Interacting Bose gas on the Lattice

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Abstract

A Lower bound for the density of a Bose gas in $\mathbb{Z}^d$ in the grand canonical ensemble is calculated for general dimension d, and for $l^1(\mathbb{Z}^d)$ potential interactions. The rigorous formalism of the measure on continuous time random walks, analogous to the Wiener measure, is briefly explained, along with justification of the Feynman-Kac formula using this. These are then used to bound the density, although with a flawed lemma. Throughout, attempts to further understand the discrete Gaussian are made.

Contents

1 Introduction

In 1995 the first Bose-Einstein condensate was produced by Eric Cornell and Carl Wieman, using a gas of Rubidium atoms. A Bose-Einstein condensate is a state of matter occurring in Bosonic gases with temperatures near absolute zero in which quantum effects become apparent on a macroscopic scale. This state, predicted by Einstein and Bose in 1925, is characterised, upon cooling, by the atoms falling into the lowest energy state. This happens at a critical temperature, T_c, below which all the particles are in the lowest energy state, and above, none are.

In this work I consider a bosonic gas that is able to move on the integer lattice $\mathbb{Z}^d$. The interactions are given by the Bose-Hubbard model, which is a simplistic model, but nonetheless useful. It can be used to describe motion on an optical lattice. These are typically created as an interference pattern of laser beams, giving a spatially periodic pattern. This potential field then traps the atoms, and resembles the atom distribution of a crystalline solid.

Atoms near absolute zero in an optical lattice are considered as a standard realisation of the Bose-Hubbard model, and this model can also be used in quantum computing.

One can relate the critical temperature to a critical fugacity z_c. Then it suffices to find a lower bound on the density of the system in order to find bounds on the critical fugacity.

This work finds a lower bound on this density as follows:

Theorem 1.0.1 *For a Bose gas in $\Lambda \subset \mathbb{Z}^d$ in the grand canonical ensemble with Hamiltonian $H_{\Lambda,N} = \Delta_D + U$ with $U \in l^1(\mathbb{Z}^d)$ the density is bounded below by*

$$\rho(z) \geq \rho^{(0)}(z) - 2z^2 g_{2\beta}^2(0) a_0 \beta - a_0 \beta z^2 [3q_0(z)q_1(z) + q_0^2(z) + 2q_1(z)M + 3q_0(z)M]$$

where

$$M = \frac{1}{\left| \frac{1}{2\pi} \int\limits_{[-\pi,\pi]^d} \left(1 - \frac{4\beta}{d} \sum_{j=1}^d (1 - \cos k_j)\right) e^{ik(x-y)} dk \right|}$$

2 Introductory Material

In this section we introduce the necessary tools to be used throughout the next section.

2.1 Discrete Gaussian

For background work on random walks and Markov chains, the reader should see Lawler [4] or Grimmett and Stirzaker [3]. We have a Markov chain with increment distribution $p : \mathbb{Z}^d \to \mathbb{R}$ where $p(e) = \frac{1}{2d}$, for e a unit vector in $\mathbb{Z}^d$, and $p(x) = 0$ for x not a unit vector. In other words we assign the probability of $1/2d$ to move to a neighbouring point and probability of 0 for anywhere else. We define $p(x, y) := p(y-x)$. We then have, for $x, y \in \mathbb{Z}^d$ $x \neq y$ and Δt small, that

$$\mathbb{P}(S_{t+\Delta t} = y | S_t = x) := p(x, y)\Delta t + o(\Delta t) \tag{2.1.1}$$

by definition. In other words, we have that the probability that the state space at time $t + \Delta t$ is y given that the state space was x at time t is linear in the time period Δt up to small perturbations. One also has, following immediately from the above, that

$$\mathbb{P}(S_{t+\Delta t} = x | S_t = x) = 1 - \sum_{y \neq x} p(x, y)\Delta t + o(\Delta t) \tag{2.1.2}$$

Henceforth, we denote by $g_t(x)$ the discrete Gaussian in space variable x and time t. This is the transition probability for the simplest type of random walk, the simple random walk. Observe here that one could combine the above into one line with a definition

$$\mathbb{P}(S_{t+\Delta t} = y | S_t = x) = \delta_{x,y} + \Delta t \tilde{p}(x, y) + O(\Delta t^2)$$

although here $\tilde{p}$ is defined slightly differently to the above p.

Definition 2.1.1 *Suppose that $\mathbb{P}$ is the probability as above. Then we define the discrete Gaussian as*

$$g_{t_2-t_1}(x_2 - x_1) := \mathbb{P}(\omega(t_2) = x_2 | \omega(t_1) = x_1) \tag{2.1.3}$$

By computing the above probability explicitly we get.

Theorem 2.1.2 *The discrete Gaussian in $\mathbb{Z}^d$, for $x_1, x_2 \in \mathbb{Z}^d$ and $t_1, t_2 \in T$, two "times" is given by*

$$g_{t_2-t_1}(x_2 - x_1) = e^{-(t_2-t_1)} \sum_{N \geq 0} \frac{(t_2 - t_1)^N}{(2d)^N N!} \sum_{y_1,\ldots,y_N} \mathbb{1}_{y_1=x_1} \mathbb{1}_{y_N=x_2} \prod_{i=2}^{N} \mathbb{1}_{||y_i-y_{i-1}||=1}$$

While it is not used in this work, it should be noted that it is expected that, for fixed x, $g_t(x) \approx \pi_t(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2t}}$ for large times t.

Lemma 2.1.3 *For all $x \in \mathbb{Z}^d$.*

$$g_t(x) \leq g_t(0)$$

Proof

The Fourier transform of the discrete Gaussian (see subsection 2.2) is

$$\widehat{g}_t(k) = e^{-\sum_{j=1}^{d}(1-\cos k_j)t}$$

and then using this we get

$$\begin{aligned} g_t(x) &= |g_t(x)| \\ &= \left| \frac{1}{2\pi} \int\limits_{[-\pi,\pi]^d} e^{-\frac{1}{d}\sum_{j=1}^{d}(1-\cos k_j)t} e^{ikx} dk \right| \\ &\leq \frac{1}{2\pi} \int\limits_{[-\pi,\pi]^d} \left| e^{-\frac{1}{d}\sum_{j=1}^{d}(1-\cos k_j)t} \right| dk \\ &= g_t(0) \end{aligned}$$

as the exponential term is real and positive so we can remove the modulus. *Q.E.D.*

Lemma 2.1.4

$$\sum_{x\in\mathbb{Z}^d} g_t(x) = 1$$

Lemma 2.1.5

$$g_{t+s}(x) = \sum_{y\in\mathbb{Z}^d} g_s(y) g_t(x-y)$$

The proofs of the previous two lemmas are just an exercise in rearranging sums.

Definition 2.1.6 *I define the* **discrete Laplacian** *to be*

$$\Delta_D f(x) := \sum_{y\tilde{x}} (f(y) - f(x))$$

2.2 Discrete Gaussian and Fourier Transform

For our random walk, the generator is the discrete Laplacian, up to a constant, as seen below.

It was mentioned to me that it would be possible to find a nicer formula for the discrete Gaussian using Fourier analysis. To this end I introduce the Fourier transform and do some work with this.

Definition 2.2.1 *The discrete Fourier transform of a function $\phi : \mathbb{Z}^d \to \mathbb{C}$ is defined to be a function $\mathcal{F}(\phi) : [-\pi,\pi]^d \to \mathbb{C}$ such that*

$$\hat{\phi}(k) = \mathcal{F}(\phi)(k) := \sum_{x\in\mathbb{Z}^d} \phi(x) e^{-ikx}$$

In analogy with the Fourier transform on $[-\pi,\pi]$ where the Fourier coefficients are sums over $\mathbb{Z}$, our inverse Fourier transform will be

$$\mathcal{F}^{-1} f(k) = \frac{1}{2\pi} \int\limits_{[-\pi,\pi]^d} f(k) e^{ikx} dk \qquad (2.2.1)$$

Lemma 2.2.2

$$-\frac{1}{2}\Delta_D = \mathcal{F}^{-1}(\sum_{i=1}^{d}(1-\cos k_i))\mathcal{F}$$

Proof

The statement above is equivalent to $-\frac{1}{2}\mathcal{F}\Delta_D = \sum_{i=1}^{d}(1-\cos k_i)\mathcal{F}$ and we proceed to prove this statement.

$$\begin{aligned}
-\frac{1}{2}\mathcal{F}\Delta_D\phi(k) &= -\frac{1}{2}\mathcal{F}\sum_{y\sim x}(\phi(y)-\phi(x))(k) \\
&= -\frac{1}{2}\sum_{x\in\mathbb{Z}^d}e^{-ikx}\sum_{y\sim x}(\phi(y)-\phi(x)) \\
&= -\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{h}e^{-ikx}(\phi(x+h)-\phi(x)) \\
&= -\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{h}\phi(x)(e^{-ik(x-h)}-e^{-ikx}) \\
&= \frac{1}{2}\sum_{h}(1-e^{ikh})\sum_{x\in\mathbb{Z}^d}\phi(x)e^{-ikx} \\
&= \frac{1}{2}\sum_{h}(1-e^{ikh})\mathcal{F}\phi(k) \\
&= \frac{1}{2}\sum_{j=1}^{d}(2-(e^{ik_j}+e^{-ik_j}))\mathcal{F}\phi(k) \\
&= \sum_{j=1}^{d}(1-\cos k_j)\mathcal{F}\phi(k)
\end{aligned}$$

By summing over all h, we mean summing over all possible unit vectors in $\mathbb{Z}^d$ in both positive and negative directions. Then kh for each h gives the element k_j of k if h is the jth unit vector of $\mathbb{Z}^d$.

Q.E.D.

We now introduce the following:

Definition 2.2.3 *This is from [4]. We define the difference operators for a function* $f:\mathbb{Z}^d\to\mathbb{R}$ *by*

$$\nabla_x f(y) = f(y+x) - f(y)$$

$$\nabla_x^2 f(y) = \frac{1}{2}f(y+x) + \frac{1}{2}f(y-x) - f(y)$$

We define the **generator** *of a random walk, defined by increment distribution* $p:\mathbb{Z}^d\to\mathbb{R}$ *to be*

$$\mathcal{L}_p f(y) = \sum_{x\in\mathbb{Z}^d}p(x)\nabla_x f(y)$$

One should observe that the transition probabilities, or in the simple random walk the discrete Gaussian, for the random walk satisfies a type of " Heat equation",

$$\frac{d}{dt}g_t(x) = \mathcal{L}_p g_t(x) \tag{2.2.2}$$

In our case we have $\mathcal{L}_p = \frac{1}{2d}\Delta_D$, and if one Fourier transforms this equation, one gets:

$$\mathcal{F}\left(\frac{d}{dt}g_t(x)\right)(k) = \mathcal{F}\frac{1}{2d}\Delta_D g_t(x)$$

and so, denoting $G_t(k) = \mathcal{F}(g_t(x))(k) = \hat{g}_t(k)$ one obtains the differential equation

$$\frac{d}{dt}G_t(k) = -\frac{1}{d}\sum_{j=1}^{d}(1-\cos k_j)G_t(k)$$

and this has the solution

$$G_t(k) = G_t(0)e^{-\frac{1}{d}\sum_{j=1}^{d}(1-\cos k_j)t}$$

where $G_t(0) = \mathcal{F}(g_t(x))(0) = \sum_{\mathbb{Z}^d} g_t(x) = 1$. Then we get that

$$g_t(x) = \frac{1}{2\pi}\int\limits_{[-\pi,\pi]^d} e^{-\frac{1}{d}\sum_{j=1}^{d}(1-\cos k_j)t}e^{ikx}dk \tag{2.2.3}$$

Recall the convolution theorem for Fourier transform, namely

Lemma 2.2.4

$$\mathcal{F}\left(\sum_{y\in\mathbb{Z}^d} g(x-y)f(y)\right)(k) = \mathcal{F}(g)(k)\mathcal{F}(f)(k)$$

2.3 Measure on continuous time Random walks

We first give a brief method on the construction of the analogue to the Wiener measure on continuous time random walks. This is done in [2, pp. 331-335] for the Wiener measure.

