Justification of the Boltzmann Equation

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Derivation of Boltzmann Equation from a particle system

A classical problem is the justification of the Boltzmann Equation

$$\partial_t f + v \cdot \nabla_x f = Q(f)$$

as an approximation of the one particle density function of a collection of interacting particles.

The goal is to show that
A collection of particles interacting via a long range potential,
Under the Boltzmann-Grad limit $N\varepsilon^{d-1} = 1$ converges,
to a solution of the Boltzmann equation with relevant collision operator $Q(f)$.

Linear-Problem

- We consider the evolution of a tagged particle in a background medium
- The relevant PDE is then the linear Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = L(f)$$

- Here one needs only consider velocity change between one particle and the others, so complexity of problem is much lower.

Existing Results

- [3] and [4] prove convergence for non-linear system for short times for short range potentials
- [2] shows convergence for inverse power law potentials, where one cuts off the potential
- [1] shows convergence for long range potentials, which decay like

$$|\nabla\Phi(x)| \lesssim \exp\left(-\exp\left(\exp\left(1+|x|^{2(d-1)}\right)\right)\right)$$

Particle World

$$\dot{x}_i(t) = v_i(t),\quad \dot{v}_i(t) = 0$$
$$\dot{v}_1(t) = -\sum_{i=2}^{N} \nabla\Phi(x_1 - x_i).$$

BBGKY Hierarchy

$$\partial_t f^{(1)} = Q^{(2)}(f^{(2)}),$$
$$\partial_t f^{(2)} = Q^{(3)}(f^{(3)}),$$
$$\vdots$$
$$\partial_t f^{(N)} = 0$$

Intermediary Technical Procedure

Stochastic Process World
Graphical Representation

Kinetic World

$$\partial_t f = Q(f).$$

Issues for BBGKY hierarchy

- BBGKY hierarchy represents the particle system as a PDE, thus losing information in the technical methodology
- Bounds on collision operators are coarse, and don't take into account the gain and loss structure, which is necessary to ensure existence of solutions.

Stochastic Representation

- Aim is to represent both systems via stochastic processes
- This maintains all information in the system
- The collisional structure manifests itself as jumps in the system, and a control of these jumps needs to be exacted to describe convergence.

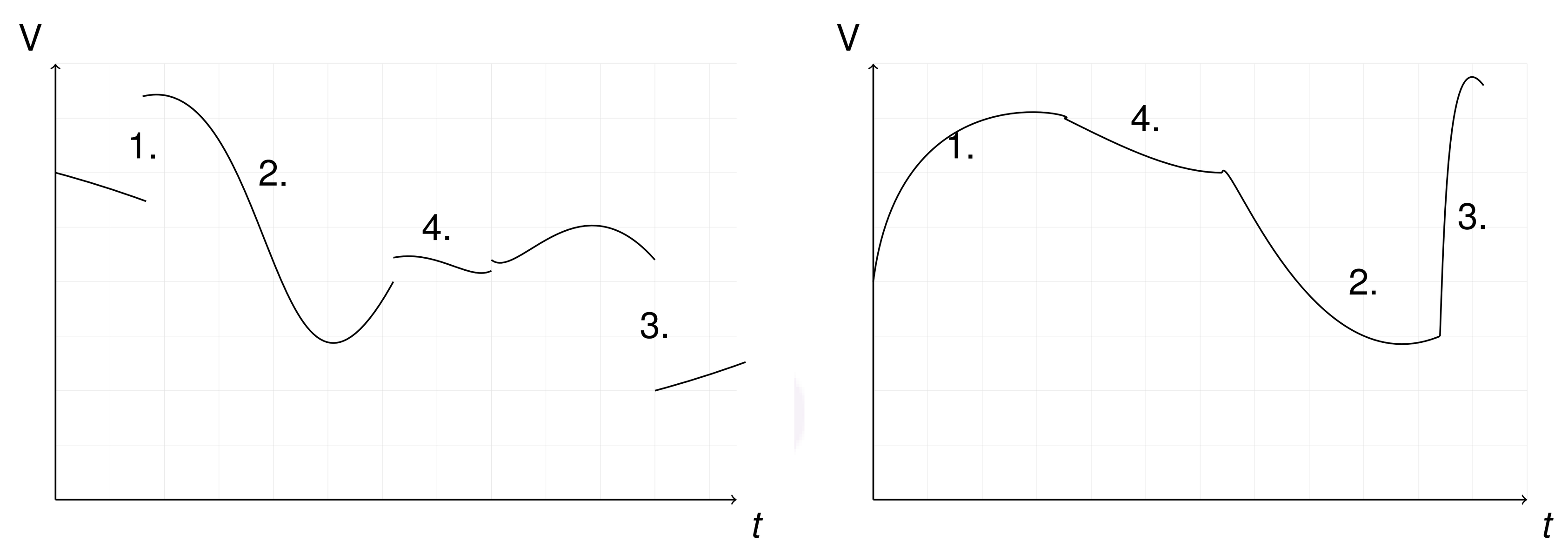

Figure : Sample paths of limiting process (left) and particle process (right)

Associated Limiting Process

- Use the linear Boltzmann equation as a generator of a Lévy process, with jump measure J. Then
- $J([0,t)) = \infty$
- $J(B_\varepsilon(0)) = \infty$
- $JB_\varepsilon(0)^C) < \infty$

References

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