omega$ *equipped with a relation* $\leq$ *satisfying*

1. $x \leq x$ *for all* $x \in \Omega$
2. *If* $x \leq y$ *and* $y \leq z$ *then* $x \leq z$.
3. *For every* $x, y \in \Omega$, *if* $x \leq y$ *and* $y \leq x$ *then* $x = y$

An example is to take $\Omega$ to be the set of all subsets of $\{1, 2, 3, 4\}$ and $A \leq B$ means $A \subseteq B$. Then $\{1\} \leq \{1, 2\}$ and $\emptyset \leq \{3\}$ but $\{1, 2\}$ is not related to $\{3\}$.

**Definition 2.13** *A* **chain** *in a partially ordered set is a subset in which every pair of elements is related.*

*An* **upper bound for a chain** $\mathcal{C} \subset \Omega$ *is an element* $u \in \Omega$ *with* $c \leq u$ *for all* $c \in \mathcal{C}$

*A* **maximal element** $m$ *of* $\Omega$ *is an element for which there is no* $v$ *with* $m \leq v$ *but* $v \neq m$.

Note that this doesnt mean that $m \geq u$ for all $u \in \Omega$.

An example of such is $\Omega = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\}$ ordered by inclusion. Then the maximal elements are $\{1,2\}$ and $\{3\}$.

**Theorem 2.14 (Zorn's Lemma)** *If $(\Omega, \leq)$ is a non-empty partially ordered set in which every chain has an upper bound, then $\Omega$ contains at least one maximal element*

If you have a 1-step extension then a maximal element must be the top, i.e. in our context the Banach space.

**Proof (Hahn Banach)** Let $\Omega$ be the partially ordered set of pairs $(Z, \phi_Z)$ where $Z$ is a subspace of $X$ containing $Y$ and $\phi_Z : Z \to \mathbb{R}$ is a linear functional extending $\phi$ and satisfying $||\phi_Z|| \leq ||\phi||$, with the ordering $(Z_1, \phi_{Z_1}) \leq (Z_2, \phi_{Z_2})$ if $Z_1 \subset Z_2$ and $\phi_{Z_1}$ and $\phi_{Z_2}$ agree on $Z_1$. It is clear that this is a partial order and $\Omega$ is non empty since $(Y, \phi) \in \Omega$.

Suppose we have a chain $\{(Z_\alpha, \phi_{Z_\alpha})\}_{\alpha \in A}$ in $\Omega$. Let $W = \cup_A Z_\alpha$ and define $\phi_W : W \to \mathbb{R}$ as follows: If $w \in W$ then $w \in Z_\alpha$ for some $\alpha$ and we set $\phi_W(w) = \phi_{Z_\alpha}(w)$ for that $\alpha$. This is well defined since if $w \in Z_\alpha$ and $w \in Z_\beta$ we have $Z_\alpha \subset Z_\beta$, say, and then $\phi_{Z_\beta}$ extends $\phi_{Z_\alpha}$ because we are in a chain. $W$ is a subspace since if $u, v \in W$ then $u \in Z_\alpha$ and $v \in Z_\beta$ say and then $Z_\alpha \subset Z_\beta$ so $u, v \in Z_\beta$ so $u + v \in Z_\beta \subset W$, and similarly if $Z_\alpha \subset Z_\beta$. In the same way $\phi_W$ is linear.

Moreover, if $w \in W$ with $w \in Z_\alpha$ then

$$|\phi_W(w)| = |\phi_{Z_\alpha}(w)| \leq ||\phi|| ||w||$$

and since this works for all $w$, $||\phi_W|| \leq ||\phi||$. Thus $(W, \phi_W) \in \Omega$ and clearly this is an upper bound for the chain. By Zorn's lemma, $\Omega$ contains a maximal element $(Z, \phi_Z)$. If $Z \neq X$ then we can find an $x \in X \setminus Z$ and by the 1-step extension theorem we can extend $\phi_Z$ to span$\{Z \cup \{x\}\}$, contradicting the maximality of $(Z, \phi_Z)$. *Q.E.D.*

Homework 4 asked to use Hahn-Banach to build a functional $\phi$ on $l_\infty$ which is bounded such that if $(x_i)$ is convergent then $\phi((x_i)) = \lim_{n\to\infty} x_n$. This is not unique though, and is rather strange.

### 2.2.3 Complex Hahn Banach

If $X$ is a complex normed space then we can regard it as a real normed space, using only the real numbers as the scalars.

**Lemma 2.15** *If $X$ is a complex normed space and $\phi : X \to \mathbb{C}$ is linear then there is a real linear functional $u : X \to \mathbb{R}$ with*

$$\phi(x) = u(x) - iu(ix)$$

*If $\phi$ is bounded then $||\phi|| = ||u||$. Conversely, given $u$ we can build $\phi$ and $||\phi|| = ||u||$.*

This is saying that for a linear functional, the real and imaginary parts are related.

An example of this is with $X = \mathbb{C}$ and $\phi(z) = z$. Thus $\phi(x + iy) = x + iy$ and so we have $u(x + iy) = x$. Then the image under $u$ of the unit disc is very small compared to the image of it under $\phi$. However, it still has norm 1.

**Proof** Let $u$ and $v$ be the real and imaginary parts of $\phi$ so that

$$\phi(x) = u(x) + iv(x) \qquad \forall x \in X$$

$u, v : X \to \mathbb{R}$ and are real linear since for $\lambda \in \mathbb{R}$,

$$u(\lambda x) + iv(\lambda x) = \phi(\lambda x) = \lambda\phi(x) = \lambda u(x) + i\lambda v(x)$$

Also we have that $u(ix) + iv(ix) = \phi(ix) = i\phi(x) = iu(x) - v(x)$ and equating real and imaginary parts gives

$$u(ix) = -v(x) \qquad\qquad u(x) = v(ix)$$

and so we have $\phi(x) = u(x) - iu(ix)$.

$$|\phi(x)| = \sqrt{u(x)^2 + u(ix)^2} \geq |u(x)|$$

and so $||\phi|| \geq ||u||$. On the other hand for any $x$ we can write $|\phi(x)| = e^{i\theta}\phi(x) = \phi(e^{i\theta}x) = u(e^{i\theta}x) - iu(ie^{i\theta}x) = u(e^{i\theta}x)$ for some $\theta \in \mathbb{R}$. Since $|\phi(x)|$ is real we have

$$|\phi(x)| = u(e^{i\theta}x) \leq ||u||||x||$$

and so $||\phi|| \leq ||u||$.

Conversely, if we build $\phi$ as $\phi(x) = u(x) - iu(ix)$ then $\phi$ is linear and its real part is $u$ so $||\phi|| = ||u||$. *Q.E.D.*

**Theorem 2.16 (Complex Hahn-Banach)** *If $Y$ is a subspace of a complex Banach space $X$ and $\phi : Y \to \mathbb{C}$ is linear then there is an extension $\tilde{\phi}$ of $\phi$ with $||\tilde{\phi}|| \leq ||\phi||$.*

**Proof** Write $\phi(x) = u(x) - iu(ix)$ for a real linear functional $u$ on $Y$. Extend $u$ to $\tilde{u} : X \to \mathbb{R}$ with $||\tilde{u}|| \leq ||u||$ by the real Hahn-Banach theorem. Set $\tilde{\phi}(x) = \tilde{u}(x) - i\tilde{u}(ix)$ for $x \in X$. Then $\tilde{\phi}$ is complex linear and extends $\phi$. Furthermore

$$||\tilde{\phi}|| \leq ||\tilde{u}|| \leq ||u|| = ||\phi||$$

*Q.E.D.*

**Corollary 2.17** *If $X$ is a real Banach space then $X$ is reflexive if and only if $X^\star$ is reflexive.*

Without the Hahn-Banach theorem, we cannot prove the $\Longleftarrow$ direction.

**Proof** We first prove the $\Longrightarrow$ direction. We want to show that each element $F$ of $X^{\star\star\star}$ is $\hat{\phi}$ for some $\phi \in X^\star$. Define $\phi : X \to \mathbb{R}$ by $\phi(x) = F(\hat{x})$. If $x \in X$, then

$$\hat{\phi}(\hat{x}) := \hat{x}(\phi) := \phi(x) := F(\hat{x})$$

and so $\hat{\phi} = F$ on all elements of $X^{\star\star}$ of the form $\hat{x}$. If $X$ is reflexive, then this is all of $X^{\star\star}$.

