MA4E0 Lie Groups

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These notes are based on the 2012 `MA4E0 Lie Groups` course, taught by John Rawnsley, typeset by Matthew Egginton.
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0 Topological Groups

Let G be a group and then write the group structure in terms of maps; multiplication becomes $m : G \times G \to G$ defined by $m(g_1, g_2) = g_1 g_2$ and inversion becomes $i : G \to G$ defined by $i(g) = g^{-1}$. If we suppose that there is a topology on G as a set given by a subset $T \subset P(G)$ with the usual rules. We give $G \times G$ the product topology. Then we require that m and i are continuous maps. Then G with this topology is a **topological group**

Examples include

1. G any group with the discrete topology
2. $\mathbb{R}^n$ with the Euclidean topology and $m(x, y) = x + y$ and $i(x) = -x$
3. $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{z \in \mathbb{C} : |z| = 1\}$ with $m(z_1, z_2) = z_1 z_2$ and $i(z) = \bar{z}$. Note that this is compact, connected and commutative
4. If G_1, G_2 are topological groups then $G_1 \times G_2$ with the product topology is a topological group.
5. $T^k = S^1 \times ... \times S^1$

We now state a first result and prove it using the following few lemmas.

Proposition 0.1 *If G is a topological group which is connected then any open set containing the identity element generates G as a group, i.e. every element of G is a finite product of elements of the open set.*

Lemma 0.2 *If G is a topological group and $g \in G$ then $L_g(h) = gh = m(g, h)$ defines a continuous map $G \to G$. In fact L_g is a homeomorphism.*

Proof

$$L_{g_1} \circ L_{g_2}(h) = g_1 g_2 h = L_{g_1 g_2}(h)$$

and also

$$L_e(h) = eh = h = Id_G(h)$$

and so $L_g \circ L_{g^{-1}} = L_e = Id_G = L_{g^{-1}} \circ L_g$ and so L_g is bijective and has a continuous inverse. *Q.E.D.*

Definition 0.3 *If G is a group and $A, B \subset G$ then we define*

1. $AB = \{ab | a \in A, b \in B\}$
2. $A^{-1} = \{a^{-1} | a \in A\}$
3. *we inductively define* $A^{k+1} = A^k A = AA^k$ *for* $k \in \mathbb{N}$ *and* $A^{-k} = (A^{-1})^k$

Lemma 0.4 *Suppose that $A \subset G$ and then*

- *if* $e \in A$ *then* $A^k \subset A^{k+1}$
- $A^{-k} = (A^k)^{-1}$
- *If* $e \in A$ *and* $A^{-1} = A$ *then* $\bigcup_{i \in \mathbb{Z}} A^k$ *is a subgroup of G, called the subgroup of G generated by A.*

Lemma 0.5 *If G is a topological group and U is an open subset of G with $e \in U$ then $V = U \cap U^{-1}$ is an open set with $V = V^{-1}$ and $e \in V$.*

Theorem 0.6 *If G is a topological group and U is an open subset such that $e \in U$ then $\cup U^k$ is an open subset of G and i fG is connected then $\cup U^k = G$.*

Proof U^k is open since it can be written as

$$\bigcup_{g \in U^{k-1}} L_g(U)$$

and since L_g is a homeomorphism so $L_g(U)$ is open hence the above is a union of open sets, and so is open.

If we set $V = U \cap U^{-1}$ and note this is a subset of U and so $V^k \subset U^k$ for all k and so $\cup V^k \subset \cup U^k$. V is open and $H := \cup V^k$ is a subgroup of G.

H has cosets $L_g(H)$ which are open sets and so $G \setminus H = \cup_{gH \neq H} gH$ is open in G. Since G is connected we have $G \setminus H = \emptyset$. Since $G \subset \cup U^k \subset G$ we have $G = \cup U^k$. *Q.E.D.*

Examples include $\mathbb{R}^\star \supset \mathbb{R}^+$ and the latter is an open subgroup and $\mathbb{R}^\star = \mathbb{R}^+ \cup (-1)\mathbb{R}^+$. and so $\mathbb{R}^+$ is closed as well. Any closed subgroup of $\mathbb{R}^+$, eg $\{1\}$ or $\{2^k k \in \mathbb{Z}\}$.

1 Manifolds

Definition 1.1 *A chart (U, ϕ) of dimension n on an open set U of a topological space $\mathcal{M}$ is a homeomorphism ϕ of U onto an open set in $\mathbb{R}^n$.*

We get functions $x_1, ..., x_n$ on U such that, for $p \in U$ $\phi(p) = \begin{pmatrix} x_1(p) \\ ... \\ x_n(p) \end{pmatrix}$. These are called the coordinate functions of the chart.

Examples include $\mathcal{M} = \mathbb{R}^n = U$ the identity map. This is called the canonical chart. One could instead have U to be an open set, and ϕ to be the inclusion map. Another example is S^1 with the angle, or with stereographic projection as the chart.

If (U, ϕ) and (V, ψ) are two charts then $\psi \circ \phi^{-1}|_{\phi(U \cap V)}$ is the change of coordinates map between the two. This is a homeomorphism $\phi(U \cap V)$ to $\psi(U \cap V)$.

Definition 1.2 *Two charts (U, ϕ) and (V, ψ) of dimension n on a topological space $\mathcal{M}$ are* **compatible** *of class C^k if either $U \cap V = \emptyset$ or if $U \cap V \neq \emptyset$ then the maps $\psi \circ \phi^{-1}|_{\phi(U \cap V)}$ and $\phi \circ \psi^{-1}|_{\psi(U \cap V)}$ are C^k as maps from open sets of $\mathbb{R}^n$.*

C^k means, for $k > 0$ that all partial derivatives of orders up to k are continuous, for $k = 0$ that the functions are continuous, and for $k = \omega$ that the function is real analytic.

Example 1.1 *INSERT FIGURE Two stereographic projections on the circle. Suppose we project from two antipodal points and set the centre of the circle to be the origin. Then we have that $\frac{y}{1+x} = \frac{\phi(x,y)}{2}$ by considering similar triangles. Thus we get that $x = \frac{4-t^2}{4+t^2}$ and $y = \frac{4t}{4+t^2}$, and thus we have found $\phi^{-1}(t)$. Also we have that $\frac{\psi(x,y)}{2} = \frac{y}{1-x}$ and so $\psi \circ \phi^{-1}(t) = \frac{4}{t}$ by direct calculation. This is a map from $\phi(U \cap V)$ which is $\mathbb{R} \setminus \{0\}$ since $U = S^1 \setminus \{(-1, 0))\}$ and $V = S^1 \setminus \{(1, 0))\}$ and so the change of coordinates map is C^∞.*

Exercise: Check other examples of charts are compatible for suitable k.

Definition 1.3 *An* **atlas** *$\mathcal{A}$ of dimension n and class C^k on a topological space M is a collection of charts which are dimension n and pairwise compatible of class C^k such that the domains of the charts cover M.*

An atlas is said to be **maximal** *if it contains all charts of dimension n which are C^k compatible with all of its charts.*

Remark Every atlas is contained in a unique maximal atlas of the same dimension and class.

Example 1.2 *On S^1 we have two charts given by projections from two distinct points covering S^1.*

Definition 1.4 *A* **manifold structure** *of dimension n and class C^k on a topological space M is a choice of maximal atlas of class C^k and dimension n.*

A manifold structure can be specified by a non maximal atlas, e.g. the atlas of two charts on S^1. We can then use charts from the maximal atlas when needed.

Example 1.3

1. *$\mathbb{R}^n$ with atlas of one chart $\{(\mathbb{R}^n, Id_{\mathbb{R}^n})\}$ of dimension n and class C^∞ (C^ω even).*
2. *Any finite dimensional real vector space V by picking a basis $b = (v_1, ..., v_n)$. We get a map $\phi_b : V \to \mathbb{R}^n$ by $\phi_b(v) = (a_1, ..., a_n)$ if $v = \sum a_i v_i$. V gets a topology by making ϕ_b a homeomorphism. Then (V, ϕ_b) is a chart of dimension n on V. This construction is independent of the choice of bases, $b_2 = (w_1, ..., w_n)$ and $b_1 = b$. Then we have $b_2 = b_1 g$ where g is an $n \times n$ real matrix that is invertible. Then we have $v = b_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = b_2 \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = b_1 g \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and so $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = g \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ and so $\phi_{b_1} = g \circ \phi_{b_2}$ and g is C^k for all k and so the charts are compatible and hence in the same maximal atlas.*
3. *S^n the the n sphere and one can generalise stereographic projection. $\phi(x) = \frac{2(x_1, ..., x_n)}{1-x_0}$ is projection onto the plane tangent at $(1, 0, ..., 0)$ from $U_1 = \{x \in S^n | x_0 \neq 1\}$.*

We can consider charts from the implicit function theorem. Let $F : \mathbb{R}^{n+k} \to \mathbb{R}^k$ be a C^∞ map then $y \in F(\mathbb{R}^{n+k}$ is called a regular value if for all $x \in \mathbb{R}^{n+k}$ with $F(x) = y$ the derivative $D_x F$ is onto as a linear map $\mathbb{R}^{n+k} \to \mathbb{R}^k$.

Theorem 1.5 *$F^{-1}(y) = \{x \in \mathbb{R}^{n+k} | F(x) = y\}$ when y is a regular value is an n dimensional manifold. Charts can be obtained by taking n of the domain coordinates on a suitable open set and the remaining k domain coordinates are C^∞ functions of these.*

Example 1.4 *$S^1 = F^{-1}(1)$ where $F(x,y) = x^2 + y^2$. $D_{(x,y)}F(h,k) = 2xh + 2yk$ and since $x^2 + y^2 = 1$ x and y cannot simultaneously be zero and so the derivative is onto so 1 is a regular value. Then we solve $x^2 + y^2 = 1$ for either x and y. This gives $x = \sqrt{1 - y^2}$ and we can do this likewise for S^n.*

Definition 1.6 *A continuous map $F : M \to N$ between two manifolds is smooth if for any pair of charts (U, ϕ) and (V, ψ) of M and N respectively with $f^{-1}(V) \cap U \neq \emptyset$ the map of open sets*

$$\psi \circ f \circ \phi^{-1} : \phi(f^{-1}(V) \cap U) \to \psi(V)$$

is C^∞ as a map between open sets of Euclidean space.

f is called a diffeomorphism if it is a homeomorphism and both f and f^{-1} are smooth. If such a map exists, then we say M and N are diffeomorphic.

When $N = \mathbb{R}$ we refer to smooth maps as smooth functions.

The set of all smooth functions on a manifold M is denoted by $C^\infty(M)$. This is a real vector space and a ring with pointwise properties

Definition 1.7 *If M and N are manifolds of dimension m and n respectively then we can form a product chart $(U \times V, \phi \times \psi)$ from charts on M and N by*

$$\phi \times \psi(x,y) = (\phi(x), \psi(y)) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$$

and then this is a homeomorphism of $U \times V$ onto an open set of $\mathbb{R}^{m+n}$. This construction gives an atlas on $M \times N$ which determines a product manifold structure of dimension $m + n$.

For example $T^n = S^1 \times ... \times S^1$ is a product manifold of dimension n and is compact.

We generally assume that our manifolds are Hausdorff topological spaces and, for Lie groups, we assume that the topology is second countable, i.e. there is a countable basis for the topology.

2 Lie Groups

Definition 2.1 *A Lie group G is a set on which there are two structures:*

1. *G is a group*
2. *G is a C^∞ manifold with Hausdorff, second countable topology such that $m : G \times G \to G$ given by $m(g,h) = gh$ and $i : G \to G$ given by $i(g) = g^{-1}$ are both smooth maps*

Remark Sometimes one map $\tilde{m}(g,h) = g^{-1}h$ is used in the definition. This is equivalent to the one given above.

Example 2.1

1. *G a finite group is a Lie group with the discrete topology*
2. *$\mathbb{R}^n$ is a Lie group under addition with the standard (canonical) smooth structure*
3. *S^1 is a group under addition of angle or complex multiplication, and is a Lie group*
4. *If G and H are Lie groups then $G \times H$ is a Lie group*
5. *T^n is a Lie group and is abelian and connected.*
6. *$\mathbb{R}^\star$ is a group under multiplication so is a 1 dimensional Lie group that is not connected*
7. *$GL(n, \mathbb{R})$ is a Lie group. It is a group under matrix multiplication, and is an open set in $M_{n\times n}(\mathbb{R})$ since it is the preimage of $\mathbb{R}^\star$ under the determinant map. It has a global chart using the n^2 matrix entries as coordinates, and the group operations are polynomials in these coordinates, and so are smooth.*
8. *$GL(n, \mathbb{C})$ is a Lie group of dimension $2n^2$*
9. *The quaternions $\mathbb{H}$. $\mathbb{H} = \mathbb{R}1 + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ where $i^2 = j^2 = k^2 = -1$ and $k = ij = -ji$ and so is a skew field. $\mathbb{H} = \mathbb{R}^4$ as a real vector space. An element in $\mathbb{H}$ is written as $q = q_0 1 + q_1 i + q_2 j + q_3 k$ and $\bar{q} = q_0 1 - q_1 i - q_2 j - q_3 k$ and $q\bar{q} = ||q||^2 1$ and $\overline{qq'} = \bar{q}\bar{q'}$. $\mathbb{H}^\star = \mathbb{H} \setminus \{0\}$ under quaternion multiplication is a group with quadratic multiplication map and invers $i(q) = \tilde{q}/||q||^2$ and so both are smooth and so it is a Lie group of dimension 4.*

   Quaternions of unit length for a subgroup $S_p(1) = S^3$ as a manifold. $S_p(1)$ is a manifold and the Euclidean coordinates are smooth functions when restricted to $S_p(1)$ and so it is a Lie group of dimension 3.

10. *$GL(n,\mathbb{H})$ is a Lie group of dimension $4n^2$*

11. *Orthogonal Group $O(n)$. This is the group of operations that preserves the inner product on $\mathbb{R}^n$.*
$$O(n) = \{g \in GL(n,\mathbb{R}) | (gx)\cdot(gy) = x.y \forall x,y \in \mathbb{R}^n\}$$
Rewrite conditions so we can use the inverse function theorem to get charts. View x,y as column vectors, and so $x\cdot y = x^T y$. Then $(gx)\cdot(gy) = x^T g^T gy \iff x\cdot(g^T gy - y) = 0 \iff g^T gy - y = 0 \iff g^T g = I_n$. Thus
$$O(n) = \{g \in GL(n,\mathbb{R}) | g^T g = I_n\} = \{g \in M_{n\times n}(\mathbb{R}) | g^T g = I_n\}$$
Set $F(g) = g^T g$ for g defined on $M_{n\times n}(\mathbb{R})$. What is the target. Note $g^T g$ is not an arbitrary $n\times n$ matrix. It is symmetric. Let $S_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) | A^T = A\}$ and note that $I_n \in S_n(\mathbb{R})$. Is I_n a regular value of F as a map $M_{n\times n}(\mathbb{R}) \to S_n(\mathbb{R})$. Take $g \in O(n)$ and consider $F(g+th)$ for $h \in M_{n\times n}(\mathbb{R})$. Then
$$D_g F(h) = \frac{d}{dt} F(g+th)|_{t=0} = g^T h + h^T g$$
For g fixed, is every element of $S_n(\mathbb{R})$ of this form for a suitable h. Take $A \in S_n(\mathbb{R})$ and try to solve $g^T h + h^T g = A$ for some h or solve $g^T h = \frac{1}{2}A$ and $h^T g = \frac{1}{2}A$, but these are the same as $A = A^T$. Thus $h = \frac{1}{2}gA$. Thus $D_g F$ is surjective for every $g \in O(n)$.

Since this holds forall $g \in O(n)$, $O(n) = F^{-1}(I_n)$ and I_n is a regular value of F and hence $O(n)$ gets a manifold structure of dimension $n^2 - dim(S_n(\mathbb{R})) = n^2 - \frac{1}{2}n(n-1)$. Also $O(n)$ has a group structure since $g_1, g_2 \in O(n)$ then $(g_1 g_2)^T g_1 g_2 = I_n$. The matrix entries are functions on $O(n)$ and are smooth. Hence multiplication and inverses are smooth in manifold structure. Thus $O(n)$ is a Lie group. $O(1) = \{\pm 1\}$. Now consider $O(2)$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $a^2 + c^2 = 1 = b^2 + d^2$ and $ab + cd = 0$. so $\begin{pmatrix} b \\ d \end{pmatrix} \perp \begin{pmatrix} a \\ c \end{pmatrix}$ and so is a multiple of $\begin{pmatrix} -c \\ a \end{pmatrix}$, e.g. $\begin{pmatrix} b \\ d \end{pmatrix} = \lambda \begin{pmatrix} -c \\ a \end{pmatrix}$ and this gives $\lambda^2 = 1$. Hence $O(2)$ is the matrices of the form $\begin{pmatrix} a & -\lambda c \\ c & \lambda a \end{pmatrix}$ with $\lambda = \pm 1$ and $a^2 + c^2 = 1$. Note that $\det(g) = \lambda$ and so λ depends continuously on g. Hence $O(2)$ has two connected components, with $\{g \in O(2) | \det(g) = 1\}$ as a subgroup and as a manifold is a circle. So $O(2)$ is a disjoint union of two circles.

