PAC Report

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1 Motivation

The aim of this research is to develop further on the idea of a "Poissonian city", that is we will try to establish a more realistic framework for this idealized model and at the same time compare this ideal city with the configuration of real transportation networks (railways and/or road networks). The Poissonian city was first introduced by Kendall on a follow up work [Ken11] to the ideas previously presented by Aldous and Kendall at [AK08].
Essentially the original idea was to construct a network to join n points in such a way that it balanced on the following criteria:

1. Short total length of the whole network. That is, considering railways for example, we are hoping to get an economic network, as the length of the network can be associated with costs (construction and/or maintenance). Remember that the shortest network is given by the Steiner Tree [PS02].

2. Small excess regarding the Euclidean distance in average. That is, however we want an economic network that still gave us, in average, short routes between either two points considered in the network. In contrast to the previous criterion, this would be achieved by the complete network (that which joins all the possible pair of points, therefore it gives the shortest route between every pair of points).

As a consequence, we can think of this problem as a trade-off between the Steiner tree network (smallest network use to join n points) and the complete network. See figure 1

The "Poissonian City" provides an idealized model that emerge from the answer to the above problem presented in [AK08]. Therefore, the main aim of this research is to set down this model to be more close to reality or practical applications. This could be achieved by different means, for example: consider segment processes instead of (whole infinite) lines processes, allow weak curvature of the roads by using fibre processes. Another important approach to be considered is to compare the roadways configuration of real cities/states (only consider principal avenues/main roads, not the whole streets network) against the one proposed by the Poissonian City. However we first need to be clear on the criterion that will be used to compare these networks.

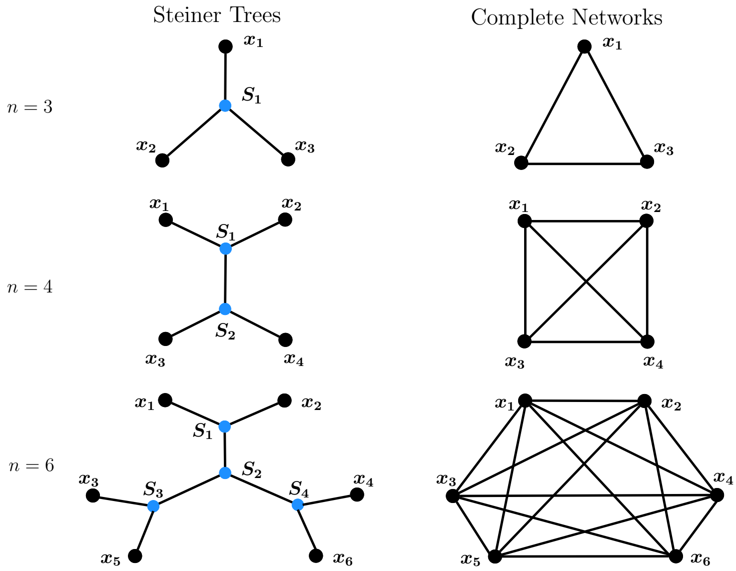

Figure 1: Examples of Steiner Trees and Complete Networks for 3, 4 and 6 points. Steiner Trees figures are based on pictures from [PS02] and [Ski09].

2 Literature Review

We have already mentioned that in the design of a network to join n points, $\boldsymbol{x^n} = \{\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}\}$, one can consider two criteria. First, the total length of the network should not be much greater than the shortest network which connects all the points (Steiner tree, $ST(\boldsymbol{x^n})$). Second, the average route length should not be much greater than the average Euclidean distance (straight-lines between source and destination points), i.e. in average the distance between two points should stay close to the one in the complete network.

The main problem addressed by Aldous and Kendall in [AK08] is regarding the order of this excesses, that is, how small can we make them? To answer this question the authors constructed a random network such as the above excesses are of order $o(n)$ and $\mathcal{O}(\log n)$, respectively.