We have a space of functions $\omega : [0,t] \to \mathbb{Z}^d$, although we can think of this space as $\prod_{[0,t]}\mathbb{Z}^d$ and we have, for $\theta = (t_1, ..., t_N)$ a projection $h_\theta(\omega) = (\omega(t_1), ..., \omega(t_N))$. We then define a process by the finite distributions

$$d\mu_{t_1,...,t_N}(x_1, ..., x_N) = g_{t_1}(x_1)g_{t_2-t_1}(x_2-x_1)...g_{t_N-t_{N-1}}(x_N-x_{N-1})g_{t-t_N}(y-x_N)$$

and then by a celebrated theorem of Kolmogorov there is a unique measure on the space of functions that agree with these distributions, given certain conditions on the distributions, that are satisfied. We have just explained existence of the following:

Definition 2.3.1 *We denote by $W^t_{x,y}$ the conditional discrete Wiener measure on the space $\Omega_{x,y}$ of paths from x to y in time t.*

In direct analogy to this we can do the same, but without the restriction of the end-point. This would give

Definition 2.3.2 *We denote by W_x^t the discrete Wiener measure on the space Ω_x of paths starting at x and finishing in time t.*

There is another way in which to formulate the existence of such a measure, and uses the Riesz-Markov representation theorem (the theorem found in [1, p.212], method again in [2, pp. 331-335]). This alternative construction gives the formula that, for a function f that depends on n points in time, we have

$$\int f(\omega(t_1), ..., \omega(t_n))dW_{x,y}^t(\omega) = \tag{2.3.1}$$
$$= \sum_{x_1,...,x_n \in \mathbb{Z}^d} f(x_1, ..., x_n)g_{t_1}(x_1)g_{t_2-t_1}(x_2 - x_1)...g_{t_n-t_{n-1}}(x_n - x_{n-1})g_{t-t_n}(y - x_n)$$

For convenience we normalise as follows

$$\int dW_{x,y}^t(\omega) = g_t(y - x) \tag{2.3.2}$$

The following lemma is stated in more generality than needed. We only need it for f a constant function.

Lemma 2.3.3 *We define $\omega \sqcup \omega'$, for $\omega \in \Omega_{x,z}^{t_1}$ and $\omega' \in \Omega_{z,y}^{t_2}$ by*

$$\omega \sqcup \omega'(t) = \begin{cases} \omega(t) & 0 \le t \le t_1 \\ \omega'(t) & t_1 \le t \le t_1 + t_2 \end{cases}$$

and so $\omega \sqcup \omega' \in \Omega_{x,y}^{t_1+t_2}$. Then we get

$$\sum_{z \in \mathbb{Z}^d} \iint f(\omega \sqcup \omega')dW_{x,z}^t(\omega)dW_{z,y}^s(\omega') = \int f(\eta)dW_{x,y}^{t+s}(\eta)$$

for f a bounded function.

Proof We use the measure theory machine. Observe that this is trivially true in the case with f a constant function. Suppose $f(\omega) = a$. Then from the normalisation above, we have

$$\sum_{z \in \mathbb{Z}^d} \iint adW_{x,z}^t(\omega)dW_{z,y}^s(\omega') = \sum_{z \in \mathbb{Z}^d} ag_t(z-x)g_s(y-z) = ag_{t+s}(y-x) = \int adW_{x,y}^{t+s}(\eta)$$

using properties of the discrete Gaussian.

Now suppose that f is a simple function, namely $f(\omega \sqcup \omega') = F(\omega \sqcup \omega'(t_1), ..., \omega \sqcup \omega'(t_N))$ where $F : \mathbb{Z}^{dN} \to \mathbb{R}$. Suppose $t_1, ..., t_i \le t$ and $t \le t_{i+1}, ..., t_N \le t + s$. We get

$$\sum_{z \in \mathbb{Z}^d} \iint F(\omega \sqcup \omega'(t_1), ..., \omega \sqcup \omega'(t_N))dW_{x,z}^t(\omega)dW_{z,y}^s(\omega') =$$
$$= \sum_{x_1,...,x_N,z \in \mathbb{Z}^d} g_{t_1}(x_1 - x)...g_{t_i-t_{i-1}}(x_i - x_{i-1})g_{t-t_i}(z - x_i)g_{t_{i+1}-t}(x_{i+1} - z)\times$$
$$...g_{t_N-t_{N-1}}(x_N - x_{N-1})g_{t+s-t_N}(y - x_N)F(x_1, ..., x_N))$$
$$= \sum_{x_1,...,x_N \in \mathbb{Z}^d} g_{t_1}(x_1 - x)...g_{t_N-t_{N-1}}(x_N - x_{N-1})g_{t+s-t_N}(y - x_N)F(x_1, ..., x_N))$$
$$= \int F(\eta(t_1), ..., \eta(t_N))dW_{x,y}^{t+s}(\eta)$$

As f is bounded, the dominated convergence theorem gives the result for general f, by taking an approximating sequence of simple functions to f. *Q.E.D.*

Lemma 2.3.4 *For* $0 \le t_1 < ... < t_N \le t$ *we have*

$$\int F(\omega(t_1), ..., \omega(t_N)) dW^t_{0,x}(\omega) = \int F(\omega(t_1), ..., \omega(t_N) \mathbb{1}_{\omega(t)=x} dW^t_0(\omega)$$

Proof

$$\begin{aligned}
&\int F(\omega(t_1), ..., \omega(t_N)) \mathbb{1}_{\{\omega(t)=x\}} dW^t_0(\omega) := \\
&\quad = \sum_{x_1,...,x_{N+1}} g_{t_1}(x_1) g_{t_2-t_1}(x_2 - x_1) ... g_{t-t_N}(x_{N+1} - x_N) F(\omega(t_1), ..., \omega(t_N)) \mathbb{1}_{\{\omega(t)=x\}} \\
&\quad = \sum_{x_1,...,x_N} g_{t_1}(x_1) g_{t_2-t_1}(x_2 - x_1) ... g_{t-t_N}(x - x_N) F(\omega(t_1), ..., \omega(t_N)) \\
&\quad =: \int F(\omega(t_1), ..., \omega(t_N)) dW^t_{0,x}(\omega)
\end{aligned}$$

Q.E.D.

Lemma 2.3.5 *The following formula holds for* f *a function on paths from* 0 *in time* t. *The discrete Wiener measure* $W^{a,t}_0$ *represents the usual measure, except the* a *denotes the variance of the paths. By* $\omega + \omega'$ *we mean pointwise summation of the paths. Then*

$$\iint f(\omega + \omega') dW^{1,t}_0(\omega) dW^{1,t}_0(\omega') = \int f(\eta) dW^{2,t}_0(\eta)$$

However, the corresponding formula for the conditional discrete Wiener measure doesn't hold, namely

$$\iint f(\omega + \omega') dW^{1,t}_{0,x}(\omega) dW^{1,t}_{0,y}(\omega') = \frac{g_t(x) g_t(y)}{g_{2t}(y + x)} \int f(\eta) dW^{2,t}_{0,x+y}(\eta)$$

is not true.

Proof

For the first formula, observe that it suffices to consider f of the form

$$F(x_1, ..., x_N) = e^{-i \sum_{j=1}^N k_j x_j}$$

as any such f can be approximated by the simple F depending on N points, and the simple F can be approximated by objects of the form of the right hand side. Then, for $N = 1$ we get

$$\begin{aligned}
\iint F(\omega(t_1) + \omega'(t_1)) dW^{1,t}_0(\omega) dW^{1,t}_0(\omega') &= \sum_{x_1,y_1} e^{-ik_1x_1 - ik_1y_1} g_{t_1}(x_1) g_{t_1}(y_1) \\
&= \hat{g}_{t_1}(k_1) \hat{g}_{t_1}(k_1) \\
&= \hat{g}_{2t_1}(k_1) \\
&= \sum_{z_1} e^{-ik_1z_1} g_{2t_1}(z_1) \\
&= \int F(\eta) dW^{2,t}_0(\eta)
\end{aligned}$$

Now for $N = 2$ we get

$$\begin{aligned}
&\iint F(\omega(t_1)+\omega'(t_1),\omega(t_2)+\omega'(t_2))dW_0^{1,t}(\omega)dW_0^{1,t}(\omega') = \\
&= \sum_{x_1,y_1,x_2,y_2} e^{-ik_1x_1-ik_1y_1-ik_2x_2-ik_2y_2} g_{t_1}(x_1)g_{t_1}(y_1)g_{t_2-t_1}(x_2-x_1)g_{t_2-t_1}(y_2-y_1) \\
&= \sum_{x_1,y_1} e^{-ik_1x_1-ik_1y_1} g_{t_1}(x_1)g_{t_1}(y_1)\hat{g}_{t_2-t_1}(k_2)e^{ik_2(x_1+y_1)}\hat{g}_{t_2-t_1}(k_2) \\
&= (\hat{g}_{t_2-t_1}(k_2))^2(\hat{g}_{t_1}(k_1-k_2))^2 \\
&= \hat{g}_{2t_2-2t_1}(k_2)\hat{g}_{2t_1}(k_1-k_2)
\end{aligned}$$

and then performing operations as in $N = 1$ case we get back to

$$\sum_{z_1,z_2} e^{-ik_1z_1-ik_2z_2} g_{2t_2-2t_1}(z_2-z_1)g_{2t_1}(z_1)$$

which is the value we want.

This suggests the general formula for this specific F of the form

$$\begin{aligned}
&\iint F(\omega(t_1)+\omega'(t_1),...,\omega(t_N)+\omega'(t_N))dW_0^{1,t}(\omega)dW_0^{1,t}(\omega') \\
&\quad = \hat{g}_{t_N-t_{N-1}}(k_N)\hat{g}_{t_{N-1}-t_{N-2}}(k_{N-1}-k_N)\ldots\hat{g}_{t_2-t_1}\left(\sum_{j=2}^{N}(-1)^j k_j\right)\times \\
&\qquad\qquad \times\hat{g}_{t_1}\left(\sum_{j=1}^{N}(-1)^{j+1}k_j\right)
\end{aligned}$$

and if true, then using the convolution properties of the Fourier transform we get the desired result. We prove this by induction. Suppose it is true for $N = L$ and all functions F of the form $e^{-\sum m_jx_j}$ for some constants m_j. Then

$$\begin{aligned}
&\iint F(\omega(t_1)+\omega'(t_1),...,\omega(t_{L+1})+\omega'(t_{L+1}))dW_0^{1,t}(\omega)dW_0^{1,t}(\omega') \\
&= \sum_{\substack{x_1,...,x_{L+1}\\ y_1,...,y_{L+1}}} e^{-i\sum_{j=1}^{L+1}k_j(x_j+y_j)} g_{t_1}(x_1)...g_{t_{L+1}-t_L}(x_{L+1}-x_L)\times \\
&\qquad\qquad \times g_{t_1}(y_1)...g_{t_{L+1}-t_L}(y_{L+1}-y_L) \\
&= \hat{g}_{t_{L+1}-t_L}(k_{L+1})\sum_{\substack{x_1,...,x_L\\ y_1,...,y_L}} e^{ik_{L+1}(x_L+y_L)}e^{-i\sum_{j=1}^{L}k_j(x_j+y_j)}g_{t_1}(x_1)... \\
&\qquad\qquad \times g_{t-t_{L-1}}(x_L-x_{L-1})g_{t_1}(y_1)...g_{t_L-t_{L-1}}(y_L-y_{L-1})
\end{aligned}$$

Then by invoking the induction hypothesis, we get the desired result. Note that in doing this we take as F the function $e^{ik_{L+1}(x_L+y_L)}e^{-i\sum_{j=1}^{L}k_j(x_j+y_j)}$ which gives the required changes of sign needed in the arguments of the Fourier transformed discrete Gaussians.

For the disproving of the second statement, take $y = -x$. Then $x + y = 0$. Also take $f(\omega) = \mathbb{1}_{\{\eta:\eta(s)=0\ \forall s\}}(\omega)$ and $t = 1$. This function is measurable with respect to the discrete Wiener measures involved. Then on the right hand side, the set of all paths that stay always at zero has strictly positive measure, and so the value of $\int f dW^1_{0,0} > 0$. On the left hand side however, the set of paths $\omega + \omega'$ that stay at zero has zero measure, as all the jumps ω makes must happen at exactly the same time the jumps in ω' happen. Thus the left hand side is zero. This is thus a counter example of the statement. *Q.E.D.*

The latter being false here is serious for this work, see lemma 2.4.2.