Now the other direction. Suppose that $X$ is not reflexive. Then the image of $X$ under $\wedge$ map is a closed subspace of $X^{\star\star}$ since it is complete. Since we are assuming that it is not the whole of $X^{\star\star}$ we can find $F \in X^{\star\star\star}$ which is not 0 but with $F(\hat{x}) = 0$ for all $x \in X$. By assumption $F = \hat{\phi}$ for some $\phi \in X^\star$. But if $x \in X$ then

$$\phi(x) = \hat{x}(\phi) = \hat{\phi}(\hat{x}) = 0$$

This contradicts $\phi \neq 0$ *Q.E.D.*

**Corollary 2.18** *$l_1$ and $l_\infty$ are not reflexive.*

**Proof** $c_0$ is not reflexive, and $l_1 \cong c_0^\star$ and $l_\infty \cong l_1^\star$ *Q.E.D.*

## 2.3 Weak and Weak-$\star$ Topologies

**Definition 2.19** *Let $X$ be a Banach space. The* **weak topology** *on $X$ consists of all possible unions of sets of the form*

$$\{x \in X : \phi_1(x) < \alpha_1, ..., \phi_n(x) < \alpha_n\}$$

*for some finite sequence $\phi_1, ..., \phi_n \in X^\star$ and $\alpha_i \in \mathbb{R}$.*

*In the same way, the* **weak-$\star$ topology** *on $X^\star$ consists of unions of sets of the form*

$$\{\phi \in X^\star : \phi(x_1) < \alpha_1, ..., \phi(x_n) < \alpha_n\}$$

*for $x_1, ..., x_n \in X$ and $\alpha_i \in \mathbb{R}$.*

The sets of the above form can be called a " basis of neighbourhoods" for the open sets.

If $X$ is reflexive, then the weak and weak-$\star$ topologies on $X^\star$ are the same.

The weak topology on $X$ makes each $\phi \in X^\star$ continuous, because $\phi^{-1}(U)$ is a union of neighbourhoods if $U$ is open in $\mathbb{R}$.

$$U = \cup_\alpha(a_\alpha, b_\alpha) = \cup_\alpha(\{x_\alpha < b_\alpha\} \cap \{-x_\alpha < -a_\alpha\})$$

and thus

$$\phi^{-1}(U) = \cup_\alpha\{x : \phi(x) < b_\alpha, -\phi(x) < -a_\alpha\}$$

The weak topology is the weakest topology which makes each element of $X^\star$ continuous. If $X$ is infinite dimensional then the weak and weak-$\star$ topologies are not metrisable, so compactness is not the same as sequentially compactness.

**Theorem 2.20 (Banach-Alaoglu)** *The unit ball $B_{X^\star}$ is compact in the weak-$\star$ topology*

The proof uses Tychonov's theorem. If $(X_\alpha)$ are topological spaces then the product $\prod_\alpha X_\alpha$ is the Cartesian product $\{(x_\alpha) : x_\alpha \in X_\alpha \forall \alpha\}$ with the topology consisting of unions of sets $\{(x_\alpha) : x_{\alpha_1} \in U_{\alpha_1}, ..., x_{\alpha_n} \in U_{\alpha_n}\}$ where $U_{\alpha_i}$ is open in $X_{\alpha_i}$ for each $i$. Then Tychonov's theorem says the product of compact spaces is compact.

**Proof** We consider the product $\prod_{x \in B_X}[-1, 1]$ with the topology from $\mathbb{R}$.

We can embed $B_{X^\star}$ into this by $\phi \mapsto (\phi(x))$ so $\phi : B_X \to [-1, 1]$. This is a 1-1 embedding. If $\phi \neq \psi$ there is an $x \in B_X$ where $\phi(x) \neq \psi(x)$. The topology inherited by $B_{X^\star}$ from the product topology is exactly the weak-$\star$ topology. Neighbourhoods restrict only finitely many values $\phi(x_1), ..., \phi(x_n)$. The product is compact by Tychonov's theorem.

It suffices to prove that the image of $B_{X^\star}$ is closed. Suppose $f$ is in the closure of the image. We want that if $x, y \in B_X$, and $\lambda, \mu \in \mathbb{R}$ with $|\lambda| + |\mu| \leq 1$ then

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

as then this will imply that $f$ is the restriction to $B_X$ of some linear functional.

Given $\varepsilon > 0$, the functions $g$ satisfying

1. $|g(x) - f(x)| < \varepsilon$
2. $|g(y) - f(y)| < \varepsilon$
3. $|g(\lambda x + \mu y) - f(\lambda x + \mu y)| < \varepsilon$

form an open set in the product, since we are restricting three values of $g$. So this set meet sthe image of $B_{X^\star}$, say $\phi$ is a linear functional whose image satisfies the inequalities, hence

$$\begin{aligned}|f(\lambda x+\mu y)-\lambda f(x)-\mu f(y)| &= |g(x)-f(x)+g(y)-f(y)+g(\lambda x+\mu y)-f(\lambda x+\mu y)| \\ &< (1+|\mu|+|\lambda|)\varepsilon \\ &< 2\varepsilon\end{aligned}$$

and letting $\varepsilon \to 0$ we have $f(\lambda x+\mu y) = \lambda f(x)+\mu f(y)$ *Q.E.D.*

# 3 The Category Theorems

If $C_1 \supset C_2 \supset ...$ is a decreasing sequence of non empty closed subsets of a compact space then $\cap_1^\infty C_i \neq \emptyset$. This is not the case in $\mathbb{R}$, since $[0,\infty) \supset [1,\infty) \supset ..$ and $\cap_{k=1}^\infty [k,\infty) = \emptyset$.

**Theorem 3.1 (Cantor)** *If $C_1 \supset C_2 \supset ...$ is a decreasing sequence of non empty closed subsets of a complete metric space $X$ and $\mathrm{diam}(C_n) \to \infty$ as $n \to 0$ then $\cap_1^\infty C_n \neq \emptyset$*

**Proof** For each $n$, choose $x_n \in C_n$. Since $x_n \in C_m$ if $n \geq m$ then $d(x_n, x_m) \leq \mathrm{diam}(C_m) \to 0$ and so the sequence $(x_n)$ is Cauchy, and let $x$ be the limit. Since $x_n \in C_m$ for all $n \geq m$ and $C_m$ is closed then $x \in C_m$ for all $m$ and so $x \in \cap C_m$. *Q.E.D.*

**Definition 3.2** *A subset $E$ of a metric space $X$ is called* **nowhere dense** *if its closure contains no non-empty open subset of $X$.*

**Theorem 3.3 (Baire)** *A complete metric space $X$ is not the countable union of nowhere dense subsets of itself.*

**Corollary 3.4** $\mathbb{R}$ *is not countable.*

This is since it is the union of all singletons $\{x\}$.
**Proof** Let $E_1, E_2, ...$ be nowhere dense in $X$. We want to find $x \notin \cup E_i$. We can assume, by taking closures, that the $E_i$s are closed and each contains no non empty open set.

Choose $x_1 \notin E_1$. There is a ball centred at $x_1$ which does not meet $E_1$, and hence a closed ball $\overline{B}(x_1, r_1) \cap E_1 = \emptyset$. We may assume that $r_1 < 1$.

The open ball $B(x_1, r_1)$ is not contained in $E_2$ since $E_2$ is nowhere dense. So choose $x_2 \in B(x_1, r_1)$ but $x_2 \notin E_2$. Since $E_2$ is closed there is a ball $\overline{B}(x_2, r_2)$ which is contained in $B(x_1, r_1)$ with $\overline{B}(X_2, r_2) \cap E_2 = \emptyset$. We may assume that $r_2 < \frac{1}{2}$.

Continuing in this way we build $\overline{B}(x_1, r_1) \supset B(x_1, r_1) \supset \overline{B}(x_2, r_2) \supset B(x_2, r_2) \supset ...$ with $r_i < \frac{1}{i}$ and $B(x_i, r_i) \cap E_i = \emptyset$.