First, it is important to be clear on how to measure the mentioned excesses in a given network, $G(\boldsymbol{x^n})$. The following notation is required to define these measures: $ST(\boldsymbol{x^n})$ denote the Steiner tree corresponding to the configuration of n points given by $\boldsymbol{x^n}$, $\ell(\boldsymbol{x_i}, \boldsymbol{x_j})$ is the shortest route length between points $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$ in the given network $G(\boldsymbol{x^n})$, $d(\boldsymbol{x_i}, \boldsymbol{x_j})$ is the Euclidean distance between the points $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$. Hence, the aim is to control the order of the following quantities:

- excess length $(G(\boldsymbol{x^n})) = \text{length}(G(\boldsymbol{x^n})) - \text{length}(ST(\boldsymbol{x^n}))$.
- excess distance $(G(\boldsymbol{x^n})) = \frac{1}{n(n-1)} \sum_{i \neq j} \left(\ell(\boldsymbol{x_i}, \boldsymbol{x_j}) - d(\boldsymbol{x_i}, \boldsymbol{x_j})\right)$

The desired network arises from considering a hierarchical construction. At small scales the idea is to use the underlying Steiner tree to connect points. However, for long distances the network will depend on the sparse collection of randomly oriented lines, that is one realisation of a stationary and isotropic Poisson line process, which turns out to be the key characteristic to achieve an excess of $\mathcal{O}(\log n)$. Finally, to simplify calculations an intermediate scale consisting of grid lines is required. With this in mind, the rest of the paper [AK08] focus mainly on the following result:

THEOREM 1 (Upper bound on minimum excess network length). *[AK08, Theorem 1]*
For any configuration $\boldsymbol{x^n}$ in square side $\sqrt{n}$ and for any $\epsilon > 0$ there are connecting networks $G(\boldsymbol{x^n})$ such that

$$\begin{aligned} \textit{excess length } (G(\boldsymbol{x^n})) &\leq \epsilon n \\ \textit{excess distance } (G(\boldsymbol{x^n})) &\leq \mathcal{O}(\log n) \end{aligned}$$

Also, this paper analyze a similar result to get lower bounds on the average excess distance as long as one had an extra condition on the configuration of the n points, $\boldsymbol{x^n}$, which are joint by the network. Basically, one desires to avoid non-random configurations on these points, that is we want *equidistributed* points.

DEFINITION 1 (L_n-equidistributed). *[AK08, Definition 3]*
Let $\boldsymbol{x^n}$, for varying n, form a sequence of configurations in the plane and let $L_n > 0$. We said that $\boldsymbol{x^n}$ is L_n-equidistributed *if a random choice $\boldsymbol{x_n}$ of a city point can be coupled to a uniformly random point $\boldsymbol{y_n}$ so that*

$$\mathbb{E}\left[\min\left\{1, \frac{|\boldsymbol{x_n} - \boldsymbol{y_n}|}{L_n}\right\}\right] \to 0 \quad \textit{as } n \to \infty$$

It turns out that a wide range of possible point patterns can be seen to be L_n-equidistributed. For this cases, we have the following lower-bound

THEOREM 2 (Lower bound on minimum excess network length). *[AK08, Theorem 2]*
Given an L_n-equidistributed configurations of cities, $\boldsymbol{x^n}$, in $[0, \sqrt{n}]^2$ with $L_n = o\big(\sqrt{\log n}\big)$, then any connecting network $G(\boldsymbol{x^n})$ with length bounded above by a multiple of n connects the city points with average connection length exceeding average Euclidean connection length by at least $\Omega(\sqrt{\log n})$, that is

$$\textit{excess distance } (G(\boldsymbol{x^n})) = \Omega(\sqrt{\log n})$$

Thus, $\liminf_{n\to\infty}$ *excess distance* $(G(\boldsymbol{x^n}))/\sqrt{\log n} > 0$.

At the same time, it is important to notice the main two branches that emerge from the network proposed by Aldous and Kendall [AK08]:

- How to define a good statistic that captures the short-routes property in a given network? As shown by the hierarchical network the ratio statistic or the excess average doesn't seem to work out completely, because this gives an example of a non-realistic transportation network. Some further thoughts on this direction are presented by Aldous and Shun [AS10] where they introduced an intermediate statistic between the average distance excess and the maximum distance excess to achieve this goal.