12. *Special Linear Group $SL(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) | \det(g) = 1\}$. We look at $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$. We have $SL(n,\mathbb{R}) = \det^{-1}(1)$ but is it a regular value of $\det$? If $g \in SL(n,\mathbb{R})$ and $h \in M_{n\times n}(\mathbb{R})$ we need to compute*
$$D_g \det(h) = \frac{d}{dt}\det(g+th)|_{t=0} = \frac{d}{dt}\det(g(I_n + tg^{-1}h))|_{t=0} = \frac{d}{dt}\det(I_n + tg^{-1}h)|_{t=0} = \operatorname{tr}(g^{-1}h) = n \neq 0$$
and so 1 is a regular value of $\det$ so $SL(n,\mathbb{R})$ is a Lie group of dimension $n^2 - 1$.

13. *$SO(n) = O(n) \cap SL(n,\mathbb{R}) = \{g \in M_{n\times n}(\mathbb{R}) | g^T g = I_n, \det(g) = 1\}$. For example $SO(1) = \{1\}$ and $SO(2) = \left\{\begin{pmatrix} a & -c \\ c & a \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) | a^2 + c^2 = 1\right\}$ which is the same as $O(2)$. Note that $g \in SO(n) \implies \det(g) = \pm 1$ and hence $SO(n)$ is an open set in $O(n)$ since $SO(n) = \det^{-1}(0,2)$ viewing $\det$ as a function on $O(n)$. $\det$ is a polynomial in the matrix entries so is smooth hence continuous and so $\det^{-1}(0,2)$ is open*

Lemma 2.2 *If G is a Lie group and $H \subset G$ is a subgroup and an open set then H is a Lie group*

Proof *Take charts from G which intersect H and take $(U \cap H, \phi|_{U\cap H})$ as a chart on H.* *Q.E.D.*

If G is a Lie group this makes G_0, the component containing the identity, into a Lie group. Also note that $O(n)_0 \subset O(n)$ and $O(n) \supset SO(n) \supset SO(n)_0$ and also $SO(2) = O(2)_0 = SO(2)_0$. We can show this holds for all n by showing $SO(n)$ is connected.

$SO(n-1)$ sits inside $SO(n)$. Given $g \in SO(n)$ with g viewed as n column vectors $g = (\underline{x}_1, ..., \underline{x}_n)$ so $x_i \cdot x_j = \delta_{ij}$, and so $\underline{x}_n \in S^{n-1}$ so rotate $\underline{x}_n$ to the vector $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ this will rotate the other vectors simultaneously and end up with a matrix of the form $\begin{pmatrix} \vdots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}$. Repeating we eventually get to $SO(2)$ which is connected. Hence $SO(n)$ is connected for all n.

We have maps $O(n) \to S^{n-1}$ by picking any column. Putting these maps together we get a map $O(n) \to \times^n S^{n-1}$ which is continuous and injective and gives $O(n)$ its usual topology. The image consists of the subset of this product of pairwise orthogonal vectors. The image is closed hence compact,so $O(n)$ is a compact Lie group. So is $SO(n)$.

14. *Pseudo -orthogonal groups $O(p,q)$ and $SO(p,q)$.*

 On $\mathbb{R}^{p+q}$ let $B_{p,q}(x,y) = \sum_1^p x_i y_i - \sum_{p+1}^{p+q} x_i y_i = x^T A_{p,q} y$. We then define

$$O(p,q) = \{g \in M_{(p+q)\times(p+q)}(\mathbb{R}) | B_{p,q}(gx, gy) = B_{p,q}(x,y)\}$$

 It is clear that $g \in O(p,q) \iff g^T A_{p,q} g = A_{p,q}$ and we can define $F : M_{(p+q)\times(p+q)}(\mathbb{R}) \to S_{p+q}(\mathbb{R})$ by $F(g) = g^T A_{p+q} g$ so $O(p,q) = F^{-1}(A_{p,q})$ and is a regular value. This makes $O(p,q)$ into a Lie group of dimension $\frac{1}{2}(p+q)(p+q-1)$.

 Observe that $O(p,0) = O(p)$ and also define $SO(p,q) = O(p,q) \cap SL(p+q,\mathbb{R})$.

 We consider $O(1,1)$. Take an element here and write it as $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and then we get that $a^2 - c^2 = 1$, $b^2 - d^2 = -1$ and $ab - cd = 0$ and so $\begin{pmatrix} b \\ -d \end{pmatrix} = -\lambda \begin{pmatrix} -c \\ a \end{pmatrix}$ and with $\lambda^2 = 1$. Now $\det(g) = -\lambda$ and so for $SO(1,1)$ we have the form $\begin{pmatrix} a & c \\ c & a \end{pmatrix}$ with $a^2 - c^2 = 1$. This is a hyperbolic function, and so $SO(1,1)$ has two connected components and $O(1,1)$ has four connected components. Now $O(1,1)_0$ corresponds to $\det(g) = 1$ and $a > 0$ and then

$$t \mapsto \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

 is a homomorphism of groups from $\mathbb{R} \to O(1,1)_0$ which is a diffeomorphism. This is an isomorphism of Lie groups.

 $O(1,3)$ is the Lorentz group, occurring in special relativity.

 Definition 2.3 *A* **homomorphism** *of Lie groups $\phi : H \to G$ is a map which is a homomorphism of groups and a smooth map of manifolds. An* **isomorphism** *of Lie groups is a bijection which is a homomorphism of Lie groups, as well as its inverse.*

15. *An antisymmetric bilinear form can only be non singular when its dimension is even. So take $\mathbb{R}^{2n}$ and set $J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and $\Omega(x,y) = x^T J_n y$. If $x = \begin{pmatrix} x' \\ x'' \end{pmatrix}$ and $y = \begin{pmatrix} y' \\ y'' \end{pmatrix}$ then $\Omega(x,y) = \begin{pmatrix} x'^T y'' - x''^T y' \\ x' y'' - x'' y' \end{pmatrix}$*

 Remark *If V is a real $2n$ dimensional vector space and w is an antisymmetric bilinear form on V which is non singular We define*

$$Sp(2n,\mathbb{R}) = \{g \in GL(2n,\mathbb{R}) | \Omega(gx, gy) = \Omega(x,y)\}$$

 Then $g \in Sp(2n,\mathbb{R}) \iff g^T J_n g = J_n$ and set $F(g) = g^T J_n g$ and then $(g^T J_n g)^T = -g^T J_n g$ and so F maps $M_{2n\times 2n}(\mathbb{R})$ into $A_{2n}(\mathbb{R}) = \{A \in M_{2n\times 2n}(\mathbb{R}) | A^T = -A\}$ which is a vector space of dimension $\frac{1}{2}2n(2n-1) = n(2n-1)$. Then one can show J_n is a regular value of F and so $Sp(2n,\mathbb{R})$ is a Lie group of dimension $(2n)^2 - n(2n-1) = n(2n+1)$. This group is called the **real symplectic group**

16. *Complex versions $GL(n,\mathbb{C})$, etc. $O(p,q,\mathbb{C})$ is the same as $O(p+q,\mathbb{C})$ as one $\mathbb{C}$ bilinear form would have all positive signs as can multiply by i.*

17. **Unitary group** *On $\mathbb{C}^n$ one can form the Hermitian inner product $\langle z, v\rangle = \bar{z}^T v$. A complex linear map $g : \mathbb{C}^n \to \mathbb{C}^n$ is unitary if $\langle gz, gv\rangle = \langle z, v\rangle$ for all $z, v \in \mathbb{C}^n$. The set of unitary matrices is denoted by $U(n)$. We can write the inner product as $\langle z, v\rangle = \bar{z}^T(\bar{g}^T g)v$ and we denote $g^\star = \bar{g}^T$. Then $g \in U(n) \iff g^\star g = I_n$. This is a group under matrix multiplication.*

 Define $F(g) = g^\star g$ and then note that $(g^\star g)^\star = g^\star g$ and let $H_n(\mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) | A^\star = A\}$ which is a real vector space of dimension n^2. If $A = X + iY$ then $A^\star == X^T - iY^T = X + iY$, and so $X = X^T$ and $Y = -Y^T$ and so corresponds to symmetric and antisymmetric and so the dimension is the sum.

 I_n is a regular value of $F : M_{n\times n}(\mathbb{C}) \to H_n(\mathbb{C})$ and so $U(n)$ is a Lie group of dimension $2n^2 - n^2 = n^2$.

 We can write $g^\star g$ in terms of the columns of g. This is that the columns of g form an orthogonal basis of $\mathbb{C}^n$ so we get a map $U(n) \to \times^n S^{2n-1}$ which is a homomorphism onto its image, which is n-tuples $z^1, ..., z^n \in \mathbb{C}^n$ with $\langle z^i, z^j\rangle = 0$ for $i \neq j$. The image is a closed set in $\times^n S^{2n-1}$ hence is compact, so $U(n)$ is a compact Lie group.

$$U(1) = \{z \in \mathbb{C} | |z| = 1\} = S^1$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is in* $U(2)$ *then* $|a|^2 + |c|^2 = 1 = |b|^2 + |d|^2$ *and* $\bar{a}b + \bar{c}d = 0$. *Thus* $b = -\lambda\bar{c}$ *and* $d = \lambda\bar{a}$ *with* $|\lambda| = 1$ *and so* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is determined by* $\lambda \in S^1$ *and* $\begin{pmatrix} a \\ c \end{pmatrix} \in S^3$ *with* $\lambda = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and so as a manifold* $U(2)$ *is diffeomorphic to* $S^1 \times S^3$.

Note that $g^\star = \bar{g}^T$ *and so* $\det g^\star = \det \bar{g} = \overline{\det g}$ *and so* $|\det g| = 1$. *If we set*

$$SU(n) = U(n) \cap SL(n, \mathbb{C}) = \overset{-1}{\det}(1)$$

by viewing $\det : U(n) \to S^1$. *Then one can check that* 1 *is a regular value of* det *and one gets* $SU(n)$ *as a manifold of dimension* $n^2 - 1$ *with matrix entries defining smooth functions, and so is a Lie group.*

$$SU(1) = \{1\}$$

We now consider $SU(2)$. *An element of* $U(2)$ *is written* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $|a|^2 + |c|^2 = 1 = |b|^2 + |d|^2$ *and* $\bar{a}b + \bar{c}d = 0$. *Then we get, using similar arguments to above, that*

$$U(2) = \left\{ \begin{pmatrix} a & -\lambda\bar{c} \\ c & \lambda\bar{a} \end{pmatrix} \middle| |a|^2 + |c|^2 = 1, |\lambda| = 1 \right\} \simeq S^3 \times S^1$$

and note that $\det g = \lambda$ *for* $g \in U(2)$ *and so* $SU(2)$ *is diffeomorphic to* S^3 *and* S^3 *is diffeomorphic to* $Sp(1)$ *and so* $SU(2)$ *and* $Sp(1)$ *are diffeomorphic.*

18. *We here consider* $U(p, q)$ *and* $SU(p, q)$. *We define*

$$\langle z, v \rangle_{p,q} = \sum_1^p \bar{z}_k v_k - \sum_1^q \bar{z}_{p+k} v_{p+k} = z^\star A_{p,q} v$$

and set

$$U(p, q) = \{g \in M_{(p+q)\times(p+q)}(\mathbb{C}) | g^\star A_{p,q} g = A_{p,q}\}$$

and observe that $A_{p,q}$ *is a regular value of* $F(g) = g^\star A_{p,q} g$ *and so* $U(p, q)$ *is a Lie group of dimension* $(p+q)^2$. *For* $g \in U(p, q)$ *we have* $|\det g| = 1$ *and* 1 *is a regular value of* $\det : U(p, q) \to S^1$ *and so* $SU(p, q)$ *is a Lie group of dimension* $(p + q)^2 - 1$.

$SU(1, 1)$ *is the isometry group of the Poincaré disc. It is three dimensional and related to* $SL(2, \mathbb{R})$. $SU(2, 2)$ *is the conformal group and is important in Penrose's twister theory.*

19. $Sp(n)$ *are called the quaternary unitary groups. Defined like* $U(n)$ *but using quaternions* $\mathbb{H}^n$.

$$Sp(n) = \{g \in M_{n\times n}(\mathbb{H}) | \bar{g}^T g = I_n\}$$

Now I_n *is a regular value of* $F : M_{n\times n}(\mathbb{H}) \to H_n(\mathbb{H})$ *and* $Sp(n) = F^{-1}(I_n)$. *The dimension of* $H_n(\mathbb{H})$ *is* $n + \frac{1}{2}n(n-1)$ *and the dimension of* $M_{n\times n}(\mathbb{H})$ *is* $4n^2$ *and so the dimension of* $Sp(n) = n(2n + 1)$ *which is the same dimension as that of* $Sp(2n, \mathbb{R})$.

If $g \in Sp(n)$ *then the columns of* g *are unit vectors in* $\mathbb{H}^n$ *so are in* S^{4n-1} *and* $Sp(n)$ *maps into* $\times^n S^{4n-1}$ *with image a closed set of pairwise orthogonal vectors, and so is compact. Now note* $O(n) \subset U(n) \subset Sp(n)$ *since* $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$.

These are the compact exceptional groups, together with their $\det = 1$ *versions. There is no determinant function on* $\mathbb{H}$ *as it makes no sense on non commutative rings.*

There are five more compact simple groups G_2, F_4, E_6, E_7, E_8. $Sp(p, q)$ *can also be defined by* $g^\star A_{p,q} g = A_{p,q}$ *with* $g \in M_{(p+q)\times(p+q)}(\mathbb{H})$

3 The Lie Algebra

Definition 3.1 *A real* **Lie algebra** *is a real vector space* V *with binary operation* $[\cdot, \cdot] : V \times V \to V$ *satisfying*

1. *it is bilinear*

2. $[x, y] = -[y, x]$
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *called the Jacobi identity*

Observe that these relations are homogeneous.

Example 3.1 1. *We can take any real vector space and set* $[x, y] = 0$. *This is the trivial Lie algebra, or also called the abelian Lie algebra*

2. $\mathcal{A}$ *an associate Lie algebra over* $\mathbb{R}$ *and set* $[a, b] = ab - ba$. *We make* $\mathcal{A}$ *into a Lie algebra, for example* $\mathcal{A} = \mathrm{End}V$ *the set of linear maps* $V \to V$. *This is called the commutator bracket. It measures how non zero the commutativity of the algebra is. For instance if* $V = \mathbb{R}^n$ *and then* $\mathrm{End}(V) = M_{n\times n}(\mathbb{R})$ *and so this is the Lie algebra with the commutator.*

3. *If* $(V, [,])$ *is a Lie algebra then a subspace* U *of* V *becomes a Lie algebra so long as* $[x, y] \in U$ *for* $x, y \in U$. *This is called a Lie subalgebra. For instance if* $\mathcal{A}$ *a commutator algebra, then* $\mathrm{End}(\mathcal{A})$ *is a Lie algebra. Denote by* $\mathrm{Der}(\mathcal{A})$ *the space of linear maps* $D : \mathcal{A} \to \mathcal{A}$ *which satisfy the Leibniz rule,*

$$D(ab) = D(a)b + aD(b)$$

This is obviously a subspace of $\mathrm{End}(\mathcal{A})$ *and we claim that it is a Lie subalgebra. Let* $D_1, D_2 \in \mathrm{Der}(\mathcal{A})$. *Then*

$$\begin{aligned}[D_1, D_2](a, b) &= D_1(D_2(ab)) - D_2(D_1(ab))\\ &= D_1(D_2(a)b + aD_2(b)) - D_2(D_1(a)b + aD_1(b))\\ &= ... =\\ &[D_1, D_2](a)b + a[D_1, D_2](b)\end{aligned}$$

and so is in $\mathrm{Der}(\mathcal{A})$.