By Cantor's theorem, there is a point in all the $\overline{B}(x_i, r_i)$ and hence in none of the $E_i$s. *Q.E.D.*

We shall use this to prove the "analogue" of the 1-1 implies onto theorem for a finite dimension vector space. This is the Open mapping theorem.

Baire's theorem allows you to construct nasty things. There are too few nice things so there must be some nasty ones. At the end of the section we shall show that there is a continuous $2\pi$ periodic function on $[-\pi, \pi]$ whose Fourier series diverges at 0. This uses the Uniform Boundedness Principle.

**Theorem 3.5 (UBP)** *Suppose $\phi_1, \phi_2, \ldots$ are bounded linear functionals on a Banach space $X$. If for each $x \in X$ the values $\{\phi_n(x)\}$ form a bounded set, then there is some $M$ with $||\phi_n|| \leq M$ for all $n$.*

This at first seems strange, if not simply wrong. Take for example the following.

Suppose that $X = l_1$. Then take $\phi_1$ big in direction $e_1$, and zero in the rest, $\phi_2$ huge in $e_2$ and zero in the rest, etc. Then at $e_7$ $\phi_7$ is gigantic, but everything else is zero. However the functionals are not uniformly bounded. This is not a contradiction as we can find some clever point $x$ which picks up some bad behaviour from lots of the $\phi$ s.

Suppose that $\phi_i : l_1 \to \mathbb{R}$ is the coordinate functional $\phi_i(x) = x_i$. Then if $x \in l_1$, we have $\sum |\phi_i(x)| = ||x||_1 < \infty$ which is essentially the opposite to the above. Thus clever means very clever.

This example is as bad as it gets.

**Theorem 3.6 (Sharp UBP/Plank)** *If $(\phi_i)$ are unit functionals (operator norm 1) and $(w_i)$ are positive numbers with $\sum w_i < 1$ then there is a unit vector $x \in X$ with $|\phi_n(x)| > w_n$ for all $n$.*

**Proof (Plank $\implies$ UBP)** If $(||\phi_n||)$ is unbounded we can choose a subsequence $\phi_{n_k}$ with $||\phi_{n_k}|| \geq 4^k$. Let $\psi_k = \phi_{n_k}/||\phi_{n_k}||$ which is a unit functional. Choose $x$ with $|\psi_k(x)| \geq \frac{1}{2^{k+1}}$ and $||x|| \geq 1$. Then $|\phi_{n_k}(x)| \geq 2^{k-1} \to \infty$. *Q.E.D.*

**Proof (UBP)** Suppose $(\phi_n(x))$ is bounded for each $x \in X$. For each $k \in \mathbb{N}$, let $C_k = \{x \in X : |\phi_n(x)| \leq k, \forall n\}$. These sets are closed and they cover $X$. Thus by Baire's theorem at least one is not nowhere dense, and hence includes a ball of positive radius. In fact $C_k = kC_1$, so $C_1$ includes such a ball, say the ball of radius $r > 0$ around $y$.

$C_1$ is convex and symmetric so if $u \in C_1$ then $-u \in C_1$. If $||x|| \leq r$ we have $y + x$ and $-y + x$ in $C_1$ and so the average $x = \frac{1}{2}(y + x) + \frac{1}{2}(-y + x) \in C_1$ by convexity. Thus if $||x|| \leq r$ then $|\phi_n(x)| \leq 1$ for all $n$. Hence $||\phi_n|| \leq \frac{1}{r}$ for all $n$. *Q.E.D.*

## 3.1 Open Mapping Theorem

Recall that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear then it is one to one if and only if it is onto.

**Theorem 3.7 (Open mapping)** *If $T : X \to Y$ is a linear map from a Banach space $X$ to a Banach space $Y$ which is onto then $T$ is an open map, i.e. $T(B_X) \supset rB_Y$ for some $r > 0$.*

**Corollary 3.8 (Inverse mapping theorem)** *If $X$ and $Y$ are Banach spaces and the linear map $T : X \to Y$ is one to one and onto and bounded then $T^{-1}$ is bounded.*

This looks like a cheat since since we assume one to one and onto, but it isnt because we conclude that $T$ is invertible in the sense that it has a bounded inverse (in the sense of analysis).

The proof of the OMT has two steps. The first uses Baire and completeness of $Y$ and the second uses an iterative argument and completeness of $X$. They are linked by a clever remark.

**Proof (OMT theorem 3.7)** We shall start by showing that for some positive $r$, the closure $\overline{T(B_\alpha)}$ includes $rB_Y$.

For each $k$ let $C_k = \overline{T(kB_X)}$ for $k = 1, 2, \ldots$ These closed sets cover $Y$ since $T$ is onto and hence by Baire's theorem at least one has non empty interior. Since $C_k = kC_1$ in fact $C_1$ has non empty interior. Suppose $C_1$ includes the ball of radius $r > 0$ around $y$. Since $C_1$ is convex and symmetric, it includes the ball of radius $r$ around 0, $rB_Y$. Thus $\overline{T(B_\alpha)} \supset rB_Y$.

In particular, if $||y|| \leq r$ there is an $x \in X$ with $||x|| \leq 1$ so that

$$||y - Tx|| \leq \frac{r}{2} \tag{3.1}$$

We shall now prove that $T(2B_X) \supset rB_Y$, and so $T(B_X) \supset \frac{r}{2}B_Y$.

Suppose that $y \in Y$ and $||y|| \leq r$. Choose $x_1 \in B_X$ with $||y - Tx_1|| \leq \frac{r}{2}$ by (3.1). Now by applying (3.1) to the vector $y - Tx_1$ we can find $x_2$ with norm $||x_2|| \leq \frac{1}{2}$ with

$$||y - Tx_1 - Tx_2|| \leq \frac{r}{4}$$

Continuing we obtain $(x_j)$ with $||x_i|| \leq \frac{1}{2^{i-1}}$ and $||y - T(x_1 + ... + x_i)|| \leq \frac{r}{2^i}$. Since $X$ is complete, the sum $\sum_1^\infty x_j$ converges to x and $||y - Tx|| = \lim ||y - T(\sum_1^i x_j)|| = 0$ so $y = Tx$ and $||x|| \leq \sum_1^\infty ||x_j|| \leq \sum_1^\infty \frac{1}{2^{i-1}} = 2$ *Q.E.D.*

We could try to find $u_i$ such that $||y - Tu_i|| \to 0$. If the $u_i$ have a convergent subsequence then its limit is in $B_X$ and has image $Y$. This though requires compactness.
**Proof (Inverse mapping theorem 3.8)** By the OMT, we know that if $||y|| \leq r$ there is an $x$ with $||x|| \leq 1$ and $y = Tx$ for some $r > 0$. Thus $||T^{-1}y|| = ||x|| \leq 1$ and so $||T^{-1}|| \leq \frac{1}{r}$. *Q.E.D.*

## 3.2 Closed Graph Theorem

If $T : X \to Y$ is a linear map, its **graph** is the set $\{(x, Tx) : x \in X\} \subset X \times Y$.

If $X$ and $Y$ are Banach spaces then $X \times Y$ is a Banach space with

$$||(x, y)|| = ||x|| + ||y||$$

but note that this isn't the only norm we can use. We can thus ask whether the graph is closed in $X \times Y$. The answer is yes if whenever $x_n \to x$ and $Tx_n \to y$ then $y = Tx$.

If $T$ is continuous, the graph is closed since in this case $x_n \to x \implies Tx_n \to Tx$, even if we do not know that $Tx_n$ converges.

**Theorem 3.9 (Closed Graph)** *If $T$ is a linear map between Banach spaces $X$ and $Y$ then $T$ is bounded if and only if its graph is closed in $X \times Y$.*

**Proof** Let $G = \{(x, Tx) : x \in X\}$ be the graph of $T$. It is a closed subspace of $X \times Y$ so is a Banach space in the inherited norm. Define

$$\pi_1 : G \to X \qquad \text{by } \pi_1(x, Tx) = x$$

$$\pi_2 : X \times Y \to Y \qquad \text{by } \pi_2(x, y) = y$$

These are obviously linear. $\pi_2$ is continuous because

$$||y|| \leq ||x|| + ||y|| = ||(x, y)||$$

$\pi_1$ is continuous because

$$||x|| \leq ||x|| + ||Tx|| = ||(x, Tx)||$$

$\pi_1$ is one to one and onto, since if $u = x$ we have $(u, Tu) = (x, Tx)$. By the IMT (or OMT), $\pi_1$ has a bounded inverse. Since $T = \pi_2 \circ \pi_1^{-1}$, it is bounded. *Q.E.D.*

### 3.3 Application to Fourier Analysis

There is a continuous $2\pi$ periodic function on $[-\pi, \pi]$ whose Fourier series diverges at 0.