- What else can be said about a network consisting of one realization of an stationary and isotropic Poisson line process? Specifically what are its geometric properties and how the semi-perimeter route (near-geodesic) compare to actual geodesic in this network. This and other questions are solved in [Ken11], which will be the approach to be followed along this research.

The concept of a *Poissonian city* was first introduced in [Ken11], that is a random network generated by a unit intensity stationary and isotropic Poisson line process, Π. The idea is to use such a network to connect n points in a disk of radius n through the semi-perimeter algorithm that was already introduced in [AK08] to establish the upper bound result, see figure 2. The results presented in this work explores the following ideas:

- What is the shape of near-geodesics in this network?
- Given a good trade-off between the excess length ($G(\boldsymbol{x^n})$) and excess distance ($G(\boldsymbol{x^n})$), how might the variance of these near-geodesics behave?
- The upper-bound is obtained by controlling these near-geodesic paths. How might true geodesics behave in comparison to the semi-perimeter route?
- Consider we attach certain amount of traffic to this network. What can be said about flows of traffic in this network? In particular, how does the traffic behave at the centre of the Poissonian city?

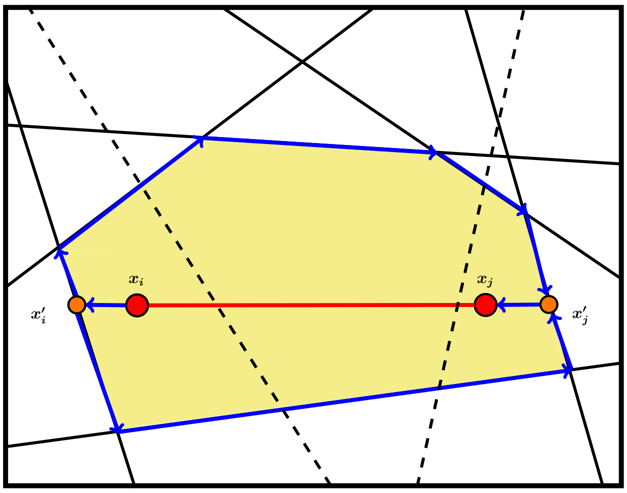

Figure 2: Construction of the semi-perimeter paths.

Specifically we will focus on the properties concerning the last two questions.

The main result regarding the shape of true geodesics is actually an improvement to the lower bound presented by Aldous and Kendall [AK08].

Theorem 3 (Lower Bound from the Poissonian city)**.** *[Ken11, Theorem 4]*
In the Poissonian city network, consider any path from $\boldsymbol{p}^-$ to $\boldsymbol{p}^+$ that is contained in the cell $\mathcal{C}(\boldsymbol{p}^-, \boldsymbol{p}^+)$ (determined by the semi-perimeter route). If $d(\boldsymbol{p}^-, \boldsymbol{p}^+) = n$, then the path must have mean excess exceeding

$$2\left(\log 4 - \frac{5}{4}\right)\log n + o(\log n) = 0.27258872\ldots\log n + o(\log n)$$

One of the specific aims of my research will be to generalize this result to the case where the connections are made by a Poisson segment process of fixed length h and intensity λ in such a way that the *length intensity* of this segment process is equivalent to the unitary Poisson line process. Alternative approaches could be to consider fibre processes with weak curvature.

On the other hand, when we consider the flow of traffic through the center of the city, the main result tell us its mean asymptotic behaviour

Theorem 4 (Mean flow through the centre). *[Ken11, Theorem 5]*
The mean flow through a line at the center of a Poissonian city that connects n points is given by the expression

$$\mathbb{E}[F_n] \quad = \quad \int_0^\pi \int_0^n \int_0^n \exp\left\{-\frac{1}{2}(r+s-\rho)\right\} r \,\mathrm{d}rs \,\mathrm{d}s\theta \,\mathrm{d}\theta \tag{1}$$

where $\rho = \sqrt{r^2+s^2+2rs\cos\theta}$. Asymptotically, as $n \to \infty$,

$$\mathbb{E}[F_n] \sim 2n^3 \tag{2}$$

Now, another specific aim for my research is to analyze how the amount of traffic fluctuates around the city, depending mainly on the Euclidean distance from the analyzed point to the centre of the city. For example, we can consider the proportion of traffic that is contained in a disk centered at the centre of the city with varying radius $r \in [0, n]$ and see if this behaviour is similar in some way to real transportation network, e.g. the british railway system, see [Boa63].