Our main use of this last example is with $\mathcal{A} = C^\infty(\mathcal{M})$ for $\mathcal{M}$ a manifold. Let $\mathcal{X}(\mathcal{M}) = \mathrm{Der}(C^\infty(M))$. If $\mathcal{M} = \mathbb{R}$ and $D \in \mathcal{X}(\mathcal{M})$ then $D(1) = D(1 \times 1) = D(1) \times 1 + 1 \times D(1) = 2D(1)$ and so $D(c) = 0$ for any constant $c \in \mathbb{R}$.

If we take $f \in C^\infty(\mathbb{R}$ and $a \in \mathbb{R}$ then we can use Taylor expansion to get

$$f(x) = f(a) + (x - a)f'(a) + (x - a)^2 g_a(x)$$

where $g_a \in C^\infty(\mathbb{R})$. We can rearrange this and then check that it is smooth

$$\begin{aligned}D(f)(x) &= 0 + D(x - a)f'(a) + D((x - a)^2 g_a(x))\\ &= D(x - a)f'(a) + D(x - a)(x - a)g_a(x) + (x - a)D(x - a)g_a(x) + (x - a)^2 D(g_a(x))\end{aligned}$$

and if we set $x = a$ we get

$$D(f)(a) = D(x - a)(a)f'(a) + 0$$

and so $D(f)(a) = h(a)f'(a)$ for some $h \in C^\infty(\mathbb{R})$ for all $a \in \mathbb{R}$ and so $D = h\frac{d}{dx}$. If $D_i = h_i\frac{d}{dx}$ for $i = 1, 2$ then

$$[D_1, D_2] = (h_1 h_2' - h_2 h_1')\frac{d}{dx}$$

and this is not the trivial Lie algebra. $\mathcal{X}(\mathcal{M})$ is the Lie algebra of vector fields on $\mathcal{M}$ and $[x, y]$ is the Lie bracket of vector fields $x, y \in \mathcal{X}(\mathcal{M})$.

If $\sigma : M \to M$ is a diffeomorphism and $X \in \mathcal{X}(M)$ then we can form an action by

$$(\sigma \cdot X)(f) = (X(f \circ \sigma)) \circ \sigma^{-1}$$

Then if $X \in \mathcal{X}(\mathcal{M})$ then $\sigma \cdot X \in \mathcal{X}(\mathcal{M})$.

If G is a Lie group define $L_g : G \to G$ for $g \in G$ by $L_g(h) = gh$. This is a C^∞ map. Note $L_{g_1} \circ L_{g_2} = L_{g_1 g_2}$ and $L_e = Id_G$ and so $L_{g^{-1}} = L_g^{-1}$ and so is invertible. We say $X \in \mathcal{X}(\mathcal{M})$ is left invariant if $L_g \cdot X = X$ for all $g \in G$. Denote by $\mathcal{X}(G)^G$ the set of left invariant vector fields. Since $X \mapsto L_g \cdot X$ is linear we have that $\mathcal{X}(G)^G$ is a subspace of $\mathcal{X}(G)$. Note that $L_g \cdot ([X, Y]) = [L_g \cdot X, L_g \cdot Y]$ for all $X, Y \in \mathcal{X}(G)$ and so $\mathcal{X}(G)^G$ is a Lie subalgebra of $\mathcal{X}(G)$.

4 The Tangent space

In $\mathbb{R}^n$ if we have a smooth curve $\gamma : (a, b) \to \mathbb{R}^n$ we can associate a tangent vector to each point on γ by taking $\frac{d\gamma(t)}{dt}$ or $\dot\gamma(t)$. How though can we define $\dot\gamma(t)$ for $\gamma : (a, b) \to M$ a smooth map?

We can take a chart on M around a point on γ and take the tangent vector in $\mathbb{R}^n$ of $\phi \circ \gamma$, say $\frac{d}{dt}(\phi \circ \gamma(t)) = v \in \mathbb{R}^n$. If we take another chart (V, ψ) around the same point then in general $\frac{d}{dt}(\psi \circ \gamma(t)) = w$ is different but $\psi \circ \gamma(t) = \psi \circ \phi^{-1} \circ \phi(\gamma(t)) = \psi \circ \phi^{-1}(\phi \circ \gamma(t))$ and so $w = D_{\phi\circ\gamma(t)}(\psi \circ \phi^{-1})v$. Fix a point $x \in M$ and consider objects of the form (U, ϕ, v) where (U, ϕ) is a chart about x and $v \in \mathbb{R}^n$. Define an equivalence relation $\sim_x$ by $(U, \phi, v) \sim_x (V, \psi, w)$ if $w = D_{\phi\circ\gamma(t)}(\psi \circ \phi^{-1})v$.

Definition 4.1 *The tangent space T_xM is the set of equivalence classes of $\sim_x$. We denote by $[U, \phi, v]_x$ the equivalence class of (U, ϕ, v) at x.*

This becomes a vector space by picking a chart (U, ϕ) at x and defining

$$[U, \phi, v]_x + [U, \phi, w]_x = [U, \phi, v + w]_x$$

and

$$a[U, \phi, v]_x = [U, \phi, av]_x$$

This is well defined as the derivative of a linear map.

Definition 4.2 *If $\gamma : (a, b) \to M$ is a smooth curve with $x = \gamma(t_0)$ we define $\dot\gamma(t_0) = \frac{d\gamma(t)}{dt}|_{t=t_0}$ by taking a chart (U, ϕ) around x and setting*

$$\dot\gamma(t_0) = \left[U, \phi, \frac{d(\phi \circ \gamma(t))}{dt}|_{t=t_0}\right]_x$$

Theorem 4.1 *If M is a manifold of dimension n then for all $x \in M$ T_xM is a vector space of dimension n. If $F : M \to N$ is smooth then there is a natural linear map $d_xF : T_xM \to T_{F(x)}N$ which satisfies the chain rule, namely if $F : M \to N$ and $E : N \to P$ and $x \in M$ then*

$$d_x(E \circ F) = d_{F(x)}E \circ d_xF$$

Furthermore if $\gamma : (a, b) \to M$ is a smooth curve then

$$d_{\gamma(t)}F(\gamma(t)) = (F \circ \gamma)^{\cdot}(t)$$

Proof Fix $x \in M$ and (U, ϕ) a chart around x. Define a map $\mathbb{R}^n \to T_xM$ by $v \mapsto [U, \phi, v]_x$ which is linear and also define $[V, \psi, w]_x \mapsto D_{\psi(x)}\phi \circ \psi^{-1}(w)$ and this is also linear. Note that these are inverses and so $\mathbb{R}^n \simeq T_xM$ if we pick a chart. Then

$$d_xF[U, \phi, v]_x = [V, \psi, D_{\phi(x)}(\psi \circ F \circ \phi^{-1})(v)]_{F(x)}$$

which is independent of charts on M and N. This formula also works to show that the chain rule holds:

$$\begin{aligned}
(F \circ \gamma)^{\cdot}(t) &= [V, \psi, \frac{d}{dt}\psi \circ F \circ \gamma(t)]_{F\circ\gamma(t)} \\
&= [V, \psi, \frac{d}{dt}\psi \circ F \circ \phi^{-1} \circ \phi \circ \gamma(t)]_{F\circ\gamma(t)} \\
&= [V, \psi, D_{\phi\circ\gamma(t)}\psi \circ F \circ \phi^{-1}\frac{d}{dt}\phi \circ \gamma(t)]_{F\circ\gamma(t)} \\
&= d_{\gamma(t)}F[U, \phi, \frac{d}{dt}\phi \circ \gamma(t)]_{\gamma(t)} \\
&= d_{\gamma(t)}F(\dot\gamma(t))
\end{aligned}$$

Q.E.D.

For example consider $\mathbb{R}^n$ with identity chart. This allows a canonical identification $T_x\mathbb{R}^n \simeq \mathbb{R}^n$ and then $\frac{d\gamma}{dt}$ is the same ordinary derivative.

Definition 4.3 *Let $f \in C^\infty(\mathcal{M})$ and let $x \in \mathcal{M}$. If $X \in T_x\mathcal{M}$ then we set $X(f) \in \mathbb{R}$ to be equal to*

$$(D_{\phi(x)}f \circ \phi^{-1})(v)$$

if (U, ϕ) is a chart about x and $X = [U, \phi, v]_x$

This is the identification of $df : T_x\mathcal{M} \to T_{f(x)}\mathbb{R}$ with $T_{f(x)}\mathbb{R} = \mathbb{R}$, and write the X to the left, $X(f) = df(X)$.

Proposition 4.4 *Fixing $x \in T_x\mathcal{M}$ the map $f \mapsto X(f)$ ($C^\infty(\mathcal{M}) \to \mathbb{R}$) is linear and satisfies the product rule, namely*

$$X(fg) = X(f)g(x) + f(x)X(g)$$

Note that this should be clear since it is the directional derivative for a fixed chart.

Remark

1. This says X is a derivation at x from $C^\infty(\mathcal{M}) \to \mathbb{R}$.
2. Suppose $D : C^\infty(\mathcal{M}) \to \mathbb{R}$ is linear and satisfies the product rule, then there is an $x \in T_x\mathcal{M}$ such that $D(f) = X(f)$ for all $f \in C^\infty(\mathcal{M})$. To see this let (U, ϕ) be a chart about x such that the coordinate functions x_i. Then $x_1, ..., x_n$ are the restrictions of globally defined $f_1, ..., f_n \in C^\infty(\mathcal{M})$. Then $v = \begin{pmatrix} D(f_1) \\ \vdots \\ D(f_n) \end{pmatrix} \in \mathbb{R}^n$ and $X = [U, \phi, v]_x$.
3. This means that we could alternatively define $T_x\mathcal{M}$ as the linear maps $C^\infty(\mathcal{M}) \to \mathbb{R}$ that satisfy the product rule.

Example 4.1 *1. $\mathcal{M}$ an open set in a vector space V (for example $GL(n, \mathbb{R})$ in $M_{n\times n}(\mathbb{R})$). When we pick a basis $b = \{e_1, ..., e_n\}$ for V we get a chart $(\mathcal{M}, \phi_b)$ as follows. Take $x \in \mathcal{M}$ and expand x in the basis so that $x = \sum_{i=1}^n x_i(x)e_i$ and put $\phi_b = \begin{pmatrix} x_1(x) \\ \vdots \\ x_n(x) \end{pmatrix}$. The $x_1, ..., x_n$ are coordinate functinos of this chart.*

$\mathcal{M}$ then becomes a manifold using the atlas $\{(\mathcal{M}, \phi_b)\}$. If $v \in V$ then $v = \sum v_i e_i$ and $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $[\mathcal{M}, \phi_b, \underline{v}]_x \in T_x\mathcal{M}$. The map $v \mapsto [\mathcal{M}, \phi_b, \underline{v}]_x$ is a linear isomorphism of vector spaces. This is independent of the basis. The corresponding derivation at x we write as v_x and for $f \in C^\infty(\mathcal{M})$, $v_x(f) = \frac{d}{dt} f(x+tv)|_{t=0}$.

2. *Suppose that $\mathcal{M} = F^{-1}(c)$ for $F : \mathbb{R}^{n+k} \to \mathbb{R}^k$ smooth and c a regular value of f, and $\mathcal{M}$ given the manifold structure by the implicit function theorem. Take $X \in T_x\mathcal{M}$ and any curve γ through x, say with $\gamma(t_0) = x$ such that $X = \dot{\gamma}(t_0)$. To do this pick a chart (U, ϕ) about x and then $X = [U, \phi, v]_x$ for $v \in \mathbb{R}^n$ and put $\gamma(t) = \phi^{-1}(\phi(x) + (t - t_0)v)$ and so $\frac{d}{dt}\phi(\gamma(t))|_{t=t_0} = v$.*

   Then we can view this as a curve in $\mathbb{R}^{n+k}$ and take its derivative at t_0 to get a $v' \in \mathbb{R}^{n+k}$. But $\gamma(t) \in \mathcal{M}$ for all t and so $F(\gamma(t)) = c$ and hence $(D_{\gamma(t_0)}F)(v') = 0$ i.e. $v' \in \mathrm{Ker} D_x F$, and $D_x F : \mathbb{R}^{n+k} \to \mathbb{R}^k$ and $D_x F$ is surjective so $\dim \mathrm{Ker} D_x F = n$. We have a map $T_x\mathcal{M} \to \mathbb{R}^{n+k}$ defined by $X \mapsto \frac{d}{dt}\gamma(t)|_{t=t_0}$ which is linear and injective and maps $T_x\mathcal{M}$ to $\mathrm{Ker}(D_x F)$ and so we can use $\mathrm{Ker}(D_x F)$ as a model for $T_x\mathcal{M}$. For example, $S^n \subset \mathbb{R}^{n+1}$ with $F(x) = x \cdot x$. For $x \in F^{-1}(1)$ we have $D_x F(v) = 2x \cdot v$ and so $\mathrm{Ker} D_x F = \{v \in \mathbb{R}^{n+1} | x \cdot v = 0\}$, i.e. all vectors perpendicular to the radius vector x.

3. *$GL(n, \mathbb{R})$ has $M_{n\times n}(\mathbb{R})$ as the tangent space at I_n.*
4. *$GL(n, \mathbb{C})$ has $M_{n\times n}(\mathbb{C})$ as the tangent space at I_n.*
5. *$GL(n, \mathbb{H})$ has $M_{n\times n}(\mathbb{H})$ as the tangent space at I_n.*
6. *$SL(n, \mathbb{R}) = \{g \in M_{n\times n}(\mathbb{R}) : \det g = 1\}$. Now the tangent space at I_n is the kernel of $D_{I_n} \det = \mathrm{tr}$ and so the space is $\{A \in M_{n\times n}(\mathbb{R}) : \mathrm{tr} A = 0\}$*
7. *$SL(n, \mathbb{C})$ has $T_{I_n} SL(n, \mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) : \mathrm{tr} A = 0\}$*
8. *$O(n) = \{g \in M_{n\times n}(\mathbb{R}) : g^T g = I_n\}$. Define $F(g) = g^T g : M_{n\times n}(\mathbb{R}) \to S_n(\mathbb{R})$ and now note that $D_{I_n} F(A) = A^T + A$ and $\ker D_{I_n} F = \{A \in M_{n\times n}(\mathbb{R}) : A^T + A = 0\} = A_n(\mathbb{R})$*
9. *$T_{I_n} U(n) = \{A \in M_{n\times n}(\mathbb{R}) : A^\star + A = 0\}$*

Recall we have diffeomorphisms $L_g : G \to G$ such that $L_g(h) = gh$ with the properties that $L_{g_1} \circ L_{g_2} = L_{g_1 g_2}$ and $L_e = Id$. Similarly for on the right with $R_g(h) = hg$ but we have $R_{g_1} \circ R_{g_2} = R_{g_2 g_1}$ and $R_e = Id$.

Lemma 4.5 *$T_g G = d_e L_g(T_e G)$ since $L_g(e) = g$.*

Definition 4.6 *A vector field on a manifold $\mathcal{M}$ is a choice of tangent vector at each point of $\mathcal{M}$, i.e. for each $x \in \mathcal{M}$ we pick an $X_x \in T_x\mathcal{M}$.*

Given a vector field X on $\mathcal{M}$ we can differentiate $f \in C^\infty(\mathcal{M})$ to give a function $x \mapsto X_x(f)$, i.e. we can define a function $X(f)$ by $X(f)(x) = X_x(f)$ and it satisfies the Leibniz rule, since

$$X(f_1f_2)(x) = X_x(f_1f_2) = X_x(f_1)f_2(x) + f_1X_x(f_2(x)) = X(f_1)f_2(x) + f_1(x)X(f_2) = (X(f_1)f_2 + f_1X(f_2))(x)$$

X is linear but X may not be a derivation since we do not know $X(f) \in C^\infty(\mathcal{M})$.

Definition 4.7 *A vector field X on $\mathcal{M}$ is said to be smooth if $X(f) \in C^\infty(\mathcal{M})$ for all $f \in C^\infty(\mathcal{M})$.*

If X is smooth then X is a derivation.

Conversely if $D : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is a derivation then consider, for a given $x \in \mathcal{M}$, the map $f \mapsto D(f)(x)$ from $C^\infty(\mathcal{M}) \to \mathbb{R}$. Then it defines a derivation at x and so a tangent vector $X_x \in T_x\mathcal{M}$ with $D(f)(x) = X_x(f)$. Then D determines a vector field X which is smooth. Thus derivations of $C^\infty(\mathcal{M})$ are smooth vector fields.

Thus $\mathcal{X}(\mathcal{M})$ is now the vector space of smooth vector fields with the bracket of derivations. This is the link between the algebraic point of view (derivations) and the geometric point of view (tangent space).