The Fourier series is $\sum_{-\infty}^{\infty} a_n e^{int}$ where

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

The $N$th partial sum is

$$\sum_{-N}^{N} a_n e^{int} = \frac{1}{2\pi}\sum_{-N}^{N} e^{int}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sum_{-N}^{N} e^{in(t-x)}dx$$

The sum inside is

$$\begin{aligned}\sum_{-N}^{N} e^{in(t-x)} &= \frac{e^{-N(t-x)} - e^{i(N+1)(t-x)}}{1 - e^{i(t-x)}} \\ &= \frac{e^{-(N+\frac{1}{2})(t-x)} - e^{i(N+\frac{1}{2})(t-x)}}{e^{-i\frac{(t-x)}{2}} - e^{i\frac{(t-x)}{2}}} \\ &= \frac{\sin((N+\frac{1}{2})(t-x))}{\sin(\frac{1}{2}(t-x))}\end{aligned}$$

and thus

$$\sum_{-N}^{N} a_n e^{int} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin((N+\frac{1}{2})(t-x))}{\sin(\frac{1}{2}(t-x))}dx$$

In particular, the value at $t = 0$ is

$$\sum_{-N}^{N} a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin((N+\frac{1}{2})(x))}{\sin(\frac{1}{2}(x))}dx$$

The bit on the left hand side is supposed to converge to $f(0)$, and so you would expect the graph of the kernel $D_N$ to spike at 0. We would like this Fourier series to approximate $f$, meaning that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_N(x)dx \approx f(0)$$

We would like the kernel to reproduce $f$. Thus the kernel should have integral 1 with its mass concentrated near $t$, it is an "approximation to the identity"

The map $f \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_N(x)dx = S_N(f)$ is a bounded linear functional on the space of continuous $2\pi$ periodic functions, with norm $||f|| = \max_{x\in[-\pi,\pi]}|f(x)|$. By UBP, we can deduce that there is an $f$ where $(S_N(f))$ is unbounded if $(||S_N||)$ is unbounded.

We claim that

$$||S_N|| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}\right|dx = ||D_N||_1$$

$S_N$ acts on $L_\infty[-\pi, \pi]$ by the same integral formula and its norm is at most $||D_N||_1$. On $L_\infty$ the norm clearly is $||D_N||_1$ because we can apply it to $\text{sgn}(D_N)$ in $L_\infty$ and $||\text{sgn}(D_N)||_\infty = 1$ but $S_N(\text{sgn}(D_N)) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\text{sgn}(D_N))D_N = ||D_N||_1$. However $\text{sgn}(D_N)$

is not continuous but we can approximate by continuous functions of $L_\infty$ norm 1 and which are $2\pi$ periodic.

Now $D_N$ vanishes at $\pm\frac{\pi}{N+\frac{1}{2}}, \pm\frac{2\pi}{N+\frac{1}{2}}, ..., \pm\frac{N\pi}{N+\frac{1}{2}}$ and

$$\begin{aligned}||D_N||_1 = \frac{1}{\pi}\int_0^\pi \left|\frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}\right| dx &\geq \frac{1}{\pi}\sum_{k=1}^{N} \int\limits_{\frac{(k-1)\pi}{N+\frac{1}{2}}}^{\frac{k\pi}{N+\frac{1}{2}}} \left|\frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}\right| dx \\ &\geq \frac{1}{\pi}\sum_{k=1}^{N} \int\limits_{\frac{(k-1)\pi}{N+\frac{1}{2}}}^{\frac{k\pi}{N+\frac{1}{2}}} \frac{\left|\sin(N+\frac{1}{2})x\right|}{\left|\sin\frac{k\pi}{2N+1}\right|} dx\end{aligned}$$

because $x \mapsto \sin x$ is increasing on $[0,\pi]$. Now if $x > 0$ we have $\sin x \leq x$ so for each $k$, $\frac{1}{\sin\frac{k\pi}{2(N+1)}} \leq \frac{2(N+1)}{k\pi}$ and so we get that

$$\begin{aligned}||D_N||_1 &\geq \frac{2(N+1)}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} \int\limits_{\frac{(k-1)\pi}{N+\frac{1}{2}}}^{\frac{k\pi}{N+\frac{1}{2}}} \left|\sin(N+\frac{1}{2})x\right| dx \\ &= \frac{2(N+1)}{\pi^2}\sum_{k=1}^{N}\frac{1}{k}\int_0^{\frac{\pi}{N+\frac{1}{2}}} \sin(N+\frac{1}{2})x dx \\ &= \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} \\ &> \frac{4}{\pi^2}\log N \to \infty\end{aligned}$$

and so we have the result that we want

# 4 Unbounded Operators

We motivate the following with a classical example. We have a taut string fixed at 0 and 1 and at time $t = 0$ it is plucked into shape $x \mapsto f(x,0)$. Thereafter its shape is $f(x,t)$. Then $f$ obeys the wave equation

$$c^2\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}$$

with the boundary conditions $f(0,t) = 0 = f(1,t)$. We now expand $f(x,t)$ as a Fourier series to get it as

$$f(x,t) = \sum_1^\infty a_n(t)\sin(n\pi x)$$

Note that $\sin n\pi x$ vanishes at 0 and 1 so we immediately satisfy the boundary conditions, and this is also the reason why we do not consider the cosine terms. By extending to an odd function on $[-1,1]$ we have an expression in $L_2$. If we can differentiate term by term then

$$\frac{\partial^2 f}{\partial t^2} = \sum_1^\infty a_n''(t)\sin(n\pi x) \qquad\qquad \frac{\partial^2 f}{\partial x^2} = -\sum_1^\infty a_n(t)n^2\pi^2\sin(n\pi x)$$

By the wave equation we have

$$a_n''(t) = -c^2 n^2 \pi^2 a_n(t)$$

and so $a_n(t) = A_n \cos(cn\pi t) + B_n \sin(cn\pi t)$. At $t = 0$ we have $f(x,0) = \sum_1^\infty C_n \sin(n\pi x)$ and if we furthermore assume that $\frac{\partial f}{\partial t}|_{t=0} = 0$, i.e. we have a slope but no speed, then we get that $B_n = 0$ for all $n$ and thus

$$a_n(t) = A_n \cos(cn\pi t)$$

and $A_n = C_n$ and so

$$f(x,t) = \sum_1^\infty A_n \cos(cn\pi t)\sin(n\pi x)$$

The method above worked because the functions $\sin n\pi x$ are eigenfunctions of the map $f \mapsto f''$ satisfying the boundary conditions $f(0) = 0 = f(1)$ and are orthogonal in $L_2[0,1]$. This is what we expect from a **self adjoint** operator. The Laplacian $f \mapsto f''$ is not a bounded operator on any useful space, so we need a theory of unbounded operators. The Laplacian is “ like ” a self adjoint operator though, since using integration by parts twice we get:

$$\begin{aligned}\langle f'', g\rangle &= \int_0^1 f''(x)g(x)dx \\ &= f'(x)g(x)|_0^1 - \int_0^1 f'(x)g'(x)dx \\ &= f'(x)g(x)|_0^1 - f(x)g'(x)|_0^1 + \int_0^1 f(x)g''(x)dx\end{aligned}$$

so if $f$ and $g$ are $C^2$ and $f(0) = 0 = f(1)$ and $g(0) = 0 = g(1)$ then we get $\langle f'', g\rangle = \langle f, g\rangle$. So $f''$ looks self adjoint on smooth functions satisfying the Dirichlet boundary conditions. The boundary conditions are thus part of the definition of the operator. Dirichlet and Neumann boundary conditions make it “like” a self adjoint operator.

The apparent self adjointness is not enough though for spectral theory.