3 Development of some ideas

Along this section, I will briefly explained some of the approaches that I have taken during this year to tackle the above problems. For the time being, the focus of the research is on the properties of Poisson segment processes. Therefore, the first question addressed in here is regarding how to define these and get an equivalent expression for the length intensity of Poisson line processes. At the same time, in the second question we analyzed the work by Baccelli, Tchoumatchenko and Zuyev [BTZ00] that studies the behaviour of a path of segments on the Delaunay graph [vL12], in particular they show that the path in question is Markovian and gave an asymptotic result regarding the excess distance of this path against the Euclidean distance.

3.1 Question 1

Consider a stationary Poisson segment process of fixed length h. This subsection addresses the following question: what intensity, ν_λ^h (for the marked point process of intensity λ defining the segments) leads to a length intensity which is the same as the unit intensity Poisson line process?

We denote by Ξ the segment process of fixed length h and intensity λ with rose of directions given by ρ (we measure the angles from the horizontal axis in the anti-clockwise direction). We seek an expression for its *length intensity*, given by

$$\nu_\lambda^h \quad = \quad \frac{\mathbb{E}\big[L\big(\Xi \cap K\big)\big]}{A(K)} \quad = \quad \frac{\mathbb{E}[\mathrm{Leb}_1([K])]}{\mathrm{Leb}_2(K)} \tag{3}$$

for $K \subseteq \mathbb{R}^2$, any compact convex set on $\mathbb{R}^2$. Here we denote by $\xi \Uparrow K$ the event "ξ hits K", meaning $\xi \cap K \neq \emptyset$, then $[K]$ stands for the hitting set, which is defined as the set of segments of Ξ that hit K, i.e. $[K] = \{\xi \in \Xi : \xi \Uparrow K\}$.
We can identify the segment process Ξ with a marked point process Φ^* in $\mathbb{R}^2 \times (0, \pi]$. Here each point $\boldsymbol{x}$ of the process represents the lower-end point of the segment ξ (in case of horizontal segments we chose the left-end point). Thus $\boldsymbol{x} = (x^*, y^*)$ where:

$$\begin{aligned} y^* \quad &:= \quad \inf\big\{y : (x, y) \in \xi \ \text{ for some } x\big\} \\ x^* \quad &:= \quad \inf\big\{x : (x, y^*) \in \xi\big\} \end{aligned}$$

We can construct the Poisson segment process by requiring that the points (x^*, y^*) form a Poisson point process of intensity λ and the angular marks θ are independent of each other and of the positions $\boldsymbol{x}$ and identically distributed with distribution given by $\rho(\,\mathrm{d}\theta)$. Viewed as a point process in (x^*, y^*, θ) space, the intensity mean of this point process is given by $\lambda(\mathrm{Leb}_2 \otimes \rho)$.

The first step in addressing this question is to reduce to the case where the mark distribution is nonrandom. To achieve this consider the following rotation

$$R(\xi) = R_\theta(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, 0) =: (\tilde{x}, \tilde{y}, 0)$$

for each segment $\xi = (x, y, \theta)$. The rotated segments will all be horizontal. The rotation $R_\theta(x, y)$ moves the segment ξ in such a way that its reference point, $\boldsymbol{x}$, is still at the same distance, $\|\boldsymbol{x}\|$, from the origin but it has been rotated θ radians in clockwise direction so that the segment is now horizontal, see figure 3. This converts Ξ into a new segment process $R\Xi$ for which all segments are horizontal, but such that the length of intersection with any disk centered at the origin is unchanged.

In particular the new point process is still Poisson.