Definition 4.8 *Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map of smooth manifolds. Vector fields X on $\mathcal{M}$ and Y on $\mathcal{N}$ are said to be F*-**related** *if $d_xF : T_x\mathcal{M} \to T_{F(x)}\mathcal{N}$ is such that*

$$d_xF(X_x) = Y_{F(x)}$$

for all $x \in \mathcal{M}$.

Theorem 4.9 *If $X_i \in \mathcal{X}(\mathcal{M})$ and $Y_i \in \mathcal{X}(\mathcal{N})$ for $i = 1, 2$ and $F : \mathcal{M} \to \mathcal{N}$ is a smooth map and X_i and Y_i are F-related for $i = 1, 2$ then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.*

Proof Take $f \in C^\infty(\mathcal{N})$ Then

$$d_xF(X_i)_x(f) = (X_i)_x(f \circ F)$$

and

$$d_xF(X_i)_x = (Y_i)_{F(x)}$$

and so

$$(Y_i)_{F(x)}(f) = (X_i)_x(f \circ F)$$

and is the same as $Y_i(f)(F(x))$ or

$$Y_i(f) \circ F = X_i(f \circ F) \tag{4.1}$$

This is the derivation version of being F-related. Now we have

$$\begin{aligned}
[X_1, X_2](f \circ F) &= X_1(X_2(f \circ F)) - X_2(X_1(f \circ F)) \\
&= X_1(Y_2(f) \circ F) - X_2(Y_1(f) \circ F) \\
&= Y_1(Y_2(f)) \circ F - Y_2(Y_1(f)) \circ F \\
&= ([Y_1, Y_2](f)) \circ F
\end{aligned}$$

Evaluating at x leads to $d_xF[X_1, X_2]_x = [Y_1, Y_2]_{F(x)}$ as required *Q.E.D.*

5 The Lie Algebra of a Lie Group

Definition 5.1 *A vector field X on a Lie group G is called* **left invariant** *if*

$$d_hL_g(X_h) = X_{gh}$$

for all $g, h \in G$.

Note that this means X is L_g related to X for all $g \in G$.

Lemma 5.2 *If X is a vector field on a Lie group which is left invariant then X is smooth*

Proof Take $f \in C^\infty(G)$ and then

$$X(f)(g) = X_g(f) = X_{ge}(f) = (d_eL_g(X_e))(f) = X_e(f \circ L_g)$$

and also note that

$$(f \circ L_g)(h) = f(L_g(h)) = f(gh) = f(m(g, h)) = (f \circ m)(g, h)$$

and so differentiating in h in the X_e direction leaves something smooth in g and so $Xf \in C^\infty(G)$. *Q.E.D.*

To find out how big $\mathcal{X}(G)^G$ is we compare it to T_eG. Define $\varepsilon : \mathcal{X}(G)^G \to T_eG$ by $\varepsilon(X) = X_e$. This is linear.

Theorem 5.3 *$\varepsilon : \mathcal{X}(G)^G \to T_eG$ defined by $\varepsilon(X) = X_e$ is a linear isomorphism.*

Proof Let $X \in \ker\varepsilon$ and so $X_e = 0$. But $X_g = d_eL_gX_e = 0$ for all g and so $X = 0$ and so ε is injective.

Now take $\xi \in T_eG$ and set $\tilde{\xi}_h = d_eL_h(\xi)$ for all $h \in G$. Then

$$d_hL_g(\tilde{\xi}_h) = d_hL_g(d_eL_h(\xi)) = d_e(L_g \circ L_h)(\xi) = d_eL_{gh}(\xi) = \tilde{\xi}_{gh}$$

Hence $\tilde{\xi}$ is left invariant and so smooth by the above lemma. Thus $\xi \in \mathcal{X}(G)^G$ and $\varepsilon(\tilde{\xi}_e) = \tilde{\xi}_e = d_eL_e(\xi) = \xi$ and so ε is surjective. *Q.E.D.*

Definition 5.4 *The left invariant vector field $\tilde{\xi}$ determined by $\xi \in T_eG$ is called the* **left invariant extension** *of ξ, with $d_eL_g(\xi) = \tilde{\xi}$.*

If $\xi, \eta \in T_eG$ then $\tilde{\xi}, \tilde{\eta}$ are in $\mathcal{X}(G)^G$ which is a Lie algebra so we can form $[\tilde{\xi}, \tilde{\eta}]$ which is left invariant and so of the form h for some $h \in T_eG$ with $h = \varepsilon([\tilde{\xi}, \tilde{\eta}])$. This makes $\varepsilon : \mathcal{X}(G)^G \to T_eG$. We call T_eG with this bracket the **Lie algebra** of G.
Remark We could equally use right invariant vector fields and get a second bracket on T_eG denoted by $[\cdot,\cdot]_{right}$ with $\hat{\xi}_h = d_eR_h(\xi)$ and $[\xi, \eta]_{right} = -[\hat{\xi}, \hat{\eta}]_e$

Corollary 5.5 *If G is an abelian Lie group then $[\xi, \eta] = 0$*

Note the bracket can be zero for non abelian Lie groups.

$$[\tilde{\xi}, \tilde{\eta}]_e(f) = \tilde{\xi}_e(\tilde{\eta}(f)) - \tilde{\eta}_e(\tilde{\xi}(f))$$

means $[\cdot,\cdot]$ only contains information about G near the identity. It gives the best information when G is connected.

Example 5.1 *Examples of abelian Lie groups: $\mathbb{R}^n$, $\mathbb{R}^\star$, $(0, \infty)$, S^1, T^n, $\mathbb{C}^\star = S^1 \times (0, \infty)$, $\mathbb{C}^n$.*

We need a technique for computing with vector fields. If (U, ϕ) is a chart with coordinates $x_1, ..., x_n$ so $x_i = t_i \circ \phi$ where t_i are Cartesian coordinates on $\mathbb{R}^n$.

If f is a smooth function on the manifold we can form $\left(\frac{\partial f \circ \phi^{-1}}{\partial t_i}\right) \circ \phi \in C^\infty(U)$ is a vector field on U denoted by $\frac{\partial}{\partial x_i}$

Lemma 5.6 *If $X \in \mathcal{X}(\mathcal{M})$ then $X|_U = \sum_{i=1}^n a^i \frac{\partial}{\partial x_i}$ with $a^i \in C^\infty(U)$.*

Example 5.2 *$GL(n, \mathbb{R}) \subset M_{n\times n}(\mathbb{R})$ is an open set. We have coordinates $x_{ij}(g) = g_{ij}$. since $GL(n, \mathbb{R})$ is open in a vector space, $T_{I_n}GL(n, \mathbb{R}) = M_{n\times n}(\mathbb{R})$ as vector spaces. $A \in M_{n\times n}(\mathbb{R})$ corresponds to $\frac{d}{dt}(I_n + tA)|_{t=0}$. $\tilde{A}$ is the left invariant extension and then*

$$\tilde{A}_g = (d_{I_n}L_g)A = (d_{I_n}L_g)\frac{d}{dt}(I_n + tA)|_{t=0} = \frac{d}{dt}L_g(I_n + tA)|_{t=0} = \frac{d}{dt}(g + tgA)|_{t=0}$$

and applying this to coordinate functions gives

$$\tilde{A}_g(x_{ij}) = \frac{d}{dt}(g + tgA)_{ij}|_{t=0} = \frac{d}{dt}g_{ij} + t\sum_k g_{ik}A_{kj}|_{t=0} = \sum_k g_{ik}A_{kj} = \sum_k x_{ik}(g)A_{kj}$$

and so $\tilde{A}(x_{ij}) = \sum_k x_{ik}A_{kj}$ Then

$$\begin{aligned}
[\tilde{A}, \tilde{B}](x_{ij}) :&= \tilde{A}(\tilde{B}(x_{ij})) - \tilde{B}(\tilde{A}(x_{ij})) \\
&= \tilde{A}(\sum_k x_{ik}B_{kj}) - \tilde{B}(\sum_k x_{ik}A_{kj}) \\
&= \sum_k A(x_{ik})B_{kj} - \sum_k \tilde{B}(x_{ik})A_{kj} \\
&= \sum_{k,l} x_{il}A_{lk}B_{kj} - \sum_{k,l} x_{il}B_{lk}A_{kj} \\
&= \sum_l x_{il}\sum_k (A_{lk}B_{kj} - B_{lk}A_{kj}) \\
&= \sum_l x_{il}(AB - BA)_{lj} \\
&= \widetilde{AB - BA}(x_{ij})
\end{aligned}$$

Using the lemma below we get $[\tilde{A}, \tilde{B}] = \widetilde{AB - BA}$ and so the Lie algebra of $GL(n, \mathbb{R})$ is $M_{n\times n}(\mathbb{R})$ with the Lie bracket as the commutator.

Lemma 5.7 *If $X, Y \in \mathcal{X}(\mathcal{M})$ and for an atlas of charts (U, ϕ) we have $X(x_i) = Y(x_i)$ for the coordinates $(x_1, ..., x_n)$ on U then $X = Y$.*

Example 5.3 *For all classical groups $SL(n, \mathbb{R})$, $O(n)$ etc, the same calculation works in the sense that $T_{I_n}G$ is identified with $\ker D_{I_n}F$ for F mapping $M_{n\times n}(\mathbb{R})$ to a suitable set, and then if $A \in \ker D_{I_n}F$ then there is a curve $I_n + tA + O(t^2)$ with A associated to $\frac{d}{dt}I_n + tA + O(t^2)|_{t=0}$. The higher order terms do not change the argument, so all brackets are the commutators.*

We denote $M_{n\times n}(\mathbb{R})$ with the commutator bracket by $gl(n, \mathbb{R})$ and likewise for the others.

If G and H are Lie groups , $F : G \to H$ is a homomorphism of Lie groups then $F(e_G) = e_H$ and so $d_{e_g}F : T_{e_G}G \to T_{e_H}H$ is there for a linear map from the Lie algebra of g to h.

Theorem 5.8 *If $F : G \to H$ is a Lie group homomorphism then $d_{e_G}F : g \to h$ is a Lie algebra homomorphism.*

Proof Take $\xi \in g$ and consider $\tilde{\xi}$ its left invariant extension:

$$d_gF\tilde{\xi}_g = d_gF(d_{e_G}L_g(\xi)) = d_{e_G}(F \circ L_g)(\xi)$$

and then

$$F \circ L_g(h) = F(L_g(h)) = F(gh) = F(g)F(h) = L_{F(g)}(F(h))$$

and then we get that

$$d_gF(\tilde{\xi}) = d_{e_G}(L_{F(g)} \circ F)(\xi) = d_{e_H}L_{F(g)}d_{e_G}F(\xi) = \widetilde{d_{e_G}F(\xi)}_{F(g)}$$

and so $\tilde{\xi}$ and $\widetilde{d_{e_G}F(\xi)}$ are F related. Take $\xi, \eta \in g$ and then $[\tilde{\xi}, \tilde{\eta}]$ is F related to $[\widetilde{d_{e_G}F(\xi)}, \widetilde{d_{e_G}F(\eta)}]$ and so

$$d_gF([\tilde{\xi}, \tilde{\eta}]_g) = [\widetilde{d_{e_G}F(\xi)}, \widetilde{d_{e_G}F(\eta)}]_{F(g)}$$

and setting $g = e_G$ we get

$$d_{e_G}F([\xi, \eta]_g) = [d_{e_G}F(\xi), d_{e_G}F(\eta)]_h$$

Q.E.D.

Definition 5.9 *We denote $d_{e_G}F$ by $F_\star : \mathfrak{g} \to \mathfrak{h}$.*

Corollary 5.10 *Homomorphic Lie groups have isomorphic Lie algebras,*

However, the converse is not true. There are a few examples of this. $\mathbb{R}$ and S^1 are abelian and one dimensional. Also $SU(2)$ and $SO(3)$ have isomorphic Lie algebras but different centres. $U(1) \times SU(2)$ and $U(2)$ are diffeomorphic as manifolds but not isomorphic as groups, and have the same Lie algebras.

If $F : G \to H$ and $E : K \to G$ are homomorphisms of Lie groups we get $F_\star : g \to h$ and $E_\star : K \to G$ and also $F \circ E : K \to H$ is a homomorphism of Lie groups so we get $(F \circ E)_\star : k \to h$ is a Lie algebra homomorphism. By the chain rule $(F \circ E)_\star = F_\star \circ E_\star$. Then $Id_G : G \to G$ has derivative $(Id_G)_\star = Id_g$.

Suppose G and H are compact connected Lie groups and $F : G \to H$ i as homomorphism such that $F_\star : g \to h$ is an isomorphism.

By the inverse function theorem there exists a $U \subset G$ and $V \subset H$ both open and $e_G \in U$, $e_H \in V$ and F a diffeomorphism of U into V. Hence F is an open map so $F(G)$ is an open subgroup of H. H is connected so $F(G) = H$ and so F is surjective.

Let $K = \ker(F)$. This is a normal subgroup of G which is closed in G. G is compact so K is compact. F is one to one on U and $e_G \in U$ and so $K \cap U = \{e_g\}$ so K is discrete in the relative topology. The sets $\{\{k\}\}_{k\in K}$ then form an open covering of K, and so K is a finite group.

Fix $k \in K$ and consider the map $G \to K$ given by $g \mapsto gkg^{-1} \in K$ which is continuous, and so it has a connected image. Hence $gkg^{-1} = k$ for all $g \in G$ and so $K \subset Z(G)$.

For example there are no trivial homomorphisms from $SO(3) \to SU(2)$. To show this, suppose that there is one, and call it F. Then $F_\star : SO(3) \to SU(2)$ is a homomorphism of Lie algebras. $F_\star$ is non zero otherwise F would be constant. Do some algebra and see that $F_\star$ has to be surjective and so a linear isomorphism. Therefore F is onto with a finite kernel in the centre of $SO(3)$ but the centre of $SO(3)$ is $\{I_3\}$ so F is an isomorphism. This is not possible since $Z(SU(2))\{\pm I_2\}$.

6 Integrating Vector fields and the exponential map

Definition 6.1 *Let $X \in \mathcal{X}(M)$ and $x \in \mathcal{M}$ then a smooth curve $\gamma : (a, b) \to \mathcal{M}$ in $\mathcal{M}$ with $\gamma(t_o) = x$ for some $t_0 \in (a, b)$ is said to be an* **integral curve** *through $x \in \mathcal{M}$ if $\dot\gamma(t) = X_{\gamma(t)}$ for all $t \in (a, b)$*

Theorem 6.2 *If $X \in \mathcal{X}(\mathcal{M})$ and $x \in \mathcal{M}$ then there exists an $\varepsilon > 0$ and $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ a smooth curve with $\gamma(0) = x$ and γ an integral curve of X.*

If $\gamma_i : (a, b) \to \mathcal{M}$ are two integral curves of X with $\gamma_1(t_0) = x = \gamma_2(t_0)$ then $\gamma_1(t) = \gamma_2(t)$ for all $t \in (a, b)$.

Proof Take a chart (U, ϕ) with $x \in U$. Then

$$X_{\phi^{-1}(y)} = [U, \phi, v(y)]_{\phi^{-1}(y)}$$

for $y \in \phi(U)$ a vector valued function v on $\phi(U) \subset \mathbb{R}^n$. X is smooth implies taht v is smooth on $\phi(U)$. $\dot\gamma(t) = [U, \phi, \Gamma(t)]_{\gamma(t)}$. with $\Gamma(t)$ a curve in $\mathbb{R}^n$. which is $\frac{d}{dt}\phi(\gamma(t))$ and so we have $\dot\gamma(t)X_{\gamma(t)}$. Thus

$$\frac{d}{dt}\phi(\gamma(t)) = v(\phi(\gamma(t))) \tag{6.1}$$

and for $\gamma(t)$ to pass through x at t_0 we need

$$\phi(\gamma(t_0)) = \phi(x)$$

and so $\phi(\gamma(t))$ is a solution of an ODE with initial value conditions at t_0 and so the existence and uniqueness theorem for ODES gives existence of integral curves for at least a short time. *Q.E.D.*

Note there is no explicit t in the RHS of (6.1) so it is autonomous. Hence $\phi(\gamma(t))$ is a solution implies that $\phi(\gamma(t + a))$ is a solution and so we can always find an integral curve through a given point at a convenient value of t. so if $\gamma_1(t_1) = x = \gamma_2(t_2)$ then we can shift one so that $\tilde\gamma_2(t) = \gamma_2(t + t_2 - t_1)$ is an integral curve with $\tilde\gamma_2(t_1) = x$ and so $\gamma_1(t) = \tilde\gamma_2(t)$ for common values of t near t_1. Thus you can get a large solution form combining γ_1 and γ_2

Definition 6.3 *An integral curve γ of a vector field X is called a* **maximal integral curve** *if any integral curve passing through a point of γ is just γ restricted to a subinterval (after shifting parameters to agree at a common point). X is said to be* **complete** *if the interval of definition of any maximal integral curve is $(-\infty, \infty)$.*

Example 6.1 *Consider $\mathbb{R}^2$ with $X_{(x,y)} = \frac{\partial}{\partial x}$. Then $v = (x, y) = (1, 0)$. If $(x(t), y(t))$ is an integral curve then $(\dot x(t), \dot y(t)) = (1, 0)$ and we want the integral cuve through (x_0, y_0) at t_0.*

This gives the integral cuve as $(t - t_0 + x_0, y_0)$ for $t \in (-\infty, \infty)$ so X is a complete vector field.