**Definition 4.1** *A linear operator $T$ is called* **densely defined** *in the Hilbert space $H$ if it is defined and linear on a dense subspace $T : D(T) \to H$.*

Note that $f''$ is defined and linear on $C^2$ which is dense in $L_2$

**Definition 4.2** *We say that $S$* **extends** *$T$ if $D(S) \supset D(T)$ and $S(x) = T(x)$ for $x \in D(T)$.*

**Definition 4.3** *We say that $T$ is* **symmetric** *if*

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

*if $x, y \in D(T)$.*

**Definition 4.4** *We define the* **adjoint** *of a densely defined $T$ as follows. We define $D(T^\star)$ to be all $y$ for which $x \mapsto \langle Tx, y\rangle$ is a bounded linear functional on $D(T)$. In this case the functional extends uniquely to $H$. It can be represented uniquely by an inner product $x \mapsto \langle x, z\rangle$ for some $z \in H$. We call this $z$ by $T^\star y$, so*

$$\langle Tx, y\rangle = \langle x, T^\star y\rangle$$

*if $x \in D(T)$ and $y \in D(T^\star)$.*

**Definition 4.5** *$T$ is called* **self adjoint** *if $D(T) = D(T^\star)$ and $T = T^\star$*

Note that $T$ is symmetric is the same as saying that $T^\star$ extends $T$.

Warning though that $D(T^\star)$ might not even be dense.

The map $f \mapsto f''$ on $C^2[0,1]$ with $f(0) = 0 = f(1)$ is not self adjoint, but it is symmetric. There are functions $g$ for which $\int_0^1 f''g = \int_0^1 fg''$ but with $g$ not in the given domain, because $g''$ is not continuous.

The integration by parts repeated works fine if $f(0) = 0 = f(1) = g(1) = g(0)$ and $f'$ and $g'$ are absolutely continuous, meaning

$$f'(x) = \int_0^x u(t)dt \qquad\qquad g'(x) = \int_0^x v(t)dt$$

and then $\int f'g'$ makes sense by Hölder and $f'' = u$ and $g'' = v$. To find the correct domain on which to study $f''$ we set $H^1[0,1]$ to be the space of absolutely continuous functions $f(x) = c + \int_0^x u(t)dt$ with $u \in L_2[0,1]$ and $H^2[0,1]$ to be the space of differentiable functions $f$ on $[0,1]$ with $f'(x) \in H^1$.

$f''$ may not exist everywhere but $f(x) = a + bx + \int_0^x \int_0^t u(s)dsdt$ and $u \in L_2$. Let $D$ be the space of functions

$$D = \{f \in H^2[0,1] : f(0) = 0 = f(1)\}$$

**Theorem 4.6** *The map $f \mapsto f''$ is self adjoint on $D$.*

**Proof** We need to show that if the map $f \mapsto \int f''g$ is a bounded linear functional on $D$ then $g \in D$ and $g(x) = a + bx + \int_0^x \int_0^t u(s)dsdt$ and $\int f''g = \int fu$, as then we would have $D(T) = D(T^\star)$ and $\langle Tx, y\rangle = \langle x, T^\star y\rangle$ as required.

If the map $f \mapsto \int f''g$ is a bounded linear functional then we can write it as $\int fh$ for some $h \in L_2$ by Riesz-Fréchet. Then let $G(x) = \int_0^x \int_0^t h(s)dsdt$. Then $G \in H^2$ and $G''(x) = h$.

We want to check that $G$ is almost $g$. Then

$$\begin{aligned}
\int_0^1 f''(x)G(x)dx &= f'(x)G(x)|_0^1 - \int_0^1 f'(x)G'(x)dx \\
&= f'(x)G(x)|_0^1 - f(x)G'(x)|_0^1 + \int_0^1 f(x)G''(x)dx \\
&= f'(x)G(x)|_0^1 + \int_0^1 f(x)G''(x)dx
\end{aligned}$$

Recall that $\int f''g = \int fh$ but also $G'' = h$ so we have

$$\int_0^1 f''(x)G(x)dx = f'(x)G(x)|_0^1 + \int_0^1 f(x)h(x)dx = f'(x)G(x)|_0^1 + \int_0^1 f''g$$

We apply to $f$ which not only satisfies $f(0) = 0 = f(1)$ but to those which satisfy as well $f'(0) = 0 = f'(1)$.

Thus $G - g$ is orthogonal to all $f'' = u \in L_2$ for functions $f \in D$ satisfying also $f'(0) = 0 = f'(1)$.

Since $f'(x) = \int_0^x u(t)dt$ the condition $f'(1) = 0$ is equivalent to the statement

$$\int_0^1 u(t)dt = 0$$

and since $f(x) = \int_0^x f'(t)dt$ so we have

$$0 = \int_0^1 f'(t)dt = xf'|_0^1 - \int_0^1 xf''(x)dx = -\int_0^1 xf''(x)dx = -\int_0^1 xu(x)dx$$

and so $G - g$ is orthogonal to all elements of $L_2$ satisfying

$$\int_0^1 u = 0 \qquad\qquad \int_0^1 xu(x)dx = 0$$

thus $G - g$ is orthogonal to all $u \in L_2$ which are themselves orthogonal to linear functions. Thus $G - g$ must be linear. This means

$$g(x) = a + bx + G(x) \in H^2$$

and $g'' = G'' = h$. This means $\int_0^1 f''g = \int_0^1 fh = \int_0^1 fg''$ but also, integrating by parts twice gives

$$\int_0^1 f''g = f'(x)g(x)|_0^1 + \int_0^1 f(x)g''(x)dx$$

and so in fact $f'(x)g(x)|_0^1 = 0$ for all $f \in D$. We can find $f \in D$ with arbitrary values of $f'(0), f'(1)$ and so we must have $g(0) = 0 = g(1)$.

We have proved that the domain of the adjoint is not too big. Conversely, if $f, g \in D$ then integration by parts shows

$$\int_0^1 f''g = \int_0^1 fg''$$

*Q.E.D.*

## 4.1 Closed Operators

**Definition 4.7** *A densely defined operator $T$ in a Hilbert space $H$ with domain $D(T)$ is called* **closed** *if its graph is closed, i.e. if $x_n \in D(T)$ and $x_n \to x$ and $T(x_n) \to y$ in $H$ then $x \in D(T)$ and $T(x) = y$.*

The closed graph theorem can thus be reworded as if $T$ is closed and $D(T) = H$ then $T$ is bounded. In other words, closed is almost bounded, if you are not defined on the whole space.

**Example 4.1** *Suppose $H = l_2$ and define $T : C_{00} \to l_2$ by*

$$T(x_1, x_2, ..., x_n, 0, 0, ...) = (x_1, 2x_2, ..., nx_n, 0, ...)$$

*where $C_{00}$ is the set of all finitely non zero sequences. This operator is not closed. Let*

$$u_1 = (1, 0, ...), \qquad u_2 = (1, 1/4, 0, ...), ... \qquad u_n = (1, 1/4, 1/9, ..., 1/n^2, 0, ...), ...$$

*and then $Tu_1 = u_1$, $Tu_n = (1, 1/2, ..., 1/n, 0, ...)$ and $u_n \to (1, 1/4, 1/9, ...) = x$ and $Tu_n \to (1, 1/2, 1/3, ...) = y$ but $x \notin C_{00} = D(T)$.*

*However, we can extend this operator to be closed. Choose*

$$D = \{(x_i)_1^\infty : \sum n^2 x_n^2 < \infty\}$$

*and observe that* $\sum n^2 x_n^2 < \infty$ *enforces* $Tx \in l_2$*. You might call this the natural domain. Then* $T$ *is defined on* $D$ *and we claim that it is closed.*

*Suppose* $u_i \to x$ *and* $Tu_i \to y = (y_1, y_2, ...)$ *in* $l_2$*. The map*

$$(\theta_1, \theta_2, ...) \mapsto (\theta_1, \theta_2/2, \theta_3/3, ...)$$

*is bounded on* $l_2$*, and we denote it by* $S$*. Then if* $u \in D$ *we have* $S(Tu) = u$*. So*

$$u_i = S(Tu_i) \to S(y)$$

*since* $S$ *is bounded and so* $x = Sy$ *and also*

$$\sum n^2 x_n^2 = \sum n^2 \left(\frac{y_n}{n}\right)^2 = \sum y_n^2 < \infty$$

*and so* $x \in D$ *and so* $Tx = y$ *since* $x_i = \frac{1}{i} y_i$*.*

In other words what we have done in the above is to define the domain of definition so that the "inverse" $S$ works as it should.