2.4 Feynman-Kac formula

Now suppose we want to consider the motion of M particles in a subset Λ of the integer lattice. This motion is governed by a Hamiltonian of the form

$$H_{\Lambda,M} = -\frac{1}{2}\Delta_D + U(x) \tag{2.4.1}$$

where $\Delta_D f(x) = \sum_{y\tilde{x}}(f(y) - f(x))$ is the discrete Laplacian and U is a potential interaction function, typically bounded and a same site type interaction. The type of interaction we consider uses $U \geq 0$ and so is repulsive. This Hamiltonian represents an interaction where two particles interact when they are on neighbouring sites (the discrete Laplacian) with a same site external potential. This can be though of as Bose-Hubbard style interaction (see [5]).

The operator $e^{-\beta H}$ where β is inverse temperature is important statistical mechanically (see the next subsection), and one aims to find a representation of this in the form

$$e^{-\beta H} f(x) = \sum_{y \in \mathbb{Z}^d} L(x,y) f(y) \tag{2.4.2}$$

Proceeding similarly to Feynman's path integral formulation, and using the Trotter product formula (found in [8]), one can find that (Ginibre [2, pp. 344-345] performs the exact method),

$$L(x,y) = \int_{\Omega_{x,y}} e^{-\int_0^\beta U(\omega(s))ds} dW^\beta_{x,y}(\omega) \tag{2.4.3}$$

We set $K(x,y)$ to be the integral kernel of the operator $e^{2\beta\Delta_D} - e^{\beta(2\Delta_D - U)}$, i.e.

$$K(x,y) = \int_{\Omega_{x,y}} \left(1 - e^{-\frac{1}{4}\int_0^{4\beta} U(\omega(s))ds}\right) dW^{4\beta}_{x,y}(\omega) \tag{2.4.4}$$

Definition 2.4.1 *[6, p.5] We introduce the notation, for $\omega, \omega' : [0, 2\beta] \to \mathbb{Z}^d$,*

$$\bar{U}(\omega, \omega') = \frac{1}{2}\int_0^{2\beta} U(\omega(s) - \omega'(s))ds \tag{2.4.5}$$

$$V(\omega) = \sum_{0 \leq l < m \leq k-1} \bar{U}(\omega_i, \omega_j) \tag{2.4.6}$$

$$V(\omega, \omega') = \sum_{l=0}^{k-1}\sum_{l'=0}^{k'-1} \bar{U}(\omega_l, \omega'_{l'}) \tag{2.4.7}$$

If the paths are enumerated $\omega_i, i \in I$ then we write $V_{ij} = V(\omega_i, \omega_j)$.

While I believe the following lemma to be false, although initially both myself and my supervisor believed it to be true (see the last lemma in the previous section), this development came too close to the deadline to enable me to change most of the results which use this. I thus leave it in as it is frequently referred to. I leave the proof in to show how the result seems so feasible.

Lemma 2.4.2 *[6, p.5] For any $x, y, x', y' \in \mathbb{Z}^d$ we have that*

$$\iint 1 - e^{-\bar{U}(\omega,\omega')} dW^{2\beta}_{x,y}(\omega) dW^{2\beta}_{x',y'}(\omega') = H(x,y,x',y')K(x-x',y-y')$$

where $H(x,y,x',y') = \frac{g_{2\beta}(y-x)g_{2\beta}(y'-x')}{g_{4\beta}(y-y'-x+x')}$

Proof The difference $\omega - \omega'$ of two Brownian bridges is again a Brownian bridge, but with double variance. Then

$$\iint \frac{1 - e^{-\bar{U}(\omega,\omega')}}{g_{2\beta}(y-x)g_{2\beta}(y'-x')} dW^{2\beta}_{x,y}(\omega) dW^{2\beta}_{x',y'}(\omega') =$$

$$= \int \frac{1 - e^{-\frac{1}{2}\int_0^{2\beta} U(\omega(2s))ds}}{g_{4\beta}(y-y'-x+x')} dW^{4\beta}_{x-x',y-y'}(\omega)$$

and then by noting that $g_{2\beta}(y-x)g_{2\beta}(y'-x')$ and $g_{4\beta}(y-y'-x+x')$ are independent of the paths in consideration, we get that

$$\iint 1 - e^{-\bar{U}(\omega,\omega')} dW^{2\beta}_{x,y}(\omega) dW^{2\beta}_{x',y'}(\omega') =$$

$$= \frac{g_{2\beta}(y-x)g_{2\beta}(y'-x')}{g_{4\beta}(y-y'-x+x')} \int 1 - e^{-\frac{1}{2}\int_0^{4\beta} U(\omega(s))ds} dW^{4\beta}_{x-x',y-y'}(\omega)$$

$$= H(y-x, y'-x')K(x-x', y-y')$$

as required. *Q.E.D.*

2.5 Statistical Mechanics

If the reader is unfamiliar with statistical mechanics, then I found [7] particularly insightful on the basics.

Statistical mechanics is the study of systems where the particle numbers are too high to consider the system by changes in momentum and position of all particles independently. One instead considers a probability distribution of position and momentum. However, one strives to express this function using macroscopic properties of the system, namely the energy of the system, number of particles and volume in consideration. Of key significance is the entropy S, which is a function dependent on these three things that is maximised in equilibrium.

For our situation, we consider a case called the Grand Canonical ensemble. This is where we assume that our system is able to exchange both energy and particles with a reservoir, and where we assume the reservoir is much larger than the system of interest. We assume that the reservoir is much larger than the system in consideration so that its properties are not significantly affected by relatively small changes in its energy or particle number. The entropy of the reservoir is given by $S = k \ln \Omega_R(E_R, N_R, V_R)$, and the probability distribution is given by

$$P(E,N,V) = \frac{\Omega(E,N,V)\Omega_R(E_T - E, N_T - N, V_R)}{\Omega_T}$$

i.e. is the chance of being in the specified state out of all states possible. Ω is the function

$$\Omega(E,V,N) = \frac{1}{h^{3N}N!}\iint \delta(E - H(q,p))\,\mathrm{d}q\,\mathrm{d}p$$

One can find that the probability in terms of the position and momentum is given by

$$P(p,q) = \frac{1}{Z}\Omega e^{-\beta H_N + \beta\mu N} \tag{2.5.1}$$

where H is the Hamiltonian of the system and $\beta = 1/T$ is inverse temperature. Z is the normalising factor, called the Grand Canonical Partition function. I define the **fugacity** to be $z = e^{\beta\mu}$. If we sum over eigenstates, we have

$$Z = \sum_{N\geq 0}\sum_{H=E}\Omega e^{-\beta H_N + \beta\mu N} = \sum_{N\geq 0} z^N \mathrm{Tr} e^{-\beta H_N} \tag{2.5.2}$$

Definition 2.5.1 *We define the* **pressure** *to be*

$$p(\beta,z) = \frac{1}{\beta|\Lambda|}\ln Z \tag{2.5.3}$$

and we define the **density** *to be*

$$\rho(z) = \beta z\frac{\partial}{\partial z}(p(\beta,z)) \tag{2.5.4}$$

The following is a summary of rewriting the partition function and the density using the Feynman-Kac formula. Due to the fact that the partition function has a trace involved, it makes sense to consider loops, i.e. functions $f : [0,t] \to \mathbb{Z}^d$ for which $f(0) = f(1)$.

We now consider winding loops, as these are important in order to rewrite the density. Let Ω_k be the set of continuous paths $[0, 2\beta k] \to \mathbb{Z}^d$ that are closed, i.e. $\omega(0) = \omega(2\beta k)$. We denote an element of Ω_k by $\underline{\omega} = (x, k, \omega)$ where $x \in \mathbb{Z}^d$ is the starting point, k is the winding number and $\omega(0) = x = \omega(2\beta k)$. For $0 \leq l \leq k-1$ we let ω^l denote the lth leg of $\underline{\omega}$. This is equal to

$$\omega^l(s) = \omega(2\beta l + s)$$

with $0 \leq s \leq 2\beta$. Then let $\Omega = \cup_{k\geq 1}\Omega_k$.

We define the measure μ^k on Ω_k by

$$d\mu_k(\underline{\omega}) = \frac{z^k}{k} dx_{\chi_\Lambda}(\omega) dW^{2\beta k}_{x,x}(\omega) e^{-\sum\limits_{0\leq l<m\leq k-1}\frac{1}{2}\int\limits_0^{2\beta} U(\omega_l(s)-\omega_m(s))ds} \tag{2.5.5}$$

One can extend this onto $\Omega = \cup_{k\geq 1}\Omega$ as follows. While here I have underlined for emphasis that $\underline{\omega}$ and ω are different, henceforth I do not distinguish between the two.

Definition 2.5.2

$$d\mu(\omega) = \sum_{k\geq 1} \frac{z^k}{k} dx_{\chi_\Lambda}(\omega) dW^{2\beta k}_{x,x}(\omega) e^{-V(\omega)}$$

where

$$e^{-V(\omega)} = e^{-\sum\limits_{0\leq l<m\leq k-1} \bar{U}(\omega_l,\omega_m)} = e^{-\sum\limits_{0\leq l<m\leq k-1} \frac{1}{2}\int\limits_0^{2\beta} U(\omega_l(s)-\omega_m(s))ds}$$

One can then rewrite the density using the Feynman-Kac formula to get

Theorem 2.5.3 *[6]*

$$Z = \sum_{n\geq 0} \frac{1}{n!} \int\limits_{\Omega^n} e^{-\sum\limits_{1\leq i<j\leq n} V(\omega_i,\omega_j)} d\mu(\omega_1)...d\mu(\omega_n) \tag{2.5.6}$$

$$\rho(z) = \frac{1}{|\Lambda|Z} \sum_{l\geq 1} \frac{1}{(l-1)!} \int_{\Omega^l} k_1 e^{-\sum\limits_{1\leq i<j\leq l} V_{ij}} d\mu(\omega_1)...d\mu(\omega_l) \tag{2.5.7}$$

It can also be shown that $\rho(z) \leq \rho^{(0)}(z)$ where the (0) signifies that the potential function is everywhere zero. This is called the ideal gas. We can show that

$$\rho^{(0)}(z) = \sum_{n\geq 1} z^n g_{2\beta n}(0) \tag{2.5.8}$$

One can calculate this using the above expression for the density, with $U = 0$, and then noting that most of the terms in the numerator give the partition function, and you are left with the formula above.

2.6 Properties of Functions

It is not clear as to why we study this function presently, but I hope the reader is patient, as all will be clear in section 3.

Definition 2.6.1 *For $\alpha \in \mathbb{R}$, I define the function*

$$q_\alpha(z) = \sum_{n\geq 1} n^\alpha z^n g_{2\beta n}(0)$$

Proposition 2.6.2 (Properties of q_α) *The following are true:*

1. *The function q_α is well defined for $0 \leq z < 1$.*
2. *For any α, as $z \to 1$ the series diverges.*

Proof

1. Clear for $z = 0$ as we have the zero sequence. We now consider the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^\alpha z^{n+1} g_{2\beta(n+1)}(0)}{n^\alpha z^n g_{2\beta n}(0)} = z\left(\frac{n+1}{n}\right)^\alpha \frac{g_{2\beta(n+1)}(0)}{g_{2\beta n}(0)}$$

We have that (at least I expect it to) $g_{2\beta n}(x) \to \infty$ as $n \to \infty$. Thus the ratio $\frac{g_{2\beta(n+1)}(0)}{g_{2\beta n}(0)}$ tends to 1. Thus there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $z\frac{g_{2\beta(n+1)}(0)}{g_{2\beta n}(0)} \leq 1$ for $m \in [0,1)$. Similarly we have an $M \in \mathbb{N}$ such that for all $n \geq M$ we have that $z\left(\frac{n+1}{n}\right)^{\alpha}\frac{g_{2\beta(n+1)}(0)}{g_{2\beta n}(0)} \leq 1$ for $z \in [0,1)$. Thus the ratio of our sequence tends to a value less than 1, and so the series converges.

2. The series diverges at $z = 1$, as then all terms in the sequence are greater than 1, and so the sequence is not null.