If we were on $\mathbb{R} \setminus \{0\}$ with the same X then the integral curves have the same formula but if $y_0 = 0$ then $t = t_0 - x_0$ gives $(0, 0)$ which is not allowed. We have two integral curves for either $t > t_0 - x_0$ or $t < t_0 - x_0$. On $\mathbb{R}^n \setminus \{0\}$ there are many complete vector fields.

Theorem 6.4 *If G is a Lie group then all left invariant vector fields are complete*

Lemma 6.5 *If $\mathcal{M}$ is a manifold and $X \in \mathcal{X}(\mathcal{M})$ and $\gamma : (a, b) \to \mathcal{M}$ is an integral curve then for $c \in \mathbb{R}$ we have $\tilde\gamma(t) = \gamma(t + c)$ is an integral curve on $(a - c, b - c)$.*

Proof Done before *Q.E.D.*

Lemma 6.6 *Let X be a left invariant vector field on a Lie group G. If $\gamma : (a, b) \to G$ is an integral curve through g at $t = t_0$ and $h \in G$ then $\bar\gamma(t) = h\gamma(t)$ is an integral curve on (a, b) passing through hg at $t = t_0$.*

Proof

$$\bar\gamma(t) = \frac{d}{dt}h\gamma(t) = \frac{d}{dt}L_h\gamma(t) = d_{\gamma(t)}L_h(\dot\gamma(t)) = X_{L_h(\gamma(t)) = X_{\bar\gamma(t)}}$$

and so $\bar\gamma$ is an integral curve and

$$\bar\gamma(t_0) = h\gamma(t_0) = hg$$

Q.E.D.

Lemma 6.7 *Let X be a left invariant vector field on a Lie group G and γ_X be the maximal integral curve of X with $\gamma_X(0) = e$. Then γ_X is defined on $(-\infty, \infty)$ and satisfies $\gamma_X(s)\gamma_X(t) = \gamma_X(s + t)$ for all s, t.*

Proof We prove that $\gamma_X(s)\gamma_X(t) = \gamma_X(s+t)$ for those s, t where both sides are defined. i.e. if γ_X is defined (a, b) then we take $s, t, s+t \in (a, b)$. Fix $s \in (a, b)$ and consider $\gamma_1(t) = \gamma_X(s)\gamma_X(t)$ on (a, b) and $\gamma_2(t) = \gamma_X(s+t)$ on $(a-s, b-s)$ for t such that $t, s+t \in (a, b)$.

By the previous lemmas, γ_1 and γ_2 are integral curves and at $t = 0$ both pass through $\gamma_X(s)$ so $\gamma_1(t) = \gamma_2(t)$ for all $t \in (a, b) \cup (a-s, b-s)$, i.e. $t, s+t \in (a, b)$.

Now suppose $b < \infty$, and we shift by an amount $s \in (a, 0)$ then $b - s > b$ and intervals (a, b) and $(0, b-a)$ overlap and we have a common solution on the overlap $\gamma_X^{-1}(s)\gamma_X(s+t)$ and $\gamma_X(t)$. The new solution is defined on $(a, b-a)$ which is a contradiction, so $b = \infty$.

The case $a = -\infty$ is similar. *Q.E.D.*

Proof (of theorem 6.4) Let X be a left invariant vector field and $\gamma_X(t)$ the maximal integral curve through e at $t = 0$. Then lemmas 6.5, 6.6 imply that $g\gamma_X(t - t_0)$ is the integral curve through g at time t_0. The property $\gamma_X(s)\gamma_X(t) = \gamma_X(t+s)$ says $\gamma_X : \mathbb{R} \to G$ is a homomorphism of groups and γ_X a smooth curve implies that $\gamma_X : \mathbb{R} \to G$ is a homomorphism of Lie groups. *Q.E.D.*

Definition 6.8 *A Lie group homomorphism $\sigma : \mathbb{R} \to G$ is called a* **1-parameter subgroup** *of G, and is denoted 1-PSG.*

If G is a Lie group and σ is a continuous homomorphism then we say σ is a continuous 1-PSG.

Example 6.2 *If $G = \mathbb{R}$ then we are looking for continuous maps $\sigma : \mathbb{R} \to \mathbb{R}$ with $\sigma(s) + \sigma(t) = \sigma(s+t)$ for all $s, t \in \mathbb{R}$. We have that $\sigma(0) = 0$. Now, for $n \in \mathcal{Z}$ we have $\sigma(n+1) = \sigma(n) + \sigma(1)$ and hence by induction $\sigma(n) = n\sigma(1)$. We can apply this argument to $\sigma(nx)$ to get $\sigma(nx) = n\sigma(x)$. Then $n\sigma(\frac{m}{n}x) = \sigma(\frac{mn}{n}x) = m\sigma(x)$ and so $\sigma(\frac{m}{n}x) = \frac{m}{n}\sigma(x)$ and since σ is continuous and rationals are dense in $\mathbb{R}$ we get that $\sigma(x) = x\sigma(1)$ for all $x \in \mathbb{R}$. This gives that σ is linear and hence smooth. We can also show that $\dot{\sigma}(0) = \sigma(1)$.*

Theorem 6.9 *The following three sets are in bijection for a Lie group G.*

1. *The set of 1-PSG of a Lie group G.*
2. *The maximal integral curves of left invariant vector fields*
3. *The Lie algebra $\mathfrak{g}$.*

Proof To map 2 to 1 note γ_X is a 1-PSG. To map 1 to 3 note $\sigma \mapsto \dot{\sigma}(0)$ gives us a map. To map 3 to 2 note $\xi \mapsto \gamma_{\tilde{\xi}}$ is one. Then we show the action of these in turn gives the identity, so they are all bijections.

Given σ a 1-PSG set $\xi = \dot{\sigma}(0)$ and form $\gamma_{\tilde{\xi}}$ It is enough to show that σ is the integral curve of $\tilde{\xi}$ through e. It should be clear that $\sigma(0) = e$. Then

$$\dot{\sigma}(t) = \frac{d}{ds}\sigma(t+s)|_{s=0} = \frac{d}{ds}\sigma(t)\sigma(s)|_{s=0} = d_eL_{\sigma(t)}\frac{d}{ds}\sigma(s)|_{s=0} = d_eL_{\sigma(t)}\xi = \tilde{\xi}_{\sigma(t)}$$

and so σ is an integral curve of $\tilde{\xi}$ so $\sigma = \gamma_{\tilde{\xi}}$ by uniqueness. *Q.E.D.*

According to the theorem, $\gamma_{\tilde{\xi}}(1) \in G$ is determined by $\xi \in \mathfrak{g}$.

Definition 6.10 *We define $\exp_G : \mathfrak{g} \to G$ by $\exp_G \xi = \gamma_{\tilde{\xi}}(1)$. This is called the exponential map of G.*

Proposition 6.11 *$\exp_G : \mathfrak{g} \to G$ is a smooth map*

Proof A smooth family of ODEs (jointly smooth in parameters and manifold variables) has solutions depending smoothly on the parameters. Hence the ODE for $\tilde{\xi}$ depends linearly on ξ. Picking a basis $\xi_1, ..., \xi_k$ for $\mathfrak{g}$ we have $\xi = a_1\xi_1 + ... + a_k\xi_k$ and so we are looking for solutions of

$$\dot{\gamma}_{\underline{a}}(t) = \tilde{\xi}_{\gamma(t)} = (a_1\tilde{\xi}_1 + ... + a_k\tilde{\xi}_k)_{\gamma_{\underline{a}}(t)}$$

where $\underline{a} = (a_1, ..., a_k)$ *Q.E.D.*

Proposition 6.12

$$\gamma_{\tilde{\xi}}(s) = \exp(s\xi)$$

Proof Consider $t \mapsto \gamma_{\tilde{\xi}}(st)$ and note this is smooth and $\gamma_{\tilde{\xi}}(st_1)\gamma_{\tilde{\xi}}(st_2) = \gamma_{\tilde{\xi}}(s(t_1+t_2))$ and so $\gamma_{\tilde{\xi}}(st) = \gamma_{\tilde{\eta}}(t)$ for some η and

$$\eta = \dot{\gamma}_{\tilde{\xi}}(0) = \frac{d}{dt}\gamma_{\tilde{\xi}}(st)|_{t=0} = \frac{d}{dt}(st)|_{t=0}\dot{\gamma}_{\tilde{\xi}}(0) = s\xi$$

and therefore $\gamma_{\tilde{\xi}}(xt) = \gamma_{\widetilde{s\xi}}(t)$ and then set $t = 1$ to conclude *Q.E.D.*

Corollary 6.13 $\exp(s\xi)\exp(t\xi) = \exp((t+s)\xi)$

It is NOT true that $\exp\xi\exp\eta = \exp(\xi+\eta)$ in general.

Corollary 6.14 *$\tilde{\xi}$ has integral curve through g at time t_0 given by $g\exp((t-t_0)\xi)$*

Example 6.3 *1. For $\mathbb{R}^n$, and 1-PSG σ has the form $\sigma(v) = v\sigma(1)$ and so $\exp v = v$ under the standard identity $T_0\mathbb{R}^n = \mathbb{R}^n$*

2. $G = \mathbb{R}^\star$ with $\mathfrak{g} = \mathbb{R}$ and then $T_1\mathbb{R}^\star = \mathbb{R}$ with the identification $\frac{d}{dt}(1+tv)|_{t=0} \leftrightarrow v$

For $\xi \in \mathbb{R}$ we want a left invariant extension to $\mathbb{R}^\star$. We use the coordinate functions x from $\mathbb{R}^\star \subset \mathbb{R}$, and we then have $\tilde{\xi}_x = f(x)\frac{d}{dx}$ and we want to know when this is left invariant.

$$\begin{aligned}
\tilde{\xi}_x = d_1L_x\xi = d_1L_x\frac{d}{dt}(1+t\xi)|_{t=0} \\
&= \frac{d}{dt}L_x(1+t\xi)|_{t=0} \\
&= \frac{d}{dt}(x+tx\xi)|_{t=0} \\
&= (x\xi)_x
\end{aligned}$$

and so $f(x) = x\xi$.

The integral curve $\sigma(t)$ through 1 at time $t=0$ will satisfy $\dot{\sigma}(t) = \tilde{\xi}_{\sigma(t)}$ and be given by a function $x(t)$ and so $\dot{x}(t) = x(t)\xi$ and so $x(t) = Ae^{\xi t}$ with $x(0) = A = 1$ and so $\exp\xi = x(1) = e^\xi$.

3. Suppose $G = GL(n,\mathbb{R})$ and $\mathfrak{g} = M_{n\times n}(\mathbb{R})$. Then $\xi \in M_{n\times n}(\mathbb{R})$ corresponds with the tangent vector $\frac{d}{dt}(I_n+t\xi)|_{t=0}$ at $I_n \in GL(n,\mathbb{R})$. We have

$$\tilde{\xi}_g = d_{I_n}L_g\frac{d}{dt}(I_n+t\xi)|_{t=0} = \frac{d}{dt}(g+tg\xi)|_{t=0} = \overline{g\xi}_g$$

where the over-line means directional derivative.

If $g(t)$ is the integral curve through I_n at $t=0$ then it satisfies

$$\dot{g}(t) = \tilde{\xi}_{g(t)} = \overline{g(t)\xi}_{g(t)}$$

and $\dot{g}(t)$ as a tangent vector is $\overline{\dot{g}(t)}_{g(t)}$ and the derivative is that of a function. Thus $\dot{g}(t) = g(t)\xi$. given a matrix ξ we form $\xi^2, \xi^3, ...$ and hence we form

$$e^\xi = I_n + \sum_{k=1}^{\infty}\frac{\xi^k}{k!}$$

which is a smooth function in ξ with $\frac{d}{dt}e^{t\xi} = 0 + \sum_{k=1}^{\infty} k\frac{t^{k-1}\xi^k}{k!} = e^{t\xi}\xi$ and so

$$\frac{d}{dt}(g(t)e^{-t\xi}) = \frac{d}{dt}(g(t))e^{-t\xi} + g(t)\frac{d}{dt}(e^{-t\xi}) = g(t)\xi e^{-t\xi} + g(t)(-e^{-t\xi}\xi) = 0$$

and so $g(t)e^{-t\xi}$ is a constant A and so $g(t) = Ae^{t\xi}$ and for $t=0$ we have $I_n = A$ and so $\exp_{GL(n,\mathbb{R})}\xi = e^\xi$.

4. We can show that, for example, if $\xi \in o(n)$ then $e^\xi \in O(n)$ and so $\exp_{O(n)}\xi = e^\xi$.

Theorem 6.15 *If $F : G \to H$ is a homomorphism of Lie groups and $F_\star : \mathfrak{g} \to \mathfrak{h}$ is the corresponding homomorphism of Lie algebras then*

$$F(\exp_G\xi) = \exp_H(F_\star\xi)$$

Proof Consider $t \mapsto F(\exp_G(t\xi))$ which is a 1-PSG of H, hence of the form $\exp_H(t\eta)$ for some $\eta \in \mathfrak{h}$ given by

$$\eta = \frac{d}{dt}F(\exp_G(t\xi))|_{t=0} = F_\star\xi$$

Q.E.D.

Corollary 6.16 *Example 4 above. Observe that if we have a matrix group, the inclusion map $O(n) \hookrightarrow GL(n,\mathbb{R})$ is a homomorphism of Lie groups whose derivative is the inclusion $o(n) \hookrightarrow gl(n,\mathbb{R})$ of Lie algebras. Hence $\exp_{O(n)}\xi = \exp_{GL(n,\mathbb{R})}\xi$ for $\xi \in o(n)$.*

Example 6.4 $\det : GL(n,\mathbb{R}) \to \mathbb{R}^\star$ *is a homomorphism of Lie groups and* $\det_\star = \mathrm{tr}$ *hence* $\det(e^\xi) = e^{\det_\star \xi} = e^{\mathrm{tr}\xi}$

$\exp : \mathfrak{g} \to G$ is a smooth map with $0 \in \mathfrak{g}$ and $e \in G$ with $\exp(0) = e$. Then

$$d_0 \exp : T_0\mathfrak{g} \to T_eG$$

is a linear map. However $T_0\mathfrak{g} = \mathfrak{g}$ (using directional derivatives) and also $T_eG = \mathfrak{g}$ and so we can view $d_0 \exp : \mathfrak{g} \to \mathfrak{g}$ as a linear map. Then we have the following.

Theorem 6.17

$$d_0 \exp = Id_{\mathfrak{g}}$$

Proof Given $\xi \in \mathfrak{g}$ there is a curve in $\mathfrak{g}$, say $\gamma(t)$ with $\gamma(0) = 0$ and $\dot\gamma(0) = \xi$, for instance $\gamma(t) = t\xi$. Then

$$d_0 \exp \dot\gamma(0) = \frac{d}{dt}\exp(\gamma(t))|_{t=0} = \frac{d}{dt}\exp(t\xi)|_{t=0} = \xi \in T_eG$$

i.e. $d_0 \exp \xi = \xi$. *Q.E.D.*

Corollary 6.18 *There are hence neighbourhoods U of $0 \in \mathfrak{g}$ and V of $e \in G$ such that $V = \exp_G U$ and $\exp U \to V$ is a diffeomorphism.*

Proof Inverse function theorem since $Id_{\mathfrak{g}}$ is an isomorphism *Q.E.D.*

Corollary 6.19 *If G is connected then every element is a finite product of exponentials, i.e. $g \in G \implies g = \exp\xi_1 ... \exp\xi_N$ for $\xi_i \in \mathfrak{g}$.*

Proof Any open set around e generates G. Take one of the form $V = \exp U$ where $\exp : U \to V$ is a diffeomorphism. *Q.E.D.*

Corollary 6.20 *If G is a connected Lie group and $F_i : G \to H$ for $i = 1, 2$ are homomorphisms of Lie groups such that $F_{i\star} : \mathfrak{g} \to \mathfrak{h}$ are equal, i.e. $F_{1\star} = F_{2\star}$ then $F_1 = F_2$*

Proof Take $g \in G$ then by the previous corollary $g = \exp\xi_1 ... \exp\xi_N$ and so

$$\begin{aligned} F_1(g) &= F_1(\exp(\xi_1)...F_1(\exp(\xi_N)) \\ &= \exp(F_{1\star}(\xi_1))...\exp(F_{1\star}(\xi_N)) \\ &= \exp(F_{2\star}(\xi_1))...\exp(F_{2\star}(\xi_N)) \\ &= F_2(\exp(\xi_1)...F_2(\exp(\xi_N)) \\ &= F_2(g) \end{aligned}$$

Q.E.D.