Consider the map $f \mapsto f'$ in $L_2[0,1]$, initially defined on $C^1[0,1]$ the space of continuously differentiable functions. The operator is not symmetric on this space, because of the change of sign when you perform integration by parts.

However, $f \mapsto if'$ is symmetric with the right boundary conditions.

The map is not closed on $C^1$; we find $f_n \to f$ in $L_2$ with $f_n' \to g$ in $L_2$ but $f \notin C^1$.

Choose $f$ to be the function 0 at 0 and 1, to be 1/2 at 1/2 and piecewise linear. Then choose $g$ to be the function 1 up to 1/2 and -1 from 1/2 to 1, as in the pictures (YET TO DRAW).

Then choose continuous approximations $g_n$ converging to $g$ in $L_2$, for example

$$g_n(x) = \begin{cases} 1 & 0 \le x \le 1/2 - 1/n \\ \text{linear} & 1/2 - 1/n \le x \le 1/2 + 1/n \\ -1 & 1/2 + 1/n \le x \le 1 \end{cases}$$

and so $g_n$ are indeed continuous and $g_n \to g$ in $L_2$. Set $f_n(x) = \int_0^x g_n(t)dt$ and so $f_n \in C^1$ and $f_n' = g_n$ by construction. Note that $f \notin C^1$. It is easy to see that $f_n \to f$ in $L_2$ because the map which takes $u$ to the function $x \mapsto \int_0^x u$ is a bounded linear operator on $L_2$. Thus since $f_n' \to g$ in $L_2$ we have $f_n \to \int g$ in $L_2$ and $\int g = f$.

We now extend the domain so that the function is closed.

**Theorem 4.8** *The map* $f \mapsto f'$ *is closed on the domain*

$$\{f \in H^1[0,1] : f(0) = f(1)\}$$

*which is called periodic* $H^1$

Note that this is the natural space for $f'$.

**Proof** Suppose $f_n$ is in periodic $H^1$ and $f_n \to f$ in $L_2$ and $f_n' \to g$ in $L_2$. Let $G(x) = \int_0^x g(t)dt$ and note that $G \in H^1$ and $G' = g$. We now want to show that $G$ is almost $f$.

The map that takes $u$ to $x \mapsto \int_0^x u(t)dt$ is bounded on $L_2$ and so

$$\int_0^x f_n'(t)dt \to \int_0^x g(t)dt = G(x)$$

in $L_2$ and also

$$f_n(x) = f_n(0) + \int_0^x f_n'(t)dt$$

and these converge to $f$. Thus $f_n(0) \to f - G$ in $L_2$ and so $f - G$ is constant. We can thus write

$$f(x) = c + G(x)$$

and so $f \in H^1$ and $f' = G' = g$. Also note that $0 = f_n(1) - f_n(0) = \int_0^1 f_n'$ for all $n$ and since $f_n' \to g$ in $L_2$ then $\int_0^1 g = 0$ and so $f(1) - f(0) = \int_0^1 f' = \int_0^1 g = 0$ and so $f$ is in periodic $H^1$ *Q.E.D.*

For $if'$, we have

$$\int_0^1 if'\bar{g} = if\bar{g}|_0^1 - \int_0^1 if\bar{g}' = \int_0^1 f\bar{ig}'$$

We now show that the integral as an operator is bounded. We are showing

$$u \mapsto \left(x \mapsto \int_0^x u(t)dt\right)$$

is bounded on $L_2$, i.e.

$$\int_0^1 \left(\int_0^x u(t)dt\right)^2 dx \leq K \int_0^1 u(t)^2 dt$$

Now

$$\begin{aligned} \int_0^1 \left(\int_0^x u(t)dt\right)^2 dx &= \int_0^1 \left(\int_0^1 u(t)\chi_{[0,x]}dt\right)^2 dx \\ &\leq \int_0^1 \left(\int_0^1 u^2\right)\left(\int_0^1 \chi_{[0,x]}\right) dx \\ &= \int_0^1 u^2 \int_0^1 x dx \\ &= \frac{1}{2}\int_0^1 u^2 \end{aligned}$$

**Theorem 4.9** *If $T$ is densely defined in $H$ then*

1. *$T^\star$ is closed*
2. *If $D(T^\star)$ is dense then $T$ has a closed extension, $T^{\star\star}$.*
3. *If $T$ is symmetric then it has closed extension $T^\star$.*

In particular, if $T$ is self adjoint then it is closed.

**Proof**

1. Suppose $x_n \to x$ in $H$ and $T^\star x_n \to y$ and $x_n \in D(T^\star)$. The map $u \mapsto \langle Tu, x_n\rangle$ is a bounded linear functional for each $n$. Then

   $$\langle Tu, x_n\rangle \to \langle Tu, x\rangle$$

   since $x_n \to x$ and also

   $$\langle Tu, x_n\rangle = \langle u, T^\star x_n\rangle \to \langle u, y\rangle$$

   since $T^\star x_n \to y$. Then $\langle Tu, x\rangle = \langle u, y\rangle$ and because the map $u \mapsto \langle u, y\rangle$ is a bounded linear functional we conclude that $x \in D(T^\star)$ and $T^\star x = y$.

2. Since $D(T^\star)$ is dense, $T^{\star\star}$ does exist. By 1 it is closed, but does it extend $T$.

   Suppose $x \in D(T)$. The map $u \mapsto \langle T^\star u, x\rangle = \langle u, Tx\rangle$ is defined on $D(T^\star)$ and is a bounded linear functional since $Tx \in H$. So $x \in D(T^{\star\star})$ and $\langle u, Tx\rangle = \langle T^\star u, x\rangle = \langle u, T^{\star\star}x\rangle$ for all $u \in D(T^\star)$. Since this is dense, $T^{\star\star}x = Tx$.

3. This is immediate from 1, since $T$ symmetric means $T^\star$ extends $T$, and $T^\star$ is closed.

*Q.E.D.*

## 4.2 The Spectrum

**Definition 4.10** *For bounded operator $T : H \to H$ on a Hilbert space, we define*

1. *The **resolvent** consists of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is invertible.*
2. *The **point spectrum** consists of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is not injective*
3. *The **continuous spectrum** consists of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is one to one, is not onto but for which $\mathrm{Im}(T - \lambda I)$ is dense.*
4. *The **residual spectrum** consists of $\lambda \in \mathbb{C}$ with $T - \lambda I$ being one to one but $\mathrm{Im}(T - \lambda I)$ is not dense.*

**Definition 4.11** *For an operator $T : H \to H$ (not necessarily bounded) on a Hilbert space, we define*

1. *The **resolvent** consists of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is one to one and $\mathrm{Im}(T - \lambda I)$ is dense in $H$ and $(T - \lambda I)^{-1}$ is bounded on the image of $D(T)$.*
2. *The **point spectrum** consists of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is not one to one on $D(T)$.*
3. *The **continuous spectrum** consists of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is one to one, $\mathrm{Im}(T - \lambda I)$ is dense but $(T - \lambda I)^{-1}$ is not bounded on $\mathrm{Im}(T - \lambda I)$.*
4. *The **residual spectrum** consists of $\lambda \in \mathbb{C}$ with $T - \lambda I$ is one to one but $\mathrm{Im}(T - \lambda I)$ is not dense.*

**Theorem 4.12** *If $T$ is a self adjoint operator in a Hilbert space then its residual spectrum is empty.*

**Proof** Suppose $\mathrm{Im}(T - \lambda I)$ is not dense. Lets use $S$ to denote $T - \lambda I$. There is a non-zero $y \in H$ with $\langle Sx, y\rangle = 0$ for all $x \in D(T) = D(S)$. So the map $x \mapsto \langle Sx, y\rangle$ is a bounded linear functional, and hence $y \in D(S^\star) = D(T^\star) = D(T)$ as $T$ is self adjoint. So $0 = \langle x, Sy\rangle$ for all $x \in D(T)$ and so $Sy = 0$. Thus $S = T - \lambda I$ is not one to one. *Q.E.D.*

Note that if $T$ is bounded then the two definitions agree. It is enough to check the resolvent since 2 and 4 are identical. We need to check that if $S$ is bounded and densely defined then its extension to $H$ is invertible if and only if $S$ is one to one, $\mathrm{Im}(S)$ is dense and $S^{-1}$ is bounded on $\mathrm{Im}(S)$.