   Since $q_\alpha(z)$ converges, we have $n^\alpha z^n g_{2\beta n}(0) \to 0$ and so these must be dominated by a constant, importantly independent of both n and z. Then the Dominated convergence theorem gives

$$\lim_{z\to 1} q_\alpha(z) = \sum_{n\geq 0} \lim_{z\to 1} n^\alpha z^n g_{2\beta n}(0) = \sum_{n\geq 0} n^\alpha g_{2\beta n}(0)$$

   and the final term diverges.

Q.E.D.

Lemma 2.6.3 *Suppose that f is a non negative function. Then*

$$e^{-\sum_{i=1}^n f(x_i)} \geq 1 - \sum_{i=1}^{n}(1 - e^{-f(x_i)})$$

Proof We show that the left hand side is greater than the right hand side initially, and also that the derivative of the LHS is greater than that of the RHS. Thus the LHS is greater than the RHS. We proceed by induction in n. For $n = 1$ we have $e^{-f(x)} \geq 1 - 1 + e^{-f(x)}$ which is obvious.

Suppose true for all $k \leq n-1$. Then we have, for x_n, that at $x_n = 0$ then the inequality holds by induction hypothesis. Now

$$\frac{\partial}{\partial x_n} e^{-\sum_{i=1}^n f(x_i)} = -\frac{\partial f}{\partial x_n} e^{-\sum_{i=1}^n f(x_i)}$$

$$\frac{\partial}{\partial x_n}\left(1 - \sum_{i=1}^{n}(1 - e^{-f(x_i)})\right) = -\frac{\partial f}{\partial x_n} e^{-f(x_n)}$$

but

$$-\frac{\partial f}{\partial x_n} e^{-f(x_n)} \leq -\frac{\partial f}{\partial x_n} e^{-\sum_{i=1}^n f(x_i)}$$

and so the inequality holds. *Q.E.D.*

Lemma 2.6.4

$$e^{-V(\omega)} = e^{-\sum_{0\leq l<m\leq k-1} \bar{U}(\omega_l,\omega_m)} \geq 1 - \sum_{0\leq l<m\leq k-1}(1 - e^{-\bar{U}(\omega_l,\omega_m)})$$

Proof This is a direct application of lemma 2.6.3, taking $f = \bar{U}$. *Q.E.D.*

Lemma 2.6.5 *Suppose that $a > 0$, then*

$$1 - e^{-a} \leq a$$

Proof The differential of the left hand side is e^{-a} and right hand side is 1 with respect to a and $e^{-a} \leq 1$ for all a, so the differential of the left hand side is less than the differential of the right hand side. At $a = 0$, we have $0 \leq 0$ and so the left hand side is always less than or equal to the right hand side. *Q.E.D.*

3 Lower Bound on the Density of the Bose gas

The methods used in this section are similar to the paper by Ueltschi and Seiringer [6, pp.6-7]. We now aim to find a lower bound on the density which can be expressed in terms of the potential function U and the density of the ideal gas, with some other position invariant terms as well. The majority of the work here still holds, but the last line or two of each calculation uses lemma 2.4.2 and so the final results are false.

We now proceed to split up the density term into parts where we have only one path interacting and where we have two paths interacting. In doing this, the interactions for higher numbers of paths become the partition function Z, thus cancelling the Z in the denominator. Once this is achieved, we also strive to remove the dependence on $|\Lambda|$ in the denominator, so that the results hold in the infinite volume limit.

We have from before that

$$\rho(z) = \frac{1}{|\Lambda| Z} \sum_{l \geq 1} \frac{1}{(l-1)!} \int_{\Omega^l} k_1 e^{-\sum\limits_{1 \leq i < j \leq l} V_{ij}} d\mu(\omega_1)...d\mu(\omega_l) \tag{3.0.1}$$

We now bound the exponentials from below i.e. we use lemma 2.6.3 to find a lower bound of this. This is essentially splitting the interactions of the first path with the others, and only bounding those below.

Inputting this into (3.0.1) we get that

$$\rho(z) \geq \frac{1}{|\Lambda| Z} \sum_{l \geq 1} \frac{1}{(l-1)!} \int_{\Omega^l} k_1 \left[1 - \sum_{j=2}^{l} (1 - e^{-V_{1j}}) \right] e^{-\sum\limits_{2 \leq i < j \leq l} V_{ij}} d\mu(\omega_1)...d\mu(\omega_l) \tag{3.0.2}$$

Then by splitting this term we get

$$\begin{aligned} \rho(z) \geq & \frac{1}{|\Lambda| Z} \sum_{l \geq 1} \int_{\Omega^l} \frac{k_1}{(l-1)!} e^{-\sum\limits_{2 \leq i < j \leq l} V_{ij}} d\mu(\omega_1)...d\mu(\omega_l) - \\ & - \frac{1}{|\Lambda| Z} \sum_{l \geq 1} \int_{\Omega^l} \frac{k_1}{(l-1)!} \sum_{j=2}^{l} (1 - e^{-V_{1j}}) e^{-\sum\limits_{2 \leq i < j \leq l} V_{ij}} d\mu(\omega_1)...d\mu(\omega_l) \end{aligned} \tag{3.0.3}$$

We now try to eliminate the partition function from these expressions, so as to get a bound that is valid in the infinite volume case. Now using the definition of Z and separating out the ω_1 and ω_2 terms in the second term we get, using lemma 2.3.3, that

$$\begin{aligned} \rho(z) \geq & \frac{1}{|\Lambda|} \int_{\Omega} k_1 d\mu(\omega_1) - \frac{1}{|\Lambda| Z} \int_{\Omega^2} k_1 (1 - e^{-V_{12}}) d\mu(\omega_1) d\mu(\omega_2) \sum_{l \geq 2} \int_{\Omega^{l-2}} \frac{1}{(l-1)!} \times \\ & \times \sum_{j=3}^{l} (1 - e^{-V_{1j}}) e^{-\sum\limits_{2 \leq i < j \leq l} V_{ij}} d\mu(\omega_3)...d\mu(\omega_l) \end{aligned} \tag{3.0.4}$$

and so we have, since V is repulsive (positive), and thus $-(1 - e^{-V}) \geq -e^0 = -1$ and so

$$\begin{aligned} \rho(z) \geq & \frac{1}{|\Lambda|} \int_{\Omega} k_1 d\mu(\omega_1) - \frac{1}{|\Lambda| Z} \int_{\Omega^2} k_1 (1 - e^{-V_{12}}) d\mu(\omega_1) d\mu(\omega_2) \times \\ & \times \sum_{n \geq 0} \frac{1}{n!} \int_{\Omega^n} e^{-\sum\limits_{1 \leq i < j \leq n} V_{ij}} d\mu(\omega_1)...d\mu(\omega_n) \end{aligned} \tag{3.0.5}$$

and noting that the part in the last sum is again the partition function, we get:

$$\rho(z) \geq \frac{1}{|\Lambda|}\int_{\Omega} k_1 d\mu(\omega_1) - \frac{1}{|\Lambda|}\int_{\Omega^2} k_1(1-e^{-V_{12}})d\mu(\omega_1)d\mu(\omega_2) \tag{3.0.6}$$

Then the first term above becomes, using lemma 2.6.4 (where we note for $k = 0, 1$ that we have no terms),

$$\begin{aligned}
\frac{1}{|\Lambda|}\int_{\Omega} k_1 d\mu(\omega_1) &\geq \frac{1}{|\Lambda|}\sum_{k\geq 1} z^k \sum_{x\in\Lambda}\int dW_{x,x}^{2\beta k}(\omega)\left[1 - \sum_{0\leq l<m\leq k-1}(1-e^{-\bar{U}(\omega_l,\omega_m)})\right] \\
&= \frac{1}{|\Lambda|}\sum_{k\geq 1} z^k \sum_{x\in\Lambda}\int dW_{x,x}^{2\beta k}(\omega) - \\
&\qquad - \frac{1}{|\Lambda|}\sum_{k\geq 2} z^k \sum_{x\in\Lambda}\int dW_{x,x}^{2\beta k}(\omega)\sum_{0\leq l<m\leq k-1}(1-e^{-\bar{U}(\omega_l,\omega_m)}) \\
&= \frac{1}{|\Lambda|}\sum_{k\geq 1} z^k \sum_{x\in_\Lambda} g_{2\beta k}(x-x) - \\
&\qquad \underbrace{- \frac{1}{|\Lambda|}\sum_{k\geq 2} z^k \sum_{x\in\Lambda}\int dW_{x,x}^{2\beta k}(\omega)\sum_{0\leq l<m\leq k-1}(1-e^{-\bar{U}(\omega_l,\omega_m)})}_{=:A} \\
&= \rho^{(0)}(z) - A
\end{aligned} \tag{3.0.7}$$

We now note that the second term above is

$$\begin{aligned}
\frac{1}{|\Lambda|}\int_{\Omega^2} k_1(1-e^{-V_{12}})d\mu(\omega_1)d\mu(\omega_2) &\leq \frac{1}{|\Lambda|}\int_{\Omega^2} k_1(1-e^{-\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\bar{U}(\omega_{1,i},\omega_{2,j})})d\mu(\omega_1)d\mu(\omega_2) \\
&= \frac{1}{|\Lambda|}\int_{\Omega^2} k_1(1-e^{-k_1k_2\bar{U}(\omega_{1,1},\omega_{2,1})})d\mu(\omega_1)d\mu(\omega_2) \\
&= \frac{1}{|\Lambda|}\int_{\Omega^2} k_1\left(1-e^{-\sum_{i=1}^{k_1k_2}\bar{U}(\omega_{1,1},\omega_{2,1})}\right)d\mu(\omega_1)d\mu(\omega_2) \\
&\leq \frac{1}{|\Lambda|}\int_{\Omega^2} k_1\left(1-\left(1-\sum_{i=1}^{k_1k_2}(1-e^{-\bar{U}(\omega_{1,1},\omega_{2,1})})\right)\right)d\mu(\omega_1)d\mu(\omega_2) \\
&\leq \frac{1}{|\Lambda|}\int_{\Omega^2} k_1^2k_2(1-e^{-\bar{U}(\omega_{1,1},\omega_{2,1})})d\mu(\omega_1)d\mu(\omega_2) \\
&=: B
\end{aligned} \tag{3.0.8}$$

where we multiply by k_1k_2 because we can reorder the points to get $\omega_{1,1}$ and $\omega_{2,1}$ to be whichever leg we want.

We now bound A and B from above and summarise in the following two subsections.

3.1 Bounding the "A" term

We first proceed to split A up into different parts as follows:

$$\begin{aligned}
A &= \frac{1}{|\Lambda|}\sum_{k\geq 2} z^k \sum_{x\in\Lambda}\int dW^{2\beta k}_{x,x}(\omega) \sum_{0\leq l<m\leq k-1}(1-e^{-\bar{U}(\omega_l,\omega_m)}) \\
&= \frac{z^2}{|\Lambda|}\sum_{x\in\Lambda}\int dW^{4\beta}_{x,x}(\omega)\sum_{0\leq l<m\leq 1}(1-e^{-\bar{U}(\omega_l,\omega_m)})+ \\
&\qquad + \frac{1}{|\Lambda|}\sum_{k\geq 3} z^k \sum_{x\in\Lambda}\int dW^{2\beta k}_{x,x}(\omega) \sum_{0\leq l<m\leq k-1}(1-e^{-\bar{U}(\omega_l,\omega_m)}) \\
&= A_1 + \frac{1}{|\Lambda|}\sum_{k\geq 3} z^k \sum_{x\in\Lambda}\int dW^{2\beta k}_{x,x}(\omega)\times \\
&\qquad \times \sum_{0\leq l<m\leq k-1}(1-e^{-\bar{U}(\omega_l,\omega_m)})\left(\mathbb{1}_{\{l=m+1\}}+\mathbb{1}_{\{l\neq m+1\}}\right) \\
&= A_1 + \frac{1}{|\Lambda|}\sum_{k\geq 3} z^k \sum_{x\in\Lambda}\int dW^{2\beta k}_{x,x}(\omega) \sum_{0\leq l<m\leq k-1}(1-e^{-\bar{U}(\omega_l,\omega_m)})\mathbb{1}_{\{l=m+1\}}+ \\
&\qquad + \frac{1}{|\Lambda|}\sum_{k\geq 3} z^k \sum_{x\in\Lambda}\int dW^{2\beta k}_{x,x}(\omega) \sum_{0\leq l<m\leq k-1}(1-e^{-\bar{U}(\omega_l,\omega_m)})\mathbb{1}_{\{l\neq m+1\}} \\
&= A_1 + A_2 + A_3
\end{aligned}$$

In other words we have A_1 where $k = 2$, A_2 where we have at least winding number of 3 and the interactions are between neighbouring parts of the path, and A_3 where the winding number is at least 3 and the interactions are not between neighbouring parts of the path. It should be clear that these completely describe all such paths. It is not obvious why we split the terms up like this at first sight, but I hope on viewing the simplicity of the resulting bounds that the reader will see the justification of this method.