Let $\exp : U \to V$ be a diffeomorphism and $\exp^{-1} : V \to U$ be the inverse function. Then this is a diffeomorphism from a neighbourhood $e \ni V \subset G$ into $U \subset \mathfrak{g}$. If we pick a basis $\xi_1, ..., \xi_N$ for $\mathfrak{g}$ then for $g \in V$ we have $\exp^{-1} g = \sum_{i=1}^N x_i(g)\xi_i$ and this will define a smooth function on V which are the components of a smooth map into $\mathbb{R}^n$, $\psi : V \to \mathbb{R}^n$ which is a diffeomorphism onto an open set in $\mathbb{R}^n$. (V, ψ) is a chart on G compatible with the atlas and these charts depend on the choice of open set U in $\mathfrak{g}$ and the basis of $\mathfrak{g}$. These are all compatible.

If $g \in G$ and then $g \in gV$ so we can left translate to any element. Then $(gV, \psi \circ L_{g^{-1}})$ will be a chart around g. These are all compatible, so we get an atlas for G built from the exponential map (problem sheet 6).

Any real vector space is a group under addition so the Lie algebra of a Lie group is itself a Lie group. When then is $\exp : \mathfrak{g} \to G$ a homomorphism of Lie groups, i.e. $\exp(\xi + \eta) = \exp(\xi)\exp(\eta)$.

Proposition 6.21 *Let G be connected Lie group and then $\exp : \mathfrak{g} \to G$ is a homomorphism of Lie groups if and only if G is abelian.*

Proof Suppose $\exp : \mathfrak{g} \to G$ is a homomorphism. Then $\exp(\xi)\exp(\eta) = \exp(\xi + \eta)$. G is connected and so G is generated by exponentials and so G is abelian.

Now suppose that G is abelian and connected. Then $t \mapsto \exp(t\xi)\exp(t\eta)$ is a 1-PSG of G therefore $\exp(t\xi)\exp(t\eta) = \exp(t\zeta)$ for some $\zeta \in \mathfrak{g}$. We have

$$\zeta = \frac{d}{dt}\exp(t\xi)\exp(t\eta)|_{t=0} = \xi + \eta$$

using the lemma below. Thus exp is a homomorphism of Lie groups. *Q.E.D.*

Lemma 6.22 *If $f:(a,b)\to\mathcal{M}$ is given by a function $F:(a,b)\times...\times(a,b)\to\mathcal{M}$ where $f(t)=F(t,t,...,t)$ and $0\in(a,b)$ then $f(0)=\sum_{i=1}^{N}f_i(0)$ where $f_i(t)=F(0,...,t,...,0)$ with the t in the i-th spot.*

Proof The chain rule for $f=F\circ\Delta$ where Δ is the diagonal map, $\Delta(t)=(t,t,...,t)$. *Q.E.D.*

Corollary 6.23 *IF $g\in G$ is a connected and abelian Lie group then $\exp:\mathfrak{g}\to G$ is surjective.*

Proof IF $g\in G$ then $g=\exp(\xi_1)...\exp(\xi_N)=\exp(\xi_1+...+\xi_N)$ *Q.E.D.*

Examples of connected abelian Lie groups are $\mathbb{R}^n,T^n,\mathbb{R}^k\times T^{n-k}$ of dimension n.

Theorem 6.24 *Up to isomorphism, the above list are the only connected abelian Lie groups.*

Lemma 6.25 *Let $\Gamma\subset\mathbb{R}^n$ be an additive subgroup which is closed and for which there is an open ball B around 0 with $\Gamma\cap B=\{0\}$. Then either $\Gamma=\{0\}$ or there are k linearly independent vectors $v_1,...,v_k\in\mathbb{R}^n$ such that $\Gamma=\{\sum_{i=1}^{k}m_iv_i|m_i\in\mathbb{Z}\}$*

Proof We proceed by induction. Take $n=1$. Then if $\Gamma\neq\{0\}$ there exists a non zero element $v\in\Gamma$ of minimal length. Since Γ is closed and $||v||\geq R$ (radius of B), we claim that $\Gamma=\mathbb{Z}v$. We know that $0\neq w\in\Gamma$ then $w=\lambda v$ for $\lambda\in\mathbb{R}$ and assume that $\lambda>0$, and then we can write $\lambda=p+r$ for $p\in\mathbb{N}$ and $0\leq r<1$. Then $v\in\Gamma\implies pv\in\Gamma$ and therefore $w-pv\in\Gamma$ and now

$$||w-pv||=r||v||<||v||$$

and so, since v was minimal, we have that $r=0$ and so $w=pv\in\mathbb{Z}v$.

The general case.

Suppose we know the result for $n-1$ and $\Gamma\neq\{0\}$. Then there is some non-zero $v_1\in\Gamma$ of minimal length. Then $\mathbb{R}v_1\supset\mathbb{R}v_1\cap\Gamma$ is the case of $n=1$ and so $\mathbb{R}v_1\cap\Gamma=\mathbb{Z}v_1$.

Consider $\mathbb{R}^n/\mathbb{R}v_1\supset\Gamma/\mathbb{Z}v_1$. We need to show that $\Gamma/\mathbb{Z}v_1$ satisfies the same assumptions as Γ. If it doesn't then there is a non zero element sequence in $\Gamma/\mathbb{Z}v_1$ tending to 0. Using coset representation there is a sequence $\gamma_m\in\Gamma$ such that $\gamma_m\notin\mathbb{Z}v_1$ and $\gamma_m+\mathbb{Z}v_1\to0$ in $\mathbb{R}^n/\mathbb{R}v_1$, i.e. $\gamma_m+\mathbb{R}v_1\to0$ in $\mathbb{R}^n/\mathbb{R}v_1$.

To tend to zero in the quotient Euclidean space means we can find a sequence in $l_m\in\mathbb{Z}$ such that $\gamma_m-l_mv_1\to0$ in $\mathbb{R}^n$. Let $l_m=h_m+r_m$ for some $h_m\in\mathbb{Z}$, and $r_m\in[0,1]$. Then $\gamma_m-l_mv_1=(\gamma_m-h_mv_1)-r_mv_1$. Then sequence $r_m\in[0,1)\subset[0,1]$ and the latter is compact and so r_m has a convergent subsequence. Hence we can suppose we have a sequence γ_m and coset representation γ_m such that there is a sequence $r_m\in[0,1]$ convergent with $\gamma_m-r_mv_1\to0$ in $\mathbb{R}^n$, but $\gamma_m\in\mathbb{R}v_1$. Let $r=\lim r_m$. γ_m is convergent in $\mathbb{R}^n$ to rv_1. Γ is a closed subgroup of $\mathbb{R}^n$ and so $rv_1\in\Gamma$. Therefore $rv_1\in\mathbb{Z}v_1\cap[0,1]v_1$ and so $r=0$ or $r=1$.

If $r=0$ then $\gamma_m\to0$ in $\mathbb{R}^n$.

If $r=1$ then $\gamma_m-v_1\to0$ in $\mathbb{R}^n$ and both are sequence in Γ. Since $\Gamma\cap B=\{0\}$ there is an M such that $m\geq M$ implies that $\gamma_m=0$ or $\gamma_m-v_1=0$. But γ_m is a representative of a non-zero coset.

Hence there exists a B' in $\mathbb{R}^n/\mathbb{R}v_1$ with $B'\cap\Gamma/\mathbb{Z}v_1=\{0\}$. Then by induction there exist $v_2,...,v_k$ in $\mathbb{R}^n/\mathbb{R}v_1$ such that $v_2+\mathbb{R}v_1,...,v_k+\mathbb{R}v_1$ are linearly independent and integer combinations of $v_2+\mathbb{Z}v_1,...,v_k+\mathbb{Z}v_k$ give $\Gamma/\mathbb{Z}v_1$. *Q.E.D.*

Proof (of theorem 6.24) G connected abelian implies that $\exp:\mathfrak{g}\to G$ is a homomorphism and exp is a surjective map. Let $\Gamma=\ker\exp$ which is a subgroup of $\mathfrak{g}$ which is closed and since exp is a diffeomorphism near $0\in\mathfrak{g}$ then there exists an open $U\subset\mathfrak{g}$ with $0\in U$ and $\Gamma\cap U=\{0\}$. It is enough to study such subgroups γ of $\mathbb{R}^n$.

We thus know from the above lemma that there exist $v_1,...,v_k$ linearly independent such that $\Gamma=\mathbb{Z}v_1+...+\mathbb{Z}v_k$. Put $\mathfrak{g}_1=\text{span}_{\mathbb{R}}\{v_1,...,v_k\}$ which has dimension k. Pick $\mathfrak{g}_2\subset\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ and then $\Gamma\subset\mathfrak{g}_1$ and $\exp:\mathfrak{g}_2\to G$ will be an isomorphism onto its image $G_2\subset G$. Put $G_1=\exp(\mathfrak{g}_1)$ which is a subgroup of G and $G=G_1\times G_2$. $G_2\cong\mathbb{R}^{n-k}$ and $G_1\cong\mathfrak{g}/\Gamma\cong\mathbb{R}^k/\mathbb{Z}^k\cong(\mathbb{R}/\mathbb{Z})^k=T^k$ and so $G\cong T^k\times\mathbb{R}^{n-k}$. *Q.E.D.*

Corollary 6.26 *Any connected compact Abelian Lie group is isomorphic to a torus.*

7 Lie Subgroups

Definition 7.1 *A smooth map of manifolds $f:\mathcal{M}\to\mathcal{N}$ is an* **immersion** *if*

$$d_xf:T_x\mathcal{M}\to T_{f(x)}\mathcal{N}$$

has zero kernel, i.e. it is injective, for all $x\in\mathcal{M}$.

A smooth map of manifolds $f:\mathcal{M}\to\mathcal{N}$ is a **submersion** *if*

$$d_xf:T_x\mathcal{M}\to T_{f(x)}\mathcal{N}$$

is surjective for all $x\in\mathcal{M}$. A smooth map of manifolds $f:\mathcal{M}\to\mathcal{N}$ is a diffeomorphism if it is a bijection and both f and f^{-1} are smooth.

A diffeomorphism is both an immersion and a submersion. Something which is both an immersion and a submersion is locally a diffeomorphism. Locally an immersion looks like an axis in a product, e.g. $\mathcal{M} \hookrightarrow \mathcal{M} \times \mathcal{N}$. A submersion is locally a projection, e.g. $\mathcal{M} \times \mathcal{N} \to \mathcal{M}$.

Definition 7.2 *A* **submanifold** *of a manifold $\mathcal{M}$ is a smooth map $f : \mathcal{M} \to \mathcal{N}$ which is an injective immersion, i.e. both f and df are injective.*

Thus crossings are not allowed in a submanifold. We often view $\mathcal{N} \subset \mathcal{M}$ and $T_x\mathcal{N} \subset T_x\mathcal{M}$.

Definition 7.3 *A submanifold $f : \mathcal{N} \to \mathcal{M}$ is called* **embedded** *if f is a homeomorphism to $f(\mathcal{N})$ with the relative topology.*

Example 7.1 *1. Linear subspaces of a finite dimensional vector space are embedded submanifolds*

2. $\mathbb{R} \to T^2$ by $t \mapsto (e^{iat}, e^{ibt})$ is an immersion so long as $(a, b) \neq (0, 0)$ and injective when there is no common period for e^{iat} and e^{ibt}. The periods of these are $t_1 = \frac{2\pi n}{a}$ and $t_2 = \frac{2\pi m}{b}$ and a common period would be when $t_1 = t_2$. For this we would need $\frac{b}{a} \in \mathbb{Q}$. When it isn't, then $t \mapsto (e^{iat}, e^{ibt})$ is injective. This is a submanifold and sits $\mathbb{R}$ as a subgroup of T^2, a line of irrational slope. This is dense.

3. To see S^1 as a subgroup of T^n we send $z \mapsto (z^{m_1}, ..., z^{m_n})$. So long as there are no common divisors of $m_1, ..., m_n$ then this will be injective and an immersion. Thus S^1 is a submanifold of T^n and is embedded.

Definition 7.4 *If G is a Lie group then a* **Lie submanifold** *$f : H \to G$ is a smooth map of manifolds where H is a Lie group and f is a homomorphism.*

The above uses the properties, in Lie group terms of

1. H and G are Lie groups.
2. $f : H \to G$ is a homomorphism of Lie groups.
3. f is injective.
4. $d_hf : T_hH \to T_{f(h)}G$ is injective.

Given $\xi \in \mathfrak{h}$, $d_{e_H}L_h\xi \in T_hH$ and all arise in this way. Then observe that

$$(f \circ L_H)(h') = f(hh') = f(h)f(h') = L_{f(h)}f(h') = (L_{f(h)} \circ f)(h')$$

and so we get that

$$d_hf(d_{e_H}L_h\xi) = d_{e_H}(f \circ L_h)(\xi) = d_{e_H}(L_{f(h)} \circ f)(\xi) = d_{e_G}L_{f(h)}d_{e_H}f(\xi) = d_{e_G}L_{f(h)}(f_\star(\xi))$$

Both of $d_{e_G}L_{f(h)}$ and $d_{e_H}L_h$ are invertible and so d_hf is injective if and only if $f_\star$ is injective. Thu f is an immersion if and only if $f_\star : \mathfrak{g} \to \mathfrak{h}$ is injective.

Proposition 7.5 *A homomorphism of Lie groups $f : H \to G$ is an immersion of manifolds if and only if f and $f_\star$ are injective.*

This then characterises Lie subgroups as homomorphisms f with both f and $f_\star$ injective. Then $f(H)$ is a subgroup of G and $f_\star(\mathfrak{h})$ is a Lie subalgebra of $\mathfrak{g}$.

Theorem 7.6 *If $f : H \to G$ is a Lie subgroup then*

$$f_\star(\mathfrak{h}) = \{\xi \in \mathfrak{g}| \exp_G(t\xi) \in f(H) \text{ for } t \text{ in some open interval } \}$$

Proof Firstly $f_\star(\mathfrak{h}) = \{\xi \in \mathfrak{g}|...\}$ since if $\eta \in \mathfrak{h}$ put $\xi = f_\star(\eta)$ and then $\exp_G(t\xi) = \exp_G(tf_\star(\eta)) = f(\exp_H(t\eta)) \in f(H)$ for all t.

We prove the converse, suppose $\xi \in \mathfrak{g}$ has $\exp_G(t\xi) \in f(H)$ for all $t \in (a, b)$. For $t \in (a, b)$ there exists $h(t) \in H$ with $f(h(t)) = \exp_G(t\xi)$.

We show that $h(t)$ is a smooth curve in H. We do this by (locally) finding a solution map for $f(h') = \exp_G \xi'$ for ξ' near ξ in $f_\star(\mathfrak{h})$.

$f_\star(\mathfrak{h})$ is a linear subspace of $\mathfrak{g}$ so we can find a subspace $m \subset \mathfrak{g}$ such that $\mathfrak{g} = f_\star(\mathfrak{h}) \oplus m$ and then $H \times m$ is a manifold of the same dimension as G.

Pick $t_0 \in (a, b)$ and consider $\Psi_{t_0} : H \times m \to G$ given by

$$\Psi_{t_0}(h, \zeta) = \exp_G(t_0\xi)f(h)\exp_G(\zeta)$$

and note Ψ_{t_0} is smooth.

$$\begin{aligned} d_{(e_H,0)}\Psi_{t_0}(\eta,\rho) &= \frac{d}{ds}\Psi_{t_0}(\exp_H s\eta, s\rho)|_{s=0} \\ &= \frac{d}{ds}\exp_G(t_0\xi)f(\exp_H(s\eta))expp_G(s\rho))|_{s=0} \\ &= d_{e_G}L_{\exp_G(t_0\xi)}(f_\star(\eta)+\rho)) \end{aligned}$$

therefore $d_{(e_H,0)}\Psi_{t_0}$ is surjective onto $T_{\exp_G(t_0\xi}G$ and the domain is $\mathfrak{h}\oplus m$ which has the same dimension so $d_{(e_H,0)}\Psi_{t_0}$ is a linear isomorphism.