If the extension is invertible then clearly $S$ is one to one and $S^{-1}$ is bounded on $\mathrm{Im}(S)$. Initially $S$ is defined on a dense subspace $D(S)$. If $y \in H$ the extension $\tilde{S}$ is onto so there is an $x$ with $\tilde{S}x = y$. Choose $x_n \in D(S)$ with $x_n \to x$. Then $\tilde{S}x_n \to \tilde{S}x = y$ because $\tilde{S}$ is bounded, but $\tilde{S}x_n = Sx_n$ so $y$ can be approximated from $\mathrm{Im}(S)$ so $\mathrm{Im}(S)$ is dense.

In the other direction, $S^{-1}$ has a continuous extension, call it $T$. We shall show that $\tilde{S} \circ T$ and $T \circ \tilde{S}$ are the identity so $\tilde{S}$ is invertible. Suppose $y \in H$ and choose $y_n \in \mathrm{Im}(S)$ with $y_n \to y$. Then $S^{-1}y_n \to Ty$ so $S(S^{-1}y_n) = y_n \to \tilde{S}(Ty)$ so $y = \tilde{S}(Ty)$ so $4\tilde{S} \circ T$ is the identity. Similarly $T \circ \tilde{S}$ is the identity.

# 5 The Laplacian and Quantum Harmonic Oscillator.

We have already seen that the map $f \mapsto f''$ is self-adjoint on Dirichlet $H^2[0,1]$. Thus it has no residual spectrum. We also observed that there is a complete orthonormal basis of $L_2[0,1]$ consisting of the eigenfunctions $x \mapsto \sin n\pi x$ with eigenvalue $-n^2\pi^2$.

We would like an analogue of the spectral theorem

1. The spectrum consists only of the eigenvalues (the operator has purely point spectrum).
2. The corresponding eigenvectors form a complete orthonormal basis.
3. The map $f \mapsto f''$ on $H^2$ can be expressed as follows. If $f'' = g = \sum \theta_n \sin n\pi x$ then $f(x) = \sum \frac{\theta_n}{-n^2\pi^2} \sin n\ pix$.

In other words the operator multiplies the coefficient by the eigenvalue $-n^2\pi^2$ or for each $f \in$ Dirichlet $H^2$, $f(x) = \sum \gamma_n \sin n\pi x$ where $\sum n^2\pi^2\gamma_n^2 < \infty$ and $f''(x) = -\sum n^2\pi^2\gamma_n \sin n\pi x$.

This is also true for the Laplacian on a nice domain $\Omega \subset \mathbb{R}^2$ but we shall only prove it in one dimension, by directly writing down the resolvent.

We need that if $\lambda \neq -n^2\pi^2$ then $(T - \lambda I)^{-1}$ is bounded on $\mathrm{Im}(T)$, i.e. everything that is one to one is in the resolvent, and so not in the spectrum. We know that if $f(x) = \sin n\pi x$ then $f''(x) - \lambda f(x) = (-n^2\pi^2 - \lambda)\sin n\pi x$ so the resolvent should be $\sum \theta_n \sin n\pi x \to \sum \frac{\theta_n}{-n^2\pi^2-\lambda} \sin \pi n x$

This is clearly a bounded operator since we divide the sequence of coefficients with respect to an orthonormal basis by non-zero numbers which tend to infinity. The operator is even compact. We need to check that if $f \in$ Dirichlet $H^2$ and $f'' - \lambda f = g = \sum \theta_n \sin n\pi x$ then $f = \sum \frac{\theta_n}{-n^2\pi^2-\lambda} \sin n\pi x$, i.e. if $g \in \mathrm{Im}(T - \lambda I)$ then we have $(T - \lambda I)^{-1} g$ well defined.

We shall use "boot strapping". We have

$$f'' = \lambda f + g \tag{5.1}$$

and $g \in L_2$ is nice and $f \in H^2$ is very nice. The fact that $f$ satisfies a differential equation means that smoothness of $f$ and $g$ passes to $f''$ and so that $f$ is much smoother.

To do this we start by observing that $f \in L_2$ so we may write

$$f(x) = \sum \gamma_n \sin n\pi x$$

Our aim is to prove that $\gamma_n = \frac{\theta_n}{-n^2\pi^2-\lambda}$ for each $n$.

Note that we can't state that $f''(x) = \sum -n^2\pi^2\gamma_n \sin n\pi x$ because this might not even make sense.

By the differential equation (5.1) above we have $f''(x) = \sum(\lambda\gamma_n + \theta_n)\sin n\pi x$. Integration is a bounded operator on $L_2$ so we can integrate term by term, so as to get

$$f(x) = cx + d + \sum \frac{(\lambda\gamma_n + \theta_n)}{-n^2\pi^2} \sin n\pi x$$

Then if $c = d = 0$ we have

$$\sum \frac{(\lambda\gamma_n + \theta_n)}{-n^2\pi^2} \sin n\pi x = \sum \gamma_n \sin \pi n x$$

in $L_2$ and so we get $-\gamma_n n^2\pi^2 = \lambda\gamma_n + \theta_n$ as required.

To eliminate the linear term we use the boundary conditions $f(0) = 0 = f(1)$. On the face of it this sum at 0 and 1 vanishes since we chose $\sin n\pi x$ to do this, but an $L_2$ convergent series might not even have a meaning at a particular point. We need something stronger. However, $\sum(\lambda\gamma_n + \theta_n) < \infty$ because $f, g \in L_2$ and so

$$\sum \left| \frac{\lambda\gamma_n + \theta_n}{-n^2\pi^2} \right| \leq \left(\sum(\lambda\gamma_n + \theta_n)^2\right) \left(\sum \frac{1}{n^4\pi^4}\right) < \infty$$

so the series $\sum \frac{(\lambda\gamma_n+\theta_n)}{-n^2\pi^2} \sin n\pi x$ converges uniformly.

Since $f \in H^2$, not just $L_2$ it should come as no surprise that its Fourier series converges much better than in $L_2$. Thus the sum does vanish at 0 and 1 and $f(0) = d$ and $f(1) = c+d$ so $c = d = 0$.

We have that $(T - \lambda I)^{-1}$ is bounded on $\mathrm{Im}(T - \lambda I)$ if $\lambda \neq -n^2\pi^2$. Setting $\lambda = 0$ we get that the spectral representation for $T^{-1}$ and hence $T$.

We now consider higher dimensions. The laplacian is

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

and we consider the Dirichlet Laplacian. The eigenfunctions are $(x, y) \mapsto \sin m\pi x \sin n\pi y$ and the eigenvalue is $-(n^2 + m^2)\pi^2$. In $k$ dimensions we replace $\sum \frac{1}{n^4\pi^4}$ with

$$\sum \frac{1}{(n_1^2 + ... + n_k^2)^2\pi^4} \approx \int_{\mathbb{R}^k} \frac{1}{|x|^4} dx = \infty$$

If the dimension $k$ is large enough we don't get the boundary conditions automatically. There are no tricks, on $\Omega$ a random domain we dont know the eigenfunctions. You need a big theorem.

On $\mathbb{R}$ the map $f \mapsto f''$ has many more eigenvalues $e^{itx} \mapsto -t^2 e^{itx}$ so $-t^2$ is an eigenvalue. On $[0, 1]$ we have discrete eigenvalues, but on $\mathbb{R}$ we have the whole of $(-\infty, 0]$ in the spectrum.

A Schrödinger operator is a map $L : f \mapsto -f'' + V(x)f$ where $V$ is a potential. To solve the Schrödinger equation in the same way as the vibrating string we need the spectral properties of $L$. The Dirichlet Laplacian corresponds to $V = \begin{cases} 0 & x \in [0, 1] \\ \infty & x \in (-\infty, 0) \cup (1, \infty) \end{cases}$. The particle is bound in $[0, 1]$ and this forces the discrete energy levels.