We bound each A_i separately.

Lemma 3.1.1

$$A_1 \leq z^2 \frac{g^2_{2\beta}(0)}{g_{4\beta}(0)} \sum_{x\in\mathbb{Z}^d} K(x,-x)$$

Proof We can write A_1 as follows, since we have two legs in the path ω starting and ending at x and so we have

$$A_1 = \frac{z^2}{|\Lambda|}\sum_{x\in\Lambda}\int dW^{4\beta}_{x,x}(\omega)(1-e^{-\bar{U}(\omega_1,\omega_2)}) \tag{3.1.1}$$

Then by splitting this into its two constituent parts, by introducing x_2 as an intermediary point in between the two paths over time 2β and rewriting the interaction term (i.e. we use lemma 2.3.3) we get

$$A_1 = \frac{z^2}{|\Lambda|}\sum_{x_1,x_2\in\Lambda}\int dW^{2\beta}_{x_1,x_2}(\omega_1)dW^{2\beta}_{x_2,x_1}(\omega_2)(1-e^{-\bar{U}(\omega_1,\omega_2)}) \tag{3.1.2}$$

Now by using lemma 2.4.2 we get

$$A_1 = \frac{z^2}{|\Lambda|} \sum_{x_1,x_2 \in \Lambda} K(x_1 - x_2, x_2 - x_1) \frac{g_{2\beta}(x_1 - x_2) g_{2\beta}(x_2 - x_1)}{g_{4\beta}(x_1 - x_2 - x_1 + x_2)} \tag{3.1.3}$$

Now expanding the domain of summation to Z^d (the functions are all positive so this only makes the value larger), and performing a change of variables, by setting $x = x_1 - x_2$ and choosing $y = x_2$, we get

$$A_1 \leq \frac{z^2}{|\Lambda|} \sum_{x,y \in \mathbb{Z}^d} K(x, -x) \frac{g_{2\beta}(x) g_{2\beta}(-x)}{g_{4\beta}(0)} \tag{3.1.4}$$

Now using lemma 2.1.3 we get

$$A_1 \leq z^2 \frac{g_{2\beta}^2(0)}{g_{4\beta}(0)} \sum_{x \in \mathbb{Z}^d} K(x, -x) \tag{3.1.5}$$

as required. *Q.E.D.*

Lemma 3.1.2

$$A_2 \leq z^2 [q_1(z) + 2q_0(z)] \sum_{x,y \in \mathbb{Z}^d} \frac{g_{2\beta}(x) g_{2\beta}(y)}{g_{4\beta}(x - y)} K(x, y)$$

where $q_\alpha(z) = \sum_{n \geq 1} n^\alpha z^n g_{2\beta n}(0)$.

Proof The second term is

$$A_2 = \frac{1}{|\Lambda|} \sum_{k \geq 3} z^k \sum_{x \in \Lambda} \int dW_{x,x}^{2\beta k}(\omega) \sum_{0 \leq l < m \leq k-1} \mathbb{1}_{\{m=l+1\}} (1 - e^{-\bar{U}(\omega_l, \omega_m)}) \tag{3.1.6}$$

where the $\mathbb{1}$ is forcing the neighbouring segments of the loop to interact. Now, since $k \geq 3$ we can split the loop from x to x up into a loop from $x_1 = x$ to x_2, x_2 to x_3 then x_3 to x_1, taking 2β for the first two loops and $2\beta(k-2)$ for the last. Then we have

$$\begin{aligned} A_2 = & \frac{1}{|\Lambda|} \sum_{x_1,x_2,x_3 \in \Lambda} \sum_{k \geq 3} z^k \int dW_{x_1,x_2}^{2\beta}(\omega_1) dW_{x_2,x_3}^{2\beta}(\omega_2) dW_{x_3,x_1}^{2\beta(k-2)}(\omega_3) \times \\ & \times \sum_{0 \leq l < m \leq k-1} \mathbb{1}_{\{m=l+1\}} (1 - e^{-\bar{U}(\omega_l, \omega_m)}) \\ = & \frac{1}{|\Lambda|} \sum_{x_1,x_2,x_3 \in \Lambda} \int dW_{x_1,x_2}^{2\beta}(\omega_1) dW_{x_2,x_3}^{2\beta}(\omega_2) \sum_{n \geq 1} z^{n+2} \int dW_{x_3,x_1}^{2\beta n}(\omega_3) \times \\ & \times \sum_{0 \leq l < m \leq n+1} \mathbb{1}_{\{m=l+1\}} (1 - e^{-\bar{U}(\omega_{l,1}, \omega_{m,1})}) \end{aligned} \tag{3.1.7}$$

Since the integrals around the loop are independent of the place we start, and because we integrate over the starting point, we can assume we start at x_1 always, so long as we multiply by the number of segments in the loop, which is $k = n + 2$. We thus get

$$A_2 = \frac{1}{|\Lambda|} \sum_{x_1,x_2,x_3 \Lambda} \int dW_{x_1,x_2}^{2\beta}(\omega_1) dW_{x_2,x_3}^{2\beta}(\omega_2) (1 - e^{-\bar{U}(\omega_1, \omega_2)} \sum_{n \geq 1} (n+2) z^{n+2} g_{2\beta n}(x_1 - x_3) \tag{3.1.8}$$

Then using lemma 2.1.3 and then lemma 2.4.2 we get

$$A_2 \leq \frac{1}{|\Lambda|} \sum_{x_1,x_2,x_3\in\Lambda} \frac{g_{2\beta}(x_1-x_2)g_{2\beta}(x_2-x_3)}{g_{4\beta}(x_1-x_2-(x_2-x_3))} K(x_1-x_2, x_2-x_3) \sum_{n\geq 1}(n+2)z^{n+2}g_{2\beta n}(0) \tag{3.1.9}$$

Then by changing variables by $x = x_1 - x_2$ and $y = x_2 - x_3$ and $w = x_3$ we get

$$A_2 \leq \frac{1}{|\Lambda|} \sum_{x,y\in\mathbb{Z}^d} \sum_{w\in\Lambda} \frac{g_{2\beta}(x)g_{2\beta}(y)}{g_{4\beta}(x-y)} K(x,y) \sum_{n\geq 1}(n+2)z^{n+2}g_{2\beta n}(0) \tag{3.1.10}$$

then by integrating out the w we get

$$A_2 \leq \sum_{x,y\in\mathbb{Z}^d} \frac{g_{2\beta}(x)g_{2\beta}(y)}{g_{4\beta}(x-y)} K(x,y) \sum_{n\geq 1}(n+2)z^{n+2}g_{2\beta n}(0) \tag{3.1.11}$$

Then we first evaluate the sum over n to get

$$\sum_{n\geq 1}(n+2)z^{n+2}g_{2\beta n}(0) = z^2 \left(\sum_{n\geq 1} nz^n g_{2\beta n}(0) + 2\sum_{n\geq 1} z^n g_{2\beta n}(0) \right) = z^2[q_1(z) + 2q_0(z)] \tag{3.1.12}$$

and then using this gives

$$A_2 \leq z^2 \sum_{x,y\in\mathbb{Z}^d} \frac{g_{2\beta}(x)g_{2\beta}(y)}{g_{4\beta}(x-y)} K(x,y)[q_1(z) + 2q_0(z)] \tag{3.1.13}$$

as required *Q.E.D.*

I feel that it is important to note that here there is no simple way in which to remove the quotient of Gaussians from the bound here, unless the function is bounded. We leave this for discussion later though.

Lemma 3.1.3

$$A_3 \leq 2z^2 q_0(z)[q_1(z) + q_0(z)] \sum_{x,y\in\mathbb{Z}^d} K(x,y)$$

Proof We have that

$$A_3 = \frac{1}{|\Lambda|} \sum_{k\geq 3} z^k \sum_{x\in\Lambda} \int dW^{2\beta k}_{x,x}(\omega) \sum_{0\leq l<m\leq k-1} (1 - e^{-\bar{U}(\omega_l,\omega_m)}) \mathbb{1}_{\{l\neq m-1\}} \tag{3.1.14}$$

We can rewrite this as follows. We suppose that the two legs to interact are between x_1 to x_2 and x_3 to x_4. We then have to vary the lengths of x_2 to x_3 and x_4 to x_1. Let the lengths of these two legs be k_1 and k_2 respectively. Then, since starting at any point is the same as starting at x_1, we must multiply by $k = k_1 + k_2 + 2$. We thus get

$$\begin{aligned} A_3 = \frac{1}{|\Lambda|} \sum_{k\geq 3} z^k \mathbb{1}_{\{k_1+k_2+2=k\}} \sum_{x_1,x_2,x_3,x_4\in\Lambda} \int dW^{2\beta}_{x_1,x_2}(\omega_1) \times \\ \times \int dW^{2\beta k_1}_{x_2,x_3}(\omega_2) \int dW^{2\beta}_{x_3,x_4}(\omega_3) \int dW^{2\beta k_2}_{x_4,x_1}(\omega_4)(1 - e^{-\bar{U}(\omega_1,\omega_3)})k \end{aligned} \tag{3.1.15}$$

We can rewrite this by setting k_1 the winding number of ω_2 and k_2 to be the winding number of ω_4. Then the sum over k becomes a sum over k_1 and k_2 and we get

$$A_3 = \frac{1}{|\Lambda|}\sum_{x_1,\ldots,x_4\in\Lambda}\int dW^{2\beta}_{x_1,x_2}(\omega_1)dW^{2\beta}_{x_3,x_4}(\omega_3)(1-e^{-\bar{U}(\omega_1,\omega_3)})\times$$
$$\times\sum_{k_1,k_2\geq 1} z^{k_1+k_2+2}\int dW^{2\beta k_1}_{x_2,x_3}(\omega_2)dW^{2\beta k_2}_{x_4,x_1}(\omega_4)(k_1+k_2+2) \quad (3.1.16)$$

Then by evaluating the integrals over ω_2 and ω_4 we get

$$A_3 = \frac{z^2}{|\Lambda|}\sum_{x_1,\ldots,x_4\in\Lambda}\int dW^{2\beta}_{x_1,x_2}(\omega_1)dW^{2\beta}_{x_3,x_4}(\omega_3)(1-e^{-\bar{U}(\omega_1,\omega_3)})\times$$
$$\times\sum_{k_1,k_2\geq 1}(k_1+k_2+2)z^{k_1+k_2}g_{2\beta k_1}(x_2-x_3)g_{2\beta k_2}(x_4-x_1) \quad (3.1.17)$$

now using lemma 2.1.3 we get

$$A_3 \leq \frac{z^2}{|\Lambda|}\sum_{x_1,\ldots,x_4\in\Lambda}\int dW^{2\beta}_{x_1,x_2}(\omega_1)dW^{2\beta}_{x_3,x_4}(\omega_3)(1-e^{-\bar{U}(\omega_1,\omega_3)})\times$$
$$\times\sum_{k_1,k_2\geq 1}(k_1+k_2+2)z^{k_1+k_2}g_{2\beta k_1}(0)g_{2\beta k_2}(0) \quad (3.1.18)$$

then using lemma 2.4.2 we get

$$A_3 \leq \frac{z^2}{|\Lambda|}\sum_{k_1,k_2\geq 1}(k_1+k_2+2)z^{k_1+k_2}g_{2\beta k_1}(0)g_{2\beta k_2}(0)\times$$
$$\times\sum_{x_1,\ldots,x_4\in\Lambda}\frac{g_{2\beta}(x_1-x_2)g_{2\beta}(x_3-x_4)}{g_{4\beta}((x_1-x_2)-(x_3-x_4))}K(x_1-x_3,x_2-x_4) \quad (3.1.19)$$

by extending the domain of summation and changing variables as follows $x = x_1 - x_3$, $y = x_2 - x_4$, $w = x_3 - x_4$ and another variable suitably chosen we get