Therefore Ψ_{t_0} is a diffeomorphism near $(e_H,0)$ and therefore there exist open sets $U_1\subset H$ and $U_2\subset m$ such that if $V=\Psi_{t_0}(U_1\times U_2)$ then V is open in G, $e_H\in U_1$, $s\in U_2$ and $\exp_G(t\xi)\in V$ and Ψ_{t_0} is a diffeomorphism of $U_1\times U_2$ onto V.

$\Psi_{t_0}^{-1}: V\to U_1\times U_2$ will be smooth and we put $\Phi_{t_0}=\pi_1\circ\Psi_{t_0}^{-1}$ and then Φ_{t_0} is a smooth map $G\supset V\to U_1\subset H$. There is an interval around $t_0\in(a',b')$ such that $(a',b')\subset(a,b)$ and $\exp_G(t\xi)\in V$. Then $t\in(a',b')$ we have $h(t)=\Phi_{t_0}(\exp_G(t\xi))$ and so $h(t)$ is smooth on (a',b'). t_0 is arbitrary and so $h(t)$ is smooth on (a,b).

Pick $c\in(a,b)$ such that $\dot{h}(c)\in T_{h(c)}H$ and so $\eta=d_{h(c)}L_{h(c)^{-1}}(h(c))\in\mathfrak{h}$ and then

$$f_\star(\eta)=f_\star(d_{h(c)}L_{h(c)^{-1}}(h(c)))=(d_{f(h(c))}L_{f(h(c)^{-1})})\circ(f_{f(h(c))}f)(\dot{h}(c))$$

but

$$d_{f(h(c))}f(\dot{h}(c))=f(h(c))^{\cdot}=\frac{d}{dt}(\exp_G(t\xi))|_{t=c}=d_{e_G}L_{\exp_G(c\xi)}(\xi)$$

and $f(h(c)^{-1})=f(h(c))^{-1}=\exp_G(-c\xi)$ and so

$$f_\star(\eta)=\xi$$

Q.E.D.

This means that $f_\star(\mathfrak{h})=\{\xi\in\mathfrak{g}|\exp_G(t\xi)\in f(H)\forall t\}$ and $f(\exp_H(t\eta))=\exp tf_\star(\eta)$. This means $\exp_H$ is determined by $f_\star$ and $\exp_G$ hence an atlas for H is determined by $f_\star$ and the topology of H.

Proposition 7.7 *Let H be a subgroup of a Lie group G and $\mathfrak{h}\subset\mathfrak{g}$ a subspace such that we have open sets U of $s\in\mathfrak{g}$ and V of $e_G\in G$ with $\exp_G: U\to V$ a diffeomorphism such that $\exp_G(U\cap\mathfrak{h})=V\cap H$. Then H with the relative topology has an atlas built using $\exp_G^{-1}$ on $V\cap H$ and left translating making H into a Lie subgroup of G.*

Lemma 7.8 *Let G be a Lie group, $\xi,\eta\in\mathfrak{g}$, then there exists a $\zeta(t)\in\mathfrak{g}$ for $|t|<\delta$ for some $\delta>0$ such that*

$$\exp(t\xi)\exp(t\eta)=\exp(t\xi+t\eta+\zeta(t))$$

for $|t|<\delta$, and $\zeta(t)/t^2\to 0$ as $t\to 0$.

Proof Take $U\subset\mathfrak{g}$, $0\in U$ such that $\exp: U\to V=\exp(U)$ is a diffeomorphism then choose δ such that $\exp(t\xi)\exp(t\eta)\in V$ for $|t|<\delta$. Then set

$$\zeta(t)=\exp^{-1}(\exp(t\xi)\exp(t\eta)-t\xi-t\eta)$$

for $|t|<\delta$. Then ζ is a smooth function of t and $\zeta(0)=0$. Also

$$\exp(t\xi)\exp(t\eta)=\exp(t\xi+t\eta+\zeta(t))$$

and differentiating at $t=0$ gives

$$\xi+\eta=\xi+\eta+\dot{\zeta}(t)$$

and so $\dot{\zeta}(0)=0$. ζ is smooth in t around $t=0$ and so we have a Taylor expansion with remainder

$$\zeta(t)=\zeta(0)+t\dot{\zeta}(0)+\frac{1}{2}t^2R(t)$$

where $R(t)$ is bounded as $t\to 0$. clearly $R(t)=\frac{\zeta(t)}{t^2}$. *Q.E.D.*

Corollary 7.9 *If $\xi,\eta\in\mathfrak{g}$ then*

$$\exp(\xi+\eta)=\lim_{N\to\infty}\left(\exp\left(\frac{\xi}{N}\right)\exp\left(\frac{\eta}{N}\right)\right)^N$$

Proof When $N \geq \frac{1}{\delta}$ we have

$$\exp\left(\frac{\xi}{N}\right)\exp\left(\frac{\eta}{N}\right) = \exp\left(\frac{\xi}{N} + \frac{\eta}{N} + \zeta\left(\frac{1}{N}\right)\right)$$

and so

$$\left(\exp\left(\frac{\xi}{N}\right)\exp\left(\frac{\eta}{N}\right)\right)^N = \exp\left(\xi + \eta + N\zeta\left(\frac{1}{N}\right)\right)$$

and $N\zeta\left(\frac{1}{N}\right) \to 0$ as $N \to \infty$. *Q.E.D.*

Theorem 7.10 *Let H be a closed subgroup of a Lie group G then H with the relative topology has a unique manifold structure making the inclusion map $i : H \to\hookrightarrow G$ a Lie subgroup.*

Proof Take a subset of $\mathfrak{g}$ which would be a Lie algebra of H if the result were true, i.e. set

$$\mathfrak{h} = \{\xi \in \mathfrak{g} | \exp_G(t\xi) \in H \forall t \in \mathbb{R}\}$$

We show $\mathfrak{h}$ is a linear subspace:

Take $\xi \in \mathfrak{h}$ adn $c \in \mathbb{R}$, and consider $t \mapsto \exp_G(t(c\xi)) = \exp_G((ct)\xi) \in H$ for all t and so $c\xi \in \mathfrak{h}$.

Take $\xi, \eta \in \mathfrak{h}$ and consider

$$\left(\exp_G\left(\frac{\xi}{N}\right)\exp_G\left(\frac{\eta}{N}\right)\right)^N \in H \text{ for all } N$$

and this converges in G as $N \to \infty$. Since H is closed the limit $\exp_G(\xi + \eta)$ is in H. Replacing ξ by $t\xi$ and η by $t\eta$ results in $\exp_G(t\xi + t\eta) \in H$ for all t and so $\xi + \eta \in \mathfrak{h}$.

We now prove that conditions for proposition 7.7 hold for this subspace, i.e. there exists open sets $U \subset \mathfrak{g}$ and $V = \exp_G(U)$ such that $\exp_G : U \to V$ is a diffeomorphism and $\exp_G(U \cap \mathfrak{h}) = \exp_G(U) \cap H$.

We proceed by contradiction. Suppose no such U exists, i.e. we never have $\exp_G(U \cap \mathfrak{h}) = \exp_G(U) \cap H$ for a $U \subset \mathfrak{g}$.

Fix $W \subset \mathfrak{g}$ where $\exp_G : W \to \exp_G(W)$ is a diffeomorphism, and fix a norm on $\mathfrak{g}$. Consider open balls $B_{\frac{1}{k}}(0)$ for large enough $k \in \mathbb{N}$ to ensure $B_{\frac{1}{k}}(0) \subset W$. We have

$$\exp_G(B_{\frac{1}{k}}(0) \cap \mathfrak{h}) \neq \exp_G(B_{\frac{1}{k}}(0)) \cap H$$

If $\xi \in B_{\frac{1}{k}}(0) \cap \mathfrak{h}$ then $\exp_G \xi \in \exp_G(B_{\frac{1}{k}}(0)) \cap H$ and so $\exp_G(B_{\frac{1}{k}}(0)) \cap H$ is strictly larger than $\exp_G(B_{\frac{1}{k}}(0) \cap \mathfrak{h})$

Let $h_k \in H \cap \exp_G(B_{\frac{1}{k}}(0))$, but $h_k \notin \exp_G(B_{\frac{1}{k}}(0) \cap \mathfrak{h})$. Then $h_k = \exp_G \xi_k$ for $\xi_k \in B_{\frac{1}{k}}(0)$ and so $(\xi_k) \to 0$ in $\mathfrak{g}$ as $k \to \infty$ and therefore $(h_k) \to e$ in G.

We <u>claim</u> that $h_k \notin \exp_G(W \cap \mathfrak{h})$. Otherwise $h_k = \exp_G \eta_k$ with $\eta_k \in W \cap \mathfrak{h}$ and $h_k = \exp_G \xi_k$ with $\xi_k \in B_{\frac{1}{k}}(0) \subset W$. Due to bijectivity of $\exp_G$ on W we have $\eta_k = \xi_k$ and so $\eta_k \in B_{\frac{1}{k}}(0) \cap \mathfrak{h}$ and so $h_k \in \exp_G(B_{\frac{1}{k}}(0) \cap \mathfrak{h})$ which contradicts our choice of h_k, and so the claim holds.

Thus we have a sequence h_k in H with $(h_k) \to e$ in G and an open set $W_1 = W \cap \mathfrak{h}$ containing 0 in $\mathfrak{h}$ but $h_k \notin \exp_G(W_1)$ for all k large.

Pick a subspace $m \subset \mathfrak{g}$ supplementary to $\mathfrak{h}$ in $\mathfrak{g}$ making $\mathfrak{g} = \mathfrak{h} \oplus m$. Define $\alpha : \mathfrak{h} \times m \to G$ by

$$\alpha(\xi, \eta) = \exp_G(\xi)\exp_G(\eta)$$

Then α is smooth and has derivative at $(0, 0)$ given by

$$d_{(0,0)}\alpha(\xi, \eta) = \xi + \eta$$

which is the identification of $\mathfrak{h} \oplus m$ with $\mathfrak{g}$ and so is a linear isomorphism. Hence α is a diffeomorphism near $(0, 0)$ so we have open sets $W_{\mathfrak{h}}$ and W_m around the origin in $\mathfrak{h}$ and m respectively so that α is a diffeomorphism of $W_{\mathfrak{h}} \times W_m$ with $\alpha(W_{\mathfrak{h}} \times W_m)$ which is an open set in G containing e.

We <u>claim</u>(to prove the claim above!!) that we can shrink W_m until H and $\exp_G(W_m)$ meet only at e.

Suppose not, then we can find a sequence η_k in W_m so that $\exp_G \eta_k \in H \setminus \{e\}$ and $\eta_k \to 0$. Put $t_k = ||\eta_k|| > 0$ and then η_k/t_k is a vector of norm 1 for each k and hence we have a sequence η_k/t_k in the unit sphere in W_m which is compact, and so this sequence has a convergent subsequence. We throw away the other terms and relabel the subsequence η_k/t_k.

Let η be the limit of this sequence. Let $t > 0$, $n_k(t) = \lfloor t/t_k \rfloor$ and so

$$t/t_k - 1 \leq n_k(t) \leq t/t_k$$

and thus we have that

$$t - t_k \leq t_k n_k(t) \leq t$$

but since $t_k \to 0$ as $k \to \infty$ we have $t_k n_k(t) \to t$ as $k \to \infty$.

Multiplying the left side by η_k / t_k and the right side by η we obtain

$$n_k(t)\eta_k \to t\eta$$

as $k \to \infty$. Applying $\exp_G$ we have that

$$\lim_{k\to\infty} \exp_G(n_k(t)\eta_k) = \exp_G(t\eta)$$

and so

$$\lim_{k\to\infty} (\exp_G(\eta_k))_k^n(t) = \exp_G(t\eta)$$

but $\exp_G(\eta_k) \in H \implies (\exp_G(\eta_k))_k^n(t) \in H \implies \exp_G(t\eta) \in H$ and also from group property of H, $\exp_G(-t\eta) \in H$ for $t > 0$. Thus $\exp_G(t\eta) \in H$ for $t \in \mathbb{R}$ and thus ηinH but since $\eta \neq 0$ and $\eta \in m$ we have a contradiction. Thus the claim holds, i.e. we can shrink W_m so that $\exp_G(W_m) \cap H = \{e\}$.

Now to conclude the proof of the first claim (using the second claim!!). Since $h_k \to e$ in G and $\exp_G(W_\mathfrak{h} \times W_m)$ is open and contains e then $h_k \in \exp_G(W_\mathfrak{h} \times W_m)$ for all $k \geq k_0$, and hence

$$h_k = \exp_G(\xi'_k)\exp_G(\eta'_k)$$

with $\xi'_k \in W_\mathfrak{h}$ and $\eta'_k \in W_m$, but $h_k \notin \exp_G W_1$ and so $\eta'_k \neq 0$ for all k. Also $\exp_G \eta'_k \in H$ since $h_k, \exp_G \xi'_k \in H$ and then we have $\exp_G \eta'_k \in \exp_G W_m \cap H = \{e\}$ and so $\eta'_k = 0$ This is a contradiction.

Thus the first claim is established, i.e. we can find U open in $\mathfrak{g}$ and $V = \exp_G U$ such that $\exp_G : U \to V$ is a diffeomorphism and $\exp_G(U \cap \mathfrak{h}) = V \cap H$.

By proposition 7.7 we then have a manifold structure on H making it into a Lie group with Lie algebra $\mathfrak{h}$. Then $i : H \hookrightarrow G$ is a Lie subgroup. *Q.E.D.*

Corollary 7.11 *H has Lie algebra $\mathfrak{h} = \{\xi \in \mathfrak{g} | \exp_G(t\xi) \in H \forall t\}$*

The following is an example of how this is used

Theorem 7.12 *If $F : G \to H$ is a homomorphism of Lie groups then*

$$\ker(F) = \{g \in G | F(g) = e_H\}$$

is a Lie subgroup of G with Lie algebra given by $\ker(F_\star)$

Proof $\ker(F) = F^{-1}(\{e_H\})$ and by continuity of F and Hausdorff property of H and G we have this set is closed, and it is also a subgroup of G. Thus it is a Lie subgroup of G with Lie algebra

$$\{\xi \in \mathfrak{g} | \exp_G(t\xi) \in \ker(F) \forall t\} = \{\xi \in \mathfrak{g} | F(\exp_G(t\xi)) = e_H \forall t\}$$

but since $F(\exp_G(t\xi)) = \exp_H(tF_\star(\xi))$ we have $F_\star(\xi) = 0$ as required. *Q.E.D.*

Example 7.2 $\det : GL(n, \mathbb{R}) \to \mathbb{R}^\star$ *and then* $\ker \det = SL(n, \mathbb{R})$ *and* $\det_\star = \operatorname{tr}$ *and so* $sl(n, \mathbb{R}) = \{\xi \in gl(n, \mathbb{R}) | \operatorname{tr}\xi = 0\}$

Theorem 7.13 *If $\sigma : G \to G$ is an automorphism, then the fixed point set $G^\sigma = \{g \in G | \sigma(g) = g\}$ is a Lie subgroup with Lie algebra $\mathfrak{g}^{\sigma_\star}$*

Example 7.3

1. *$G = GL(n, \mathbb{R})$ with $\sigma(g) = (g^{-1})^T$ is an automorphism and $G^\sigma = O(n)$, and $g^{\sigma_\star} = \{\xi \in M_{n\times n}(\mathbb{R}) | -\xi^T = \xi\} = A_n(\mathbb{R})$.*
2. *$G = GL(n, \mathbb{C})$ with $\sigma(g) = (\overline{g}^{-1})^T$ then $G^\sigma = U(n)$.*
3. *$\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ and $G = GL(2n, \mathbb{R})$ and write $2n \times 2n$ matrices as 2×2 matrices of $n \times n$ blocks. $J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and so $J_n^2 = -I_n$ and define $\sigma_1(g) = J_n g J_n^{-1}$, which is an inner automorphism. Then $G^{\sigma_1} = \{\begin{pmatrix} U & -V \\ V & U \end{pmatrix} | U + iV \in GL(n, \mathbb{C})\}$ and so $GL(n, \mathbb{C})$ is a Lie subgroup of $GL(2n, \mathbb{R})$. If instead $\sigma_2(g) = J_n(g^{-1})^T J_n^{-1}$ then $G^{\sigma_2} = Sp(2n, \mathbb{R})$.*

8 Continuous homomorphisms are smooth

The basic idea is to look at continuous 1-PSG of $\mathbb{R}^n$. These are linear $\sigma(t) = tv$ with $v = \sigma(1)$, and hence smooth.