## 5.1 Quantum Harmonic Oscillator

The simplest example of a Schrödinger operator on $\mathbb{R}$ is

$$Tf(x) = -f''(x) + \frac{x^2}{4} f(x)$$

where we have taken our potential $V$ to be $x^2/4$. This potential tends to $\infty$ fairly rapidly. Particles can wander over all of $\mathbb{R}$ but typically they are at distance about 1 from 0. The states are bound. If we put $f(x) = p(x)e^{-x^2}/4$ then

$$-f''(x) + \frac{x^2}{4} f(x) - \lambda f(x) = e^{-x^2}/4(p''(x) + xp'(x) - (\lambda - \frac{1}{2})p(x))$$

and we claim that for $n = 0, 1, 2, ...$ the map $p \mapsto -p''(x) + xp'(x)$ has eigenvalue $n$ with eigenfunction a polynomial of degree $n$. This gives rise to the eigenvalues $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$ for the original $T$.

The map $L$ which is $p \mapsto -p''(x) + xp'(x)$ acts on polynomials of degree $n$ as follows:

$$L(1) = 0, \qquad L(x) = x, \qquad L(x^2) = -2 + 2x, \qquad L(x^3) = -6x + 3x^3$$

and can be represented by

$$\begin{pmatrix} 0 & 0 & -2 & 0 & \dots \\ 0 & 1 & 0 & -6 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \end{pmatrix}$$

and this matrix is upper triangular and so its eigenvalues are $0, 1, 2, ...$ and the eigenfunction corresponding to $n$ is a polynomial of degree at most $n$. These are related to the Hermite polynomials. Let

$$F(x,t) = e^{xt - t^2/2} = \sum_0^\infty \frac{p_n(x)}{n!} t^n$$

We can show that $p_n$ is a polynomial of degree n in x. This is left as an exercise. We want that $-p_n''(x) + xp'(x) = np_n(x)$ or equivalently

$$\sum \frac{(-p_n''(x) + xp'(x))}{n!} = \sum \frac{np_n(x)}{n!}$$

or alternatively

$$-\frac{\partial^2 F}{\partial x^2} + x\frac{\partial F}{\partial x} = t\frac{\partial F}{\partial t}$$

which is true upon differentiating $F$. We call such an $F$ a generating function.

$T$ is supposed to be self adjoint so we would like the eigenfunctions $p_n(x)e^{-x^2/4}$ to be orthogonal. We would like to know that if $m \neq n$ then

$$\int_{-\infty}^\infty p_n(x)p_m(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = 0 \qquad \int_{-\infty}^\infty p_n^2(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = ||p_n||^2$$

or in other formulation

$$\int_{-\infty}^\infty \sum_{m\geq 0}\frac{p_m(x)}{m!}s^m \sum_{n\geq 0}\frac{p_n(x)}{n!}t^n \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = \int_{-\infty}^\infty \sum_{n\geq 0}\frac{p_n(x)}{n!}(st)^n\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = W(st)$$

Lets see if this holds:

$$\begin{aligned}\int_{-\infty}^\infty F(x,s)F(x,t)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx &= \int_{-\infty}^\infty e^{xs-s^2/2}e^{xt-t^2/2}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\ &= \int_{-\infty}^\infty e^{\frac{1}{2}(x-s-t)^2}e^{st}\frac{dx}{\sqrt{2\pi}} \\ &= e^{st}\int_{-\infty}^\infty e^{-\frac{1}{2}(x-s-t)^2}\frac{dx}{\sqrt{2\pi}} \\ &= e^{st} = \sum\frac{(st)^n}{n!}\end{aligned}$$

and so the $p_n$ are orthogonal and $||p_n||^2 = n!$. In order to show that $T$ has pure point spectrum $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$ the main thing we need is that the $p_n(x)e^{-x^2/4}$ form a complete orthonormal basis in $L_2(\mathbb{R})$. We want that the $p_n$ form a complete orthonormal basis in $L_2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}})$.

We know that polynomials are dense in $L_2[a,b]$ because they are dense in $C[a,b]$ with the much stronger uniform norm. On $\mathbb{R}$ with weight $\frac{e^{-x^2/2}}{\sqrt{2\pi}}$ we can approximate in $L_2$ by continuous functions with bounded support. We can approximate these on the bounded support by polynomials but the polynomials will explode elsewhere and not approximate on the line.

The usual proof of the density in $L_2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}})$ uses the invertibility of the Fourier transform. We shall use a different approach.

If $q \in L_2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}})$ then its coefficients with respect to the $p_n$ are

$$a_n = \int_{-\infty}^{\infty} q(x) \frac{p_n(x)}{n!} \frac{e^{-x^2/4}}{\sqrt{2\pi}} dx = \langle q, \frac{p_n}{n!} \rangle$$

and so the $N$th partial sum of the expansion is

$$y \mapsto \sum_0^N a_n \frac{p_n(y)}{\sqrt{n!}} = \int_{-\infty}^{\infty} q(x) \underbrace{\sum_0^N \frac{p_n(x)p_n(y)}{n!} \frac{e^{-x^2/2}}{\sqrt{2\pi}}}_{K_N(y,x)} dx$$

and so $K_N$ is like the Dirichlet kernel.

It gives the best $L_2$ approximations, but potentially bad with uniform approximations. We shall use a better kernel built with a smooth cut of. The kernel given above cuts off sharply at $N$, and so we take

$$\tilde{K}_t(y,x) = \sum_0^\infty \frac{p_n(y)p_n(x)}{n!} t^n \frac{e^{-x^2/4}}{\sqrt{2\pi}}$$

for $0 \le t < 1$ and we will let $t \to 1$. We try to rewrite this kernel in a better manner. We know that

$$F(y, tu) = \sum_0^\infty \frac{p_n(y)}{n!} t^n u^n$$

but instead of $u$ we have $p_n(x)$ and so the idea is to make $p_n$ like $u$. We do this as follows:

**Lemma 5.1**

$$i^n p_n(x) e^{-x^2/2} = \int_{-\infty}^{\infty} u^n e^{ixu} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}$$

**Proof** We first multiply by $t^n/n!$ and sum. On the left we get

$$\sum_0^\infty i^n p_n(x) e^{-x^2/2} \frac{t^n}{n!} = e^{-x^2/2} F(x, it) = e^{-x^2/2} e^{xit + t^2/2} = e^{-\frac{1}{2}(x-it)^2}$$

On the right we get

$$\begin{aligned} \int_{-\infty}^{\infty} \sum_0^\infty \frac{t^n}{n!} u^n e^{ixu} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du &= \int_{-\infty}^{\infty} e^{tu + ixu - u^2/2} \frac{du}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u-ix-t)^2} e^{\frac{1}{2}(ix+t)^2} \frac{du}{\sqrt{2\pi}} \\ &= e^{\frac{1}{2}(ix+t)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u-ix-t)^2} \frac{du}{\sqrt{2\pi}} \end{aligned}$$

The integral is a contour integral along $v \mapsto v - ix$ of $\frac{e^{-\frac{1}{2}(z-t)^2}}{\sqrt{2\pi}}$. By Cauchy's integral theorem, the integral around a rectangle is 0. We thus need the integral on the vertical sides to tend to zero. This does as we are integrating $e^{-z^2}$. *Q.E.D.*

We thus have

$$\begin{aligned}
\tilde{K}_t(y,x) &= \sum_0^\infty \frac{p_n(y)}{n!} t^n \int_{-\infty}^{\infty} (-iu)^n e^{ixu} e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(y, -itu) e^{ixu} e^{-u^2/2} du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iytu + t^2u^2/2} e^{ixu - u^2/2} du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-t^2)\left(u - \frac{ix - iyt}{1-t^2}\right)^2} e^{-\frac{1}{2}\frac{(x-yt)^2}{1-t^2}} du \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-yt)^2}{1-t^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(u - \text{fish})^2}{\text{shark}}} \frac{du}{\sqrt{2\pi}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-yt)^2}{1-t^2}} \frac{1}{\sqrt{1-t^2}}
\end{aligned}$$

This kernel as a function of $v$ is the density of a Gaussian with mean $yt$ and variance $1 - t^2$ and so

$$\int q(x)\tilde{K}_t(y,x)dx \approx q(yt) \approx q(y)$$