$$A_3 \leq \frac{|\Lambda|z^2}{|\Lambda|}\sum_{k_1,k_2\geq 1}(k_1+k_2+2)z^{k_1+k_2}g_{2\beta k_1}(0)g_{2\beta k_2}(0)\times$$
$$\times\sum_{w,x,y\in\mathbb{Z}^d}\frac{g_{2\beta}(x-y-w)g_{2\beta}(w)}{g_{4\beta}(x-y)}K(x,y) \quad (3.1.20)$$

and then using lemma 2.3.3 and then evaluating the sum over k_1, k_2 as follows,

$$\sum_{k_1,k_2\geq 1}(k_1+k_2+2)z^{k_1+k_2}g_{2\beta k_1}(0)g_{2\beta k_2}(0)$$
$$= \sum_{k_1,k_2\geq 1} k_1 z^{k_1+k_2}g_{2\beta k_1}(0)g_{2\beta k_2}(0) + k_1 z^{k_1+k_2}g_{2\beta k_1}(0)g_{2\beta k_2}(0) + 2z^{k_1+k_2}g_{2\beta k_1}(0)g_{2\beta k_2}(0)$$
$$= \sum_{k_1\geq 1}\left(k_1 z^{k_1}g_{2\beta k_1}(0)\left(\sum_{k_2\geq 1} z^{k_2}g_{2\beta k_2}(0)\right)\right) + + \sum_{k_2\geq 1}\left(k_2 z^{k_2}g_{2\beta k_2}(0)\left(\sum_{k_1\geq 1} z^{k_1}g_{2\beta k_1}(0)\right)\right) + 2\sum_{k_1\geq 1}\left(z^{k_1}g_{2\beta k_1}(0)\left(\sum_{k_2\geq 1} z^{k_2}g_{2\beta k_2}(0)\right)\right)$$
$$= 2q_0(z)[q_1(z)+q_0(z)]$$

we get

$$A_3 \leq 2z^2 q_0(z)[q_1(z)+q_0(z)]\sum_{x,y\in\mathbb{Z}^d}\frac{g_{4\beta}(x-y)}{g_{4\beta}(x-y)}K(x,y) \tag{3.1.21}$$

and simplifying we get

$$A_3 \leq 2z^2 q_0(z)[q_1(z)+q_0(z)]\sum_{x,y\in\mathbb{Z}^d}K(x,y) \tag{3.1.22}$$

as required.

Q.E.D.

3.2 Bounding the "B" term

We now proceed to split up B as follows:

$$B = \frac{1}{|\Lambda|}\int_{\Omega^2} k_1^2 k_2 (1-e^{-\bar{U}(\omega_{1,1},\omega_{2,1})})d\mu(\omega_1)d\mu(\omega_2)$$
$$= \frac{1}{|\Lambda|}\sum_{k_1,k_2\geq 1} k_1 z^{k_1+k_2}\sum_{x_1,x_2\in\Lambda}\int dW^{2\beta k_1}_{x_1,x_1}(\omega_1)\times \times\int dW^{2\beta k_2}_{x_2,x_2}(\omega_2)e^{-V(\omega_1)-V(\omega_2)}(1-e^{-\bar{U}(\omega_{1,1},\omega_{2,1})})$$
$$= \frac{z^2}{|\Lambda|}\sum_{x_1,x_2\in\Lambda}\int dW^{2\beta}_{x_1,x_1}(\omega_1)\int dW^{2\beta}_{x_2,x_2}(\omega_2)(1-e^{-\bar{U}(\omega_{1,1},\omega_{2,1})}) +$$
$$+ \frac{1}{|\Lambda|}\sum_{k_1,k_2\geq 2} k_1 z^{k_1+k_2}\sum_{x_1,x_2\in\Lambda}\int dW^{2\beta k_1}_{x_1,x_1}(\omega_1)\times \times\int dW^{2\beta k_2}_{x_2,x_2}(\omega_2)e^{-V(\omega_1)-V(\omega_2)}(1-e^{-\bar{U}(\omega_{1,1},\omega_{2,1})})$$

$$= B_1 + \frac{1}{|\Lambda|}\sum_{k_1\geq 2} k_1 z^{k_1+1} \sum_{x_1,x_2\in\Lambda} \int dW^{2\beta k_1}_{x_1,x_1}(\omega_1)\times$$
$$\times \int dW^{2\beta}_{x_2,x_2}(\omega_2) e^{-V(\omega_1)}(1-e^{-\bar{U}(\omega_{1,1},\omega_{2,1})})+$$
$$+\frac{1}{|\Lambda|}\sum_{k_1,k_2\geq 2} k_1 z^{k_1+k_2} \sum_{x_1,x_2\in\Lambda} \int dW^{2\beta k_1}_{x_1,x_1}(\omega_1)\times$$
$$\times \int dW^{2\beta k_2}_{x_2,x_2}(\omega_2) e^{-V(\omega_1)-V(\omega_2)}(1-e^{-\bar{U}(\omega_{1,1},\omega_{2,1})})$$
$$= B_1 + B_2 + B_3$$

Here we have split B up into parts with both paths having winding number 1, with the second path having winding number 1 and then both having winding number at least 2.

We now bound each B_i separately.

Lemma 3.2.1

$$B_1 \leq z^2 \frac{g^2_{2\beta}(0)}{g_{4\beta}(0)} \sum_{x\in\mathbb{Z}^d} K(x,x)$$

Proof We have that B_1 is equal to the following, as $k_1 = 1 = k_2$.

$$B_1 = \frac{z^2}{|\Lambda|}\sum_{x_1,x_2\in\Lambda}\int dW^{2\beta}_{x_1,x_1}(\omega_1) dW^{2\beta}_{x_2,x_2}(\omega_2)(1-e^{-\bar{U}(\omega_1,\omega_2)}) \tag{3.2.1}$$

Now using lemma 2.4.2 we get that

$$B_1 = \frac{z^2}{|\Lambda|}\sum_{x_1,x_2\in\Lambda} \frac{g_{2\beta}(x_1-x_1)g_{2\beta}(x_2-x_2)}{g_{4\beta}(x_1-x_2-x_1+x_2)} K(x_1-x_2, x_1-x_2) \tag{3.2.2}$$

Then from lemma 2.1.3, and a change of variables, we get

$$B_1 \leq \frac{z^2}{|\Lambda|}\frac{g_{2\beta}(0)g_{2\beta}(0)}{g_{4\beta}(0)} \sum_{x\in\mathbb{Z}^d}\sum_{y\in\Lambda} K(x,x) \tag{3.2.3}$$

and integrating with respect to y gives

$$B_1 \leq z^2 \frac{g^2_{2\beta}(0)}{g_{4\beta}(0)} \sum_{x\in\mathbb{Z}^d} K(x,x) \tag{3.2.4}$$

as required. *Q.E.D.*

Lemma 3.2.2

$$B_2 \leq z^2(q_1(z)+q_0(z)) \sum_{x,y\in\mathbb{Z}^d} \frac{g_{2\beta}(0)g_{2\beta}(x-y)}{g_{4\beta}(x+y)} K(x,y)$$

Proof Suppose that $k_2 = 1$ and that k_1 is the loop that varies over larger winding numbers. Then we have

$$B_2 = \frac{1}{|\Lambda|}\sum_{x_1,x_2\in\Lambda}\int zdW^{2\beta}_{x_2,x_2}(\omega_2)e^{-V(\omega_2)}\sum_{k\geq 2}kz^k dW^{2\beta k}_{x_1,x_1}(\omega_1)(1-e^{-\bar{U}(\omega_{1,1},\omega_2)})e^{-V(\omega_1)} \tag{3.2.5}$$

We can bound the self interaction terms by one to get

$$B_2 \leq \frac{1}{|\Lambda|}\sum_{x_1,x_2\in\Lambda}\int zdW^{2\beta}_{x_2,x_2}(\omega_2)\sum_{k\geq 2}kz^k dW^{2\beta k}_{x_1,x_1}(\omega_1)(1-e^{-\bar{U}(\omega_{1,1},\omega_2)}) \tag{3.2.6}$$

We observe that the interaction term depends only on the first leg of ω_1 and so we isolate that from the rest of ω_1. We use lemma 2.3.3 so we get

$$B_2 \leq \frac{1}{|\Lambda|}\sum_{x_1,x_2\in\Lambda}\int zdW^{2\beta}_{x_2,x_2}(\omega_2)\sum_{k\geq 2}kz^k\sum_{x_3\in\Lambda}\int dW^{2\beta}_{x_1,x_3}(\omega_{1,1})\times$$
$$\times\int dW^{2\beta(k-1)}_{x_3,x_1}(\omega_3)(1-e^{-\bar{U}(\omega_{1,1},\omega_2)}) \tag{3.2.7}$$

Rearranging gives

$$B_2 \leq \frac{1}{|\Lambda|}\sum_{x_1,x_2,x_3\in\Lambda}\int zdW^{2\beta}_{x_2,x_2}(\omega_2)dW^{2\beta}_{x_1,x_3}(\omega_{1,1})(1-e^{-\bar{U}(\omega_{1,1},\omega_2)})\times$$
$$\times\sum_{k\geq 2}kz^k\int dW^{2\beta(k-1)}_{x_3,x_1}(\omega_3) \tag{3.2.8}$$

Using the normalisation of Wiener integrals we have chosen and reordering the sum gives

$$B_2 \leq \frac{1}{|\Lambda|}\sum_{x_1,x_2,x_3\in\Lambda}\int dW^{2\beta}_{x_2,x_2}(\omega_2)dW^{2\beta}_{x_1,x_3}(\omega_{1,1})(1-e^{-\bar{U}(\omega_{1,1},\omega_2)})\times$$
$$\times\sum_{k\geq 1}(k+1)z^{k+2}g_{2\beta k}(x_1-x_3) \tag{3.2.9}$$

Then by using lemma 2.1.3 we get that

$$B_2 \leq \frac{1}{|\Lambda|}\sum_{x_1,x_2,x_3\in\Lambda}\int dW^{2\beta}_{x_2,x_2}(\omega_2)dW^{2\beta}_{x_1,x_3}(\omega_{1,1})(1-e^{-\bar{U}(\omega_{1,1},\omega_2)})\sum_{k\geq 1}(k+1)z^{k+2}g_{2\beta k}(0) \tag{3.2.10}$$

Using lemma 2.4.2 we get that

$$B_2 \leq \frac{1}{|\Lambda|}\sum_{x_1,x_2,x_3\in\Lambda}\sum_{k\geq 1}(k+1)z^{k+2}g_{2\beta k}(0)\frac{g_{2\beta}(0)g_{2\beta}(x_3-x_1)}{g_{4\beta}(x_2-x_1-x_2+x_3)}K(x_2-x_1,x_2-x_3) \tag{3.2.11}$$

Now by setting $x = x_2 - x_1$ and $y = x_2 - x_3$ and $w = x_3$ we get

$$B_2 \leq z^2\sum_{x,y\in\mathbb{Z}^d}\sum_{k\geq 1}(k+1)z^k g_{2\beta k}(0)\frac{g_{2\beta}(0)g_{2\beta}(y-x)}{g_{4\beta}(x-y)}K(x,y) \tag{3.2.12}$$

Thus evaluating the sum over k we get

$$B_2 \leq z^2[q_1(z) + q_0(z)] \sum_{x,y\in\mathbb{Z}^d} \frac{g_{2\beta}(0)g_{2\beta}(x-y)}{g_{4\beta}(x+y)} K(x,y) \tag{3.2.13}$$

as required. *Q.E.D.*

Lemma 3.2.3

$$B_3 \leq z^2 q_0(z)[q_0(z) + q_1(z)] \sum_{x,y\in\mathbb{Z}^d} K(x,y)$$

Proof

$$B_3 = \frac{1}{|\Lambda|} \sum_{k_1\geq 2} k_1 z^{k_1} \sum_{x_1\in\Lambda} \int dW^{2\beta k_1}_{x_1,x_1}(\omega_1) e^{-V(\omega_1)} \times \\ \times \sum_{k_2\geq 2} z^{k_2} \sum_{x_2\in\Lambda} \int dW^{2\beta k_2}_{x_2,x_2}(\omega_2) e^{-V(\omega_2)} (1 - e^{-\bar{U}(\omega_{1,1},\omega_{2,1})}) \tag{3.2.14}$$

then by bounding the self interaction terms by 1 we get

$$B_3 \leq \frac{1}{|\Lambda|} \sum_{k_1\geq 2} k_1 z^{k_1} \sum_{x_1\in\Lambda} \int dW^{2\beta k_1}_{x_1,x_1}(\omega_1) \sum_{k_2\geq 2} z^{k_2} \sum_{x_2\in\Lambda} \int dW^{2\beta k_2}_{x_2,x_2}(\omega_2)(1 - e^{-\bar{U}(\omega_{1,1},\omega_{2,1})}) \tag{3.2.15}$$