Lemma 8.1 *If G is a Lie group and H is a subgroup then the topological closure $\bar{H}$ is a subgroup.*

Proof Take $g, h \in \bar{H}$ and suppose $g_k \to g$ and $h_k \to h$ for sequences $h_k, g_k \in H$. Then

$$gh = m(g,h) = \lim_{k\to\infty} m(g_k, h_k)$$

and $m(g_k, h_k) \in H$ and so $gh \in \bar{H}$. Continuity of the inverse gives $i(\bar{H}) \subset \bar{H}$ so $\bar{H}$ is a subgroup. *Q.E.D.*

Lemma 8.2 *If $\sigma : \mathbb{R} \to G$ is a continuous 1-PSG then $\overline{\sigma(\mathbb{R})}$ is a connected closed abelian Lie subgroup of G.*

Proof By the previous lemma, it is a closed subgroup, and hence a Lie subgroup. $\sigma(\mathbb{R})$ is connected henve $\overline{\sigma(\mathbb{R})}$ is connected. $\sigma(\mathbb{R})$ is abelian and so $\overline{\sigma(\mathbb{R})}$ is abelian. *Q.E.D.*

Corollary 8.3 *If $\sigma : \mathbb{R} \to G$ is a continuous 1-PSG then σ is a continuous 1-PSG of $\overline{\sigma(\mathbb{R})}$. Thus it is enough to prove that σ is smooth when G is a connected abelian Lie group.*

Theorem 8.4 *Any continuous 1-PSG of a Lie subgroup is smooth*

Proof If G is connected and abelian then G is isomorphic to $\mathbb{R}^k \times T^{n-k}$ for some k and $n = \dim G$.

Since maps into products are continuous or smooth or homomorphism if and only if the components are continuous or smooth or homomorphisms we need only consider the cases where $G = \mathbb{R}$ and $G = S^1$.

Continuous homomorphisms $\mathbb{R} \to \mathbb{R}$ are smooth since they are linear. for $\mathbb{R} \to S^1$ let $\sigma : \mathbb{R} \to S^1 \subset \mathbb{C}$ be continuous and $\sigma(s) + \sigma(t) = \sigma(s+t)$ as a complex valued function. Use the result from complex function theory so there exists a unique $g : \mathbb{R} \to \mathbb{R}$ continuous with $\sigma(t) = e^{ig(t)}$ with $g(0) = 0$. We then have

$$e^{i(g(s)+g(t))} = e^{ig(s+t)}$$

and so

$$g(s+t) - g(s) - g(t) \in 2\pi\mathbb{Z}$$

so we have a continuous map $\mathbb{R} \times \mathbb{R} \to 2\pi\mathbb{Z}$ and so the image must be connected which must contain the image of $(0,0)$ which is 0. Hence

$$g(s+t) - g(s) - g(t) = 0$$

for all s, t. This implies $g(t) = tg(1)$ so σ is smooth. *Q.E.D.*

Theorem 8.5 *If G and H are Lie groups and $F : G \to H$ is a continuous homomorphism then F is smooth.*

Proof Let $\exp_G(t\xi)$ be a smooth 1-PSG of G. Then $F(\exp_G(t\xi))$ is a continuous 1-PSG of H and by theorem 8.4 is smooth.

We claim that F is smooth on G if and only if it is smooth on an open set V around e_H.

If F is smooth on V and $g \in G$ then gV is an open set around g. If $g' \in gV$ then $g^{-1}g' \in V$ and

$$F(g') = F(gg^{-1}g') = F(g)F(g^{-1}g') = L_{F(g)} \circ F \circ L_{g^{-1}}(g')$$

which is a composition of smooth maps hence is smooth. Thus the claim is established.

Take a basis $\xi_1, ..., x1_n$ for $\mathfrak{g}$ and consider $\alpha : \mathbb{R}^n \to G$ defined by $\alpha(t_1, ..., t_n) = \exp_G(t_1\xi_1)...\exp_G(t_n\xi_n)$. Let $h = (h_1, ..., h_n) \in \mathbb{R}^n$ and then

$$\frac{d}{dt}\alpha(0 + th)|_{t=0} = \frac{d}{dt}\exp_G(th_1\xi_1)...\exp_G(th_n\xi_n)|_{t=0} = h_1\xi_1 + ... + h_n\xi_n$$

and so $d_0\alpha$ is the identification of $\mathbb{R}^n$ with $\mathfrak{g}$ given by a basis, therefore $d_0\alpha$ is a linear isomorphism and hence α is n onto a neighbourhood V of e_G in G. For $(t_1, ..., t_n) \in U$,

$$F(\alpha(t_1, ..., t_n)) = F(\exp_G(t_1\xi_1)...\exp_G(t_n\xi_n)) = F(\exp_G(t_1\xi_1))...F(\exp_G(t_n\xi_n))$$

which is smooth in $(t_1, ..., t_n)$ by theorem 8.4. Therefore F is smooth on V, and hence smooth everywhere. *Q.E.D.*

9 Distributions and Frobenius Theorem

Definition 9.1 *1. A k-dimensional tangent distribution $\mathcal{D}$ on a manifold $\mathcal{M}$ is a k-dimensional subspace $\mathcal{D}_x \subset T_x\mathcal{M}$ for each $x \in \mathcal{M}$.*

2. We say $\mathcal{D}$ is smooth if each point in $\mathcal{M}$ has a neighbourhood U with k-smooth vector fields $X_1, ..., X_k \in \mathcal{X}(U)$ with $(X_1)_x, ..., (X_k)_x$ forming a basis at each $x \in U$ for $\mathcal{D}_x$.

3. A vector field X on $\mathcal{M}$ is said to belong to $\mathcal{D}$ if $X_x \in \mathcal{D}_x$ for all $x \in \mathcal{M}$.

4. $\mathcal{D}$ is said to be involutive if the set of vector fields belonging to $\mathcal{D}$ is a Lie subalgebra of $\mathcal{X}(\mathcal{M})$.

5. The set of smooth vector fields belonging to $\mathcal{D}$ is denoted by $\Gamma(\mathcal{D})$.

We can rewrite involutive to be $[\Gamma(\mathcal{D}), \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$.

Example 9.1 *1. $\mathcal{M} = \mathbb{R}^n$ and put $\mathcal{D}_x = \{X \in T_x\mathbb{R}^n | X(t_{k+1}) = ... = X(t_n) = 0\} = \{X \in T_x\mathbb{R}^n | X = \sum_{i=1}^k a_i \frac{\partial}{\partial t_i}|_x\}$. Then $\Gamma(\mathcal{D}) = C^\infty(\mathbb{R}^n)\frac{\partial}{\partial t_1} + ... + C^\infty(\mathbb{R}^n)\frac{\partial}{\partial t_k}$ and so $\mathcal{D}$ is smooth and involutive.*

2. G a Lie group, $\mathfrak{h} \subset \mathfrak{g}$ a k-dimensional linear subspace. Put $\mathcal{D}_g = d_eL_g(\mathfrak{h}) \subset T_gG$. Then $\mathcal{D}$ is a k-dimensional distribution and if $\xi_1, ..., \xi_k$ is a basis for $\mathfrak{h}$ then $\mathcal{D}_g = \mathrm{span}(\tilde{\xi}_{1,g}, ..., \tilde{\xi}_{k,g})$ and so $\mathcal{D}$ is smooth. We have $\tilde{\xi}_i \in \Gamma(\mathcal{D})$ and so $\mathcal{D}$ is involutive so $[\tilde{\xi}_i, \tilde{\xi}_j] \in \Gamma(\mathcal{D})$ but $\widetilde{[\xi_i, \xi_j]} = [\tilde{\xi}_i, \tilde{\xi}_j]$ and so $[\xi_i, \xi_j] \in \mathfrak{h}$ so $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. The converse is also true.

Hence left invariant involutive smooth distributions are in bijection with Lie subalgebras.

Definition 9.2 *Let $\mathcal{D}$ be a smooth k-dimensional tangent distribution on $\mathcal{M}$, then an* **integral submanifold** *for $\mathcal{D}$ is a k-dimensional submanifold $(\mathcal{N}, \phi)$ with $\phi : \mathcal{N} \to \mathcal{M}$ with $\mathcal{N}$ a k-dimensional connected manifold and ϕ an immersion of $\mathcal{N}$ into $\mathcal{M}$ such that $d_y\phi(T_y\mathcal{N}) = \mathcal{D}_{\phi(y)}$ for all $y \in \mathcal{N}$.*

Note that we have a notion of maximal integral submanifold, analogous to maximal integral curves of a vector field.

Proposition 9.3 *If $\mathcal{D}$ is a smooth distribution on $\mathcal{M}$ such that for each $x \in \mathcal{M}$ there is an integral submanifold $\phi : \mathcal{N} \to \mathcal{M}$ with $x \in \phi(\mathcal{N})$ then $\mathcal{D}$ is involutive.*

Proof Let $X, Y \in \Gamma(\mathcal{D})$, $x \in \mathcal{M}$, $\phi : \mathcal{N} \to \mathcal{M}$ with $y \in \mathcal{N}$ and $\phi(y) = x$ and $(, \phi)$ an integral submanifold. Then there exists $\tilde{X}, \tilde{Y}$ on $\mathcal{N}$ with $d_{y'}\phi(\tilde{X}_{y'}) = X_{\phi(y')}$ and $d_{y'}\phi(\tilde{Y}_{y'}) = Y_{\phi(y')}$ and so near y, $\tilde{X}$ and $\tilde{(Y)}$ are ϕ-related to X and Y respectively. This implies $[\tilde{X}, \tilde{Y}]$ and $[X, Y]$ are ϕ-related and therefore $[X, Y]_x = [X, Y]_{\phi(Y)} = d_y\phi([\tilde{X}, \tilde{Y}]_y) \in \mathcal{D}_x$. Since x is arbitrary, $[X, Y] \in \Gamma(\mathcal{D})$ *Q.E.D.*

Theorem 9.4 (Frobenius) *Let $\mathcal{D}$ be a smooth involutive k-dimensional tangent distribution on a manifold $\mathcal{M}$ and then each $x \in \mathcal{M}$ lies on a locally unique integral submanifold.*

Definition 9.5 *A maximal integral submanifold (or leaf) $(\mathcal{N}, \phi)$ of an involutive distribution $\mathcal{D}$ on a manifold $\mathcal{M}$ is a connected integral submanifold whose image is not a proper subset of any other integral submanifold.*

Theorem 9.6 *If G is a Lie group, $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra then there is a Lie subgroup (H, ϕ) of G with $d_{e_H}\phi(T_{e_H}H) = \mathfrak{h}$ and $d_{e_H}\phi : T_{e_H}H \to \mathfrak{h}$ an isomorphism of Lie algebras.*

Proof If G is a Lie group, $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra, and $\mathcal{D}$ the left invariant extension of $\mathfrak{h}$ then $\mathcal{D}$ is involutive so we let (H, ϕ) be the corresponding leaf of $\mathcal{D}$ with $e_G \in \phi(H)$.

We know $\phi : H \to G$ is smooth, injective with injective differential. We need to show H is a group in a way which is smooth and making ϕ a Lie group homomorphism.

First step is to show $\phi(H)$ is a subgroup of G. Take $g \in \phi(H)$. The left invariance of $\mathcal{D}$ means left translates of integral manifolds are integral manifolds. Consider $(H, L_{g^{-1}} \circ \phi)$ which is a submanifold of G and an integral submanifold. Also $e = L_{g^{-1}}(g) \in L_{g^{-1}}(\phi(H))$ so $L_{g^{-1}}(\phi(H)) = \phi(H)$ hence $\phi(H)$ is a subgroup of G.

We make H into a group by defining $\phi(h_1h_2) = \phi(h_1)\phi(h_2)$. We **claim** this makes H into a Lie subgroup.

Pick a supplement m for $\mathfrak{h}$ in $\mathfrak{g}$, so $\mathfrak{g} = \mathfrak{h} \oplus m$ and define $\Phi : H \times m \to G$ by

$$\Phi(h, \xi) = \phi(h) \exp \xi$$

Then we have

$$d_{(h,0)}\Phi(X, \eta) = \frac{d}{dt}\Phi(\gamma(t), 0 + t\eta)|_{t=0} = \frac{d}{dt}\phi(\gamma(t)) \exp(t\eta)|_{t=0} = d_h\phi(x) + d_eL_{\phi(h)}\eta$$

and both maps have zero kernel so $d_{(h,0)}\Phi$ has zero kernel as a linear map $T_hH \oplus m \to \mathfrak{g}$ and hence $d_{(h,0)}\Phi$ is an isomorphism.

Hence by inverse function theorem there is a neighbourhood U^h of h in H, W^h of 0 in m and V^h of $\phi(h)$ in G with $V^h = \Phi(U^h \times W^h)$ and $\Phi : U^h \times W^h \to V^h$ a diffeomorphism.

Let Φ^{-1} on V^h be given by maps $\psi_1^h : V^h \to U^h$ and $\psi_2^h : V^h \to W^h$. These are smooth maps and $\psi_1^h(\phi(h')\exp 0)$ is the first component of $\Phi^{-1}(\phi(h')\exp 0) = \Phi^{-1}(\Phi(h',0)) = (h',0)$ and so

$$\psi_1^h(\phi(h')) = h'$$

for $h' \in U^h$. Let $h_1, h_2 \in H$ then for h_1' near h_1 and h_2' near h_2 then

$$h_1'h_2' = \psi_1^{h_1'h_2'}(\phi(h_1'h_2')) = \psi_1^{h_1'h_2'}(\phi(h_1')\phi(h_2'))$$

and this is a composition of smooth functions. Then

$$m_H(h_1', h_2') = \psi_1^{h_1'h_2'}(m_H(\phi(h_1'), \phi(h_2')))$$

on $(U^h \times U^h) \cap (m_G \circ \phi \times \phi)^{-1}(U^{h_1h_2})$. This is an open set and contains h_1h_2 so m_H is smooth on $H \times H$. Similarly $i_H : H \to H$ is smooth. Let $\phi(e_H) = e$ and so $d_{e_H}(T_{e_H}H) = \mathfrak{h} = \mathcal{D}_e$. *Q.E.D.*

An application of this is that every Lie algebra (real, finite dimensional) is the Lie algebra of a Lie group, up to isomorphism. The method to show this uses representation theory.

Definition 9.7 *A* **representation** *of a Lie algebra $\mathfrak{g}$ is a homomorphism $\sigma : \mathfrak{g} \to gl(n, \mathbb{R})$ where n is called the dimension of the representation.*

$$\sigma([\xi, \eta]) = \sigma(\xi)\sigma(\eta) - \sigma(\eta)\sigma(\xi)$$

Note $gl(n, \mathbb{R}) = M_{n\times n}(\mathbb{R}) \subset M_{n\times n}(\mathbb{C})$.

Definition 9.8 *A representation is* **faithful** *if $\ker \sigma = \{0\}$.*

Thus for a faithful representation $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $gl(n, \mathbb{R})$.

Theorem 9.9 (Ado) *Every finite dimensional real Lie algebra has a faithful representation.*

A consequence of this is every finite dimensional Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $gl(n, \mathbb{R})$ for some n. Hence there is a Lie subgroup $\phi : H \to GL(n, \mathbb{R})$ whose Lie algebra $\mathfrak{h}$ is isomorphic to $\mathfrak{g}$. Thus H is a Lie group whose Lie algebra is $\mathfrak{g}$. Lie subgroups of $GL(n, \mathbb{R})$ are called linear Lie groups. For example all classical matrix groups are linear. Also note $SL(2n, \mathbb{R})$ has the same Lie algebra as many non-linear Lie groups.

10 Lie Group topics

- Actions of Lie groups. G acts on $\mathcal{M}$ becomes a map $\sigma : G \times \mathcal{M} \to \mathcal{M}$ by $\sigma(g, x) = g \cdot x$ is a smooth map. G is a symmetry group of $\mathcal{M}$. For example rotations are the symmetry group of sphere. $O(n+1)$ acting on S^n is the symmetry group of spherical geometry. This is an example of a homogeneous geometry. $S^n = O(n+1)/O(n)$.
- Integration on Lie groups
- Representation theory
- Internal symmetry groups.