If we take out integration over the first leg of each loop and reorder the sum over k_1, k_2 to go from 1 instead of 2 (i.e. we use lemma 2.3.3) to get

$$B_3 \leq \frac{1}{|\Lambda|} \sum_{x_1,\dots,x_4\in\Lambda} \int dW^{2\beta}_{x_1,x_3}(\omega_{1,1}) dW^{2\beta}_{x_2,x_4}(\omega_{2,1})(1 - e^{-\bar{U}(\omega_{1,1},\omega_{2,1})}) \times \\ \times \sum_{k_1,k_2\geq 1} (k_1+1) z^{k_1+k_2+2} \int dW^{2\beta k_1}_{x_3,x_1}(\omega_3) dW^{2\beta k_2}_{x_4,x_2}(\omega_4) \tag{3.2.16}$$

Then by using the normalisation of the Wiener integral we get

$$B_3 \leq \frac{1}{|\Lambda|} \sum_{x_1,\dots,x_4\in\Lambda} \int dW^{2\beta}_{x_1,x_3}(\omega_{1,1}) dW^{2\beta}_{x_2,x_4}(\omega_{2,1})(1 - e^{-\bar{U}(\omega_{1,1},\omega_{2,1})}) \times \\ \times \sum_{k_1,k_2\geq 1} (k_1+1) z^{k_1+k_2+2} g_{2\beta k_1}(x_3 - x_1) g_{2\beta k_2}(x_4 - x_2) \tag{3.2.17}$$

and then by using lemma 2.1.3 we get

$$B_3 \leq \frac{1}{|\Lambda|} \sum_{x_1,\dots,x_4\in\Lambda} \int dW^{2\beta}_{x_1,x_3}(\omega_{1,1}) dW^{2\beta}_{x_2,x_4}(\omega_{2,1})(1 - e^{-\bar{U}(\omega_{1,1},\omega_{2,1})}) \times \\ \times \sum_{k_1,k_2\geq 1} (k_1+1) z^{k_1+k_2+2} g_{2\beta k_1}(0) g_{2\beta k_2}(0) \tag{3.2.18}$$

Then by taking this sum out of the integral and using lemma 2.4.2 we get

$$B_3 \leq \frac{1}{|\Lambda|} \sum_{k_1,k_2 \geq 1} (k_1+1) z^{k_1+k_2+2} g_{2\beta k_1}(0) g_{2\beta k_2}(0) \times \\ \times \sum_{x_1,\ldots,x_4 \in \Lambda} \frac{g_{2\beta}(x_1-x_3) g_{2\beta}(x_2-x_4)}{g_{4\beta(x_1-x_2-x_3+x_4)}} K(x_1-x_2, x_3-x_4) \tag{3.2.19}$$

and then by changing variables to $x = x_1 - x_2$, $y = x_3 - x_4$, $w = x_2 - x_4$ and $h = x_4$ we get

$$B_3 \leq \sum_{k_1,k_2 \geq 1} (k_1+1) z^{k_1+k_2+2} g_{2\beta k_1}(0) g_{2\beta k_2}(0) \sum_{x,y,w \in \mathbb{Z}^d} \frac{g_{2\beta}(x-y-w) g_{2\beta}(w)}{g_{4\beta(x-y)}} K(x,y) \tag{3.2.20}$$

and then by integrating out by h and w from one of the properties of the discrete Gaussian, namely $\sum_{z\in\mathbb{Z}^d} g_t(x-z) g_s(z-y) = g_{t+s}(y-x)$, we get that

$$B_3 \leq \sum_{k_1,k_2 \geq 1} (k_1+1) z^{k_1+k_2+2} g_{2\beta k_1}(0) g_{2\beta k_2}(0) \sum_{x,y \in \mathbb{Z}^d} \frac{g_{4\beta}(x-y)}{g_{4\beta(x-y)}} K(x,y) \tag{3.2.21}$$

Then by rewriting the sum as follows

$$\begin{aligned}
\sum_{k_1,k_2 \geq 1} & (k_1+1) z^{k_1+k_2+2} g_{2\beta k_1}(0) g_{2\beta k_2}(0) = \\
&= z^2 \sum_{k_1 \geq 1} \sum_{k_2 \geq 1} k_1 z^{k_1} z^{k_2} g_{2\beta k_1}(0) g_{2\beta k_2}(0) + z^2 \sum_{k_1 \geq 1} \sum_{k_2 \geq 1} z^{k_1} z^{k_2} g_{2\beta k_1}(0) g_{2\beta k_2}(0) \\
&= z^2 \sum_{k_1 \geq 1} \left(k_1 z^{k_1} g_{2\beta k_1}(0) \left(\sum_{k_2 \geq 1} z^{k_2} g_{2\beta k_2}(0) \right) \right) + \\
&\qquad + z^2 \sum_{k_1 \geq 1} \left(z^{k_1} g_{2\beta k_1}(0) \left(\sum_{k_2 \geq 1} z^{k_2} g_{2\beta k_2}(0) \right) \right) \\
&= z^2 q_0(z) q_1(z) + q_0^2(z)
\end{aligned}$$

we get that

$$B_3 \leq z^2 q_0(z)[q_1(z) + q_0(z)] \sum_{x,y \in \mathbb{Z}^d} \frac{g_{4\beta}(x-y)}{g_{4\beta}(x-y)} K(x,y) \tag{3.2.22}$$

as required. *Q.E.D.*

3.3 Lower Bound Result

Collating all of lemmas 3.1.1-3.1.3 and 3.2.1-3.2.3 we get the following:

Theorem 3.3.1 *The grand canonical density is bounded below by*

$$\begin{aligned}
\rho(z) \geq \rho^{(0)}(z) &- z^2 \frac{g_{2\beta}^2(0)}{g_{4\beta}(0)} \sum_{x\in\mathbb{Z}^d} K(x,-x) \\
&- z^2[q_1(z)+2q_0(z)] \sum_{x,y\in\mathbb{Z}^d} \frac{g_{2\beta}(x)g_{2\beta}(y)}{g_{4\beta}(x-y)} K(x,y) \\
&- 2z^2 q_0(z)[q_1(z)+q_0(z)] \sum_{x,y\in\mathbb{Z}^d} K(x,y) \\
&- z^2 \frac{g_{2\beta}^2(0)}{g_{4\beta}(0)} \sum_{x\in\mathbb{Z}^d} K(x,x) \\
&- z^2[q_1(z)+2q_0(z)) \sum_{x,y\in\mathbb{Z}^d} \frac{g_{2\beta}(0)g_{2\beta}(x-y)}{g_{4\beta}(x-y)} K(x,y) \\
&- z^2 q_0(z)[q_0(z)+q_1(z)] \sum_{x,y\in\mathbb{Z}^d} K(x,y) \\
\geq \rho^{(0)}(z) &- z^2 \frac{g_{2\beta}^2(0)}{g_{4\beta}(0)} \left[\sum_{x\in\mathbb{Z}^d} K(x,-x) + \sum_{x\in\mathbb{Z}^d} K(x,x) \right] \\
&- 3z^2 q_0(z)[q_1(z)+q_0(z)] \sum_{x,y\in\mathbb{Z}^d} K(x,y) \\
&- z^2[q_1(z)+2q_0(z)] \sum_{x,y\in\mathbb{Z}^d} \frac{g_{2\beta}(x)g_{2\beta}(y)}{g_{4\beta}(x-y)} K(x,y) \\
&- z^2[q_1(z)+2q_0(z)] \sum_{x,y\in\mathbb{Z}^d} \frac{g_{2\beta}(0)g_{2\beta}(y-x)}{g_{4\beta}(x-y)} K(x,y)
\end{aligned}$$

As one can see, two of these estimates, namely for A_2 and B_2 are not of the form that we set out to obtain. Some methods to resolve this are considered in later subsections. However, we continue blindly on and assume that the functions that appear are "nice " in the sense that they do what we want. In other words I assume that

$$\frac{g_{2\beta}(x)g_{2\beta}(y)}{g_{4\beta}(x-y)} = H(0,x,0,y) \qquad\qquad \frac{g_{2\beta}(0)g_{2\beta}(x-y)}{g_{4\beta}(x+y)} = H(0,0,x,y)$$

are position invariant, or simply bounded above by some value, so as to remove them from the summation over Λ.

3.4 Bounding the Integral Kernel

We first consider a bound that uses $a_0 = \sum_{x\in\mathbb{Z}^d} U(x)$, but it is necessary to suppose that $U \in l^1(\mathbb{Z}^d)$ and that $U(x) < \infty$ for all $x \in \mathbb{Z}^d$. We wish to evaluate

$$\sum_{x,y\in\mathbb{Z}^d} K(x,y) = \sum_{x,y\in\mathbb{Z}^d} \int \left(1 - e^{-\frac{1}{4}\int_0^{4\beta} U(\omega(s))ds}\right) dW_{x,y}^{4\beta}(\omega)$$

Theorem 3.4.1 *The integral kernel $K(x,y)$ is bounded as follows:*

$$\sum_{x,y\in\mathbb{Z}^d} K(x,y) \leq \beta a_0$$

Proof Now using lemma 2.6.5 we get that

$$\sum_{x,y\in\mathbb{Z}^d}\int\left(1-e^{-\frac{1}{4}\int_0^{4\beta}U(\omega(s))ds}\right)dW_{x,y}^{4\beta}(\omega)\leq\frac{1}{4}\sum_{x,y\in\mathbb{Z}^d}\int\int_0^{4\beta}U(\omega(s))\mathrm{d}s\mathrm{d}W_{x,y}^{4\beta}(\omega)$$

Now since $U(\omega(s))\leq||U||_\infty<\infty$ and is positive we can apply Fubini to this to get that

$$\frac{1}{4}\sum_{x,y\in\mathbb{Z}^d}\int\int_0^{4\beta}U(\omega(s))\mathrm{d}s\mathrm{d}W_{x,y}^{4\beta}(\omega)=\frac{1}{4}\sum_{x,y\in\mathbb{Z}^d}\int_0^{4\beta}\int U(\omega(s))\mathrm{d}W_{x,y}^{4\beta}(\omega)\mathrm{d}s$$

and then due to the integral on the ω only depending on the path at the point $\omega(s)$ this becomes an integral over z with gaussian kernels added in: we have

$$\frac{1}{4}\sum_{x,y\in\mathbb{Z}^d}\int_0^{4\beta}\int U(\omega(s))\mathrm{d}W_{x,y}^{4\beta}(\omega)\mathrm{d}s=\frac{1}{4}\sum_{x,y\in\mathbb{Z}^d}\int_0^{4\beta}\sum_{z\in\mathbb{Z}^d}g_s(z-x)U(z)g_{4\beta-s}(y-z)\mathrm{d}s$$

and then again noting that the integrand is bounded and positive so we can apply Fubini to get that

$$\frac{1}{4}\sum_{x,y\in\mathbb{Z}^d}\int_0^{4\beta}\sum_{z\in\mathbb{Z}^d}g_s(z-x)U(z)g_{4\beta-s}(y-z)\mathrm{d}s=$$

$$=\frac{1}{4}\int_0^{4\beta}\sum_{x,y,z\in\mathbb{Z}^d}g_s(z-x)U(z)g_{4\beta-s}(y-z)\mathrm{d}s$$

and then

$$\frac{1}{4}\int_0^{4\beta}\sum_{x,y,z\in\mathbb{Z}^d}g_s(z-x)U(z)g_{4\beta-s}(y-z)\mathrm{d}s=$$

$$=\frac{1}{4}\int_0^{4\beta}\sum_{z\in\mathbb{Z}^d}U(z)\sum_{x\in\mathbb{Z}^d}(g_s(z-x))\sum_{y\in\mathbb{Z}^d}(g_{4\beta-s}(y-z))\,\mathrm{d}s$$

and then appealing to lemma 2.1.4 we get that this is

$$\frac{1}{4}\int_0^{4\beta}\sum_{z\in\mathbb{Z}^d}U(z)\sum_{x\in\mathbb{Z}^d}(g_s(z-x))\sum_{y\in\mathbb{Z}^d}(g_{4\beta-s}(y-z))\,\mathrm{d}s=\frac{1}{4}\int_0^{4\beta}\sum_{z\in\mathbb{Z}^d}U(z)\mathrm{d}s$$

as required *Q.E.D.*

Following the same procedure as the proof of the above theorem, one can bound the following:

Lemma 3.4.2

$$\sum_{x\in\mathbb{Z}^d}K(x,x)\leq\beta g_{4\beta}(0)a_0$$

$$\sum_{x\in\mathbb{Z}^d}K(x,-x)\leq\beta a_0 g_{4\beta}(0)$$

Using these bounds, we get

Theorem 3.4.3

$$\rho(z) \geq \rho^{(0)}(z) - 2z^2 g_{2\beta}^2(0) a_0\beta - a_0\beta z^2[3q_0(z)q_1(z) + q_0^2(z) + 2q_1(z)M + 3q_0(z)M]$$

where M is a bound on the Gaussian ratio that appears several times.

We now turn to bounding the ratio of Gaussians. We use the Fourier transform as introduced in subsection 2.2.

$$\left|\frac{g_{2\beta}(x)g_{2\beta}(y)}{g_{4\beta}(x-y)}\right| = \frac{\left|\frac{1}{2\pi}\int\limits_{[-\pi,\pi]^d} e^{-\frac{1}{d}\sum_{j=1}^d(1-\cos k_j)2\beta}e^{ikx}dk\frac{1}{2\pi}\int\limits_{[-\pi,\pi]^d} e^{-\frac{1}{d}\sum_{j=1}^d(1-\cos h_j)2\beta}e^{ihy}dh\right|}{\left|\frac{1}{2\pi}\int\limits_{[-\pi,\pi]^d} e^{-\frac{1}{d}\sum_{j=1}^d(1-\cos h_j)4\beta}e^{ih(x-y)}dh\right|}$$

$$= \frac{\left|\frac{1}{(2\pi)^2}\int\limits_{[-\pi,\pi]^d}\int\limits_{[-\pi,\pi]^d} e^{-\frac{1}{d}\sum_{j=1}^d(1-\cos k_j)2\beta}e^{ikx}e^{-\frac{1}{d}\sum_{j=1}^d(1-\cos h_j)2\beta}e^{ihy}dkdh\right|}{\left|\frac{1}{2\pi}\int\limits_{[-\pi,\pi]^d} e^{-\frac{1}{d}\sum_{j=1}^d(1-\cos k_j)4\beta}e^{ik(x-y)}dk\right|}$$

$$\leq \frac{\frac{1}{(2\pi)^2}\int\limits_{[-\pi,\pi]^d}\int\limits_{[-\pi,\pi]^d} \left|e^{-\frac{1}{d}\sum_{j=1}^d(1-\cos k_j)2\beta}e^{ikx}e^{-\frac{1}{d}\sum_{j=1}^d(1-\cos h_j)2\beta}e^{ihy}\right|dkdh}{\left|\frac{1}{2\pi}\int\limits_{[-\pi,\pi]^d} \left(1-\frac{4\beta}{d}\sum_{j=1}^d(1-\cos k_j)\right)e^{ik(x-y)}dk\right|}$$

$$\leq \frac{1}{\left|\frac{1}{2\pi}\int\limits_{[-\pi,\pi]^d} \left(1-\frac{4\beta}{d}\sum_{j=1}^d(1-\cos k_j)\right)e^{ik(x-y)}dk\right|}$$

Where we note that we have integrated a holomorphic function on a compact domain, and so it has finite integral. This bound now concludes our bounding of the density, barring the error from misusing lemma 2.4.2.

4 Machinations on the Discrete Gaussian

4.1 Attempted proofs of some product rule

It should be clear from the previous part that the difficulties in getting a lower bound dependent only on some position invariant functions and on the integral kernel $K(x, y)$ were due to the lack of a nice formula for

$$g_t(x)g_t(y)$$

In the continuous case, there is an easy formula following directly from the parallelogram law, namely

$$\pi_t(x)\pi_s(y) = \pi_{t+s}(x+y)\pi_{t+s}(x-y)$$

Initially, I proceeded blindly into attempting to prove this, and I describe some of my attempts below.

I initially tried working with the formula given for the gaussian, namely

$$g_{t_2-t_1}(x_2-x_1) = e^{-(t_2-t_1)} \sum_{N\geq 0} \frac{(t_2-t_1)^N}{(2d)^N N!} \sum_{\substack{y_1,\dots,y_N \\ y_1=x_1 \\ y_N=x_2 \\ ||y_n-y_{n-1}||=1}} 1$$

and then for the product of two of these, one has a formula

$$g_t(x)g_s(y) = \left(e^{-t} \sum_{N\geq 0} \frac{t^N}{(2d)^N N!} \sum_{\substack{y_1,\dots,y_N \\ y_1=0 \\ y_N=x \\ ||y_n-y_{n-1}||=1}} 1 \right) \left(e^{-s} \sum_{M\geq 0} \frac{s^M}{(2d)^M M!} \sum_{\substack{z_1,\dots,z_M \\ z_1=0 \\ z_M=y \\ ||z_n-z_{n-1}||=1}} 1 \right)$$

and then one can rearrange this sum to get

$$= e^{-(t+s)} \sum_{N\geq 0} \sum_{M\geq 0} \frac{t^N}{(2d)^N N!} \frac{s^M}{(2d)^M M!} \left(\sum_{\substack{y_1,\dots,y_N \\ y_1=0 \\ y_N=x \\ ||y_n-y_{n-1}||=1}} 1 \right) \left(\sum_{\substack{z_1,\dots,z_M \\ z_1=0 \\ z_M=y \\ ||z_n-z_{n-1}||=1}} 1 \right)$$

$$= e^{-(t+s)} \sum_{N\geq 0} \sum_{M\geq 0} \frac{t^N s^M}{(2d)^{N+M} N!M!} \sum_{\substack{y_1,\dots,y_N \\ y_1=0 \\ y_N=x \\ ||y_n-y_{n-1}||=1}} 1 \sum_{\substack{z_1,\dots,z_M \\ z_1=0 \\ z_M=y \\ ||z_n-z_{n-1}||=1}} 1$$

$$= e^{-(t+s)} \sum_{N\geq 0} \sum_{M\geq 0} \frac{t^N s^M}{(2d)^{N+M} N!M!} \sum_{\substack{y_1,\dots,y_N,z_1,\dots,z_M \\ y_1=0 \\ y_N=x \\ ||y_n-y_{n-1}||=1 \\ z_1=0 \\ z_M=y \\ ||z_n-z_{n-1}||=1}} 1$$

$$= e^{-(t+s)} \sum_{N\geq 0} \sum_{M \geq 0} \frac{t^N s^M}{(2d)^{N+M} N! M!} \sum_{\substack{y_1,\dots,y_N,z_1,\dots,z_M \\ y_1=0 \\ y_N=x \\ ||y_n-y_{n-1}||=1 \\ z_1=x \\ z_M=x+y \\ ||z_n-z_{n-1}||=1}} 1$$

But I saw no way in which to rewrite the sum over numbers of paths, and so gave up on that method. I next tried going back to the definition of the discrete gaussian. Suppose we have the probability $\mathbb{P}$ as in section 1.1. Then by definition we have

$$g_{t_2-t_1}(x_2-x_1) = \mathbb{P}(\{\omega : \omega(t_1) = x_1\}|\{\omega : \omega(t_2) = x_2\})$$

I introduce the notation that $\Omega^t_x = \{\omega : \omega(t) = x\}$ and also $\Omega^t_{x,y} = \{\omega : \omega(0) = x, \omega(t) = y\}$ i.e. the former is the set of all paths that pass through x at time t and the latter is the set of all paths that start at x and end at y. With this notation we have that

$$g_t(x)g_s(y) = \mathbb{P}(\Omega^t_{0,x})\mathbb{P}(\Omega^s_{0,y})$$

and due to independence we have this equal to

$$= \mathbb{P} \otimes \mathbb{P}(\Omega^t_{0,x} \times \Omega^s_{0,y})$$

At this stage I made a crazy guess, i.e. I tried to establish the result similarly to the continuous case, by trying to link this with

$$g_{t+s}(x+y)g_{t+s}(x-y)$$

as this would be the natural formula I though. Doing the same as above, one gets that

$$g_{t+s}(x+y)g_{t+s}(x-y) = \mathbb{P} \otimes \mathbb{P}(\Omega^{t+s}_{0,x+y} \times \Omega^{t+s}_{0,x-y})$$

and we want to relate the two sets in this product measure space. If we could find a measurable isomorphism between the two then we would be done. However, there is no bijection between the two, so we cannot. For example, if we define a map $\Omega^t_{0,x} \times \Omega^s_{0,y} \to \Omega^{t+s}_{0,x+y} \times \Omega^{t+s}_{0,x-y}$ by $(\omega, \sigma) \mapsto (\omega \star \sigma, \omega \star (-\sigma))$ where $\omega \star \sigma(h) = \begin{cases} \omega(h) & 0 \leq h \leq t \\ \sigma(h) & t \leq h \leq t+s \end{cases}$ then this maps $\Omega^t_{0,x} \times \Omega^s_{0,y}$ each element uniquely into the subset of $\Omega^{t+s}_{0,x+y} \times \Omega^{t+s}_{0,x-y}$ that passes through the intermediary point x, x at time t. This clearly cannot be the whole of $\Omega^{t+s}_{0,x+y} \times \Omega^{t+s}_{0,x-y}$, and so there cannot be an isomorphism.

4.2 Modifications of the Random Walk

These being unsuccesful, Daniel Ueltschi suggested to me to completely change the discrete Gaussian for another similar function, by going back to the definition and using another function where I have specified the properties that I want. I shall now explain my attempts.

One can observe in the previous section that it would suffice to have the following properties in order to obtain the bounds in the form that we want:

1. $g_t(0) \geq g_t(x)$ for fixed $t \geq 0$ and $x \in \mathbb{Z}^d$.
2. $g_t(x) \leq g_s(x)$ for fixed $x \in \mathbb{Z}^d$ and $t \leq s$.

3. $g_t(x)g_t(y) = g_{2t}(x+y)g_{2t}(x-y)$ for $x, y \in \mathbb{Z}^d$ and $t \geq 0$.

I summarise the properties that these give on the increment distribution in the following lemma

We have that $\mathbb{P}(\omega(t) = x|\omega(0) = 0) = \delta_{x,0} + tp(x) + O(t^2)$ for small t. Thus we have that

$$\begin{aligned} g_t(x)g_t(y) &= (\delta_{x,0} + tp(x) + O(t^2))(\delta_{y,0} + tp(y) + O(t^2)) \\ &= \delta_{x,0}\delta_{y,0} + t\delta_{x,0}p(y) + t\delta_{y,0}p(x) + O(t^2) \end{aligned}$$

and

$$\begin{aligned} g_{2t}(x+y)g_{2t}(x-y) &= \left(\delta_{x+y,0} + tp(x+y) + O(4t^2)\right)\left(\delta_{x-y,0} + tp(x-y) + O(4t^2)\right) \\ &= \delta_{x+,0}\delta_{x-y,0} + 2t\delta_{x+y,0}p(x-y) + 2t\delta_{x-y,0}p(x+y) + O(t^2) \end{aligned}$$

and thus we want these two to be equal, i.e.

$$\delta_{x,0}\delta_{y,0} + t\delta_{x,0}p(y) + t\delta_{y,0}p(x) = \delta_{x+,0}\delta_{x-y,0} + 2t\delta_{x+y,0}p(x-y) + 2t\delta_{x-y,0}p(x+y)$$

but setting $x = y = 0$ gives

$$1 + tp(0) + tp(0) = 1 + 2tp(0) + 2t(p(0)$$

which is true only if $p(0) = 0$ which is very undesirable, as we have no reason for the paths to always jump. We thus have that no such useful Gaussian with this formula exists, and so we had to stick with this and work with it.

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