Theorem 5. *If Φ^* is a marked Poisson point process in $\mathbb{R}^2 \times (0, \pi]$, with intensity λ and rose of directions ρ then $\tilde{\Phi} = \{(\tilde{x}, \tilde{y}) : (\tilde{x}, \tilde{y}, 0) = R_\theta(x, y) \text{ with } (x, y, \theta) \in \Phi^*\}$ is a Poisson point process of intensity λ.*

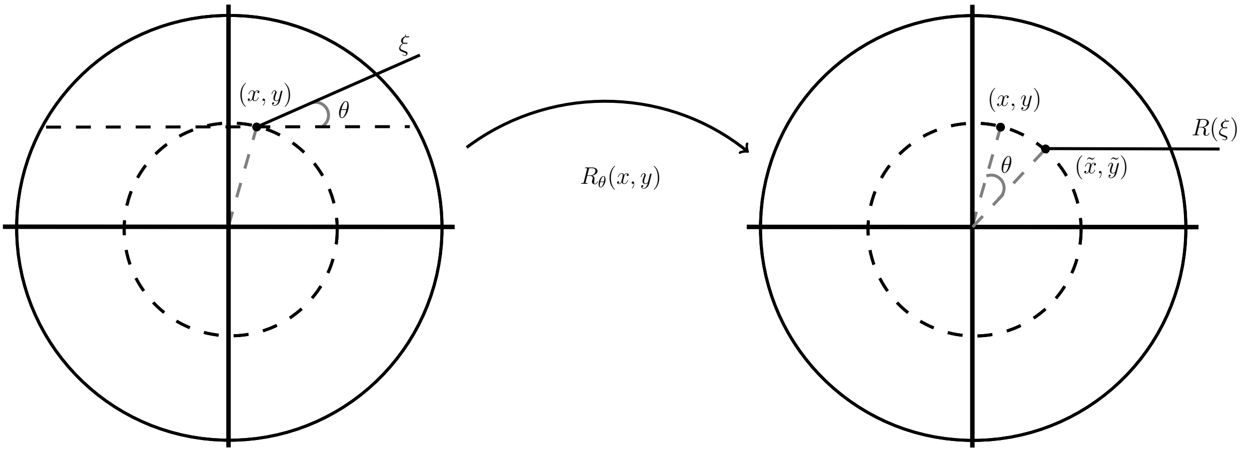

Figure 3: Illustration of the rotation $R_\theta(x, y)$.

Proof.
The family of avoidance probabilities determines the distribution of a point process [Kin92], so it is enough to show that:

$$\mathbb{P}[\tilde{\Phi} \cap \tilde{E} = 0] \quad = \quad \exp\left(-\lambda \operatorname{Leb}_2(\tilde{E})\right)$$

for all measurable $\tilde{E} \subseteq \mathbb{R}^2$.
For $\tilde{E}$ any measurable subset in $\mathbb{R}^2$, we define $E \subseteq \mathbb{R}^2 \times (0, \pi]$ as follows:

$$E \quad = \quad \left\{(x, y, \theta) : (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta) \in \tilde{E}\right\}$$

Therefore $\{\tilde{\Phi} \cap \tilde{E} = 0\} = \{\Phi^* \cap E = 0\}$. By the rotation invariance of the Leb_2 measure and the fact that ρ is a probability measure we can compute the intensity measure for $\tilde{\Phi}$:

$$\begin{aligned}
\lambda(\operatorname{Leb}_2 \otimes \rho)(E) \quad &= \quad \lambda \int_0^\pi \left(\iint\limits_{\mathbb{R}^2} \mathbb{1}_E(x, y, \theta) \operatorname{Leb}_2(\,\mathrm{d}x\,\mathrm{d}y) \right) \rho(\,\mathrm{d}\theta) \\
= \lambda \int_0^\pi \left(\iint\limits_{\mathbb{R}^2} \mathbb{1}_{\tilde{E}}(R_\theta(x, y)) \operatorname{Leb}_2(\,\mathrm{d}x\,\mathrm{d}y) \right) \rho(\,\mathrm{d}\theta) &= \lambda \int_0^\pi \left(\iint\limits_{\mathbb{R}^2} \mathbb{1}_{\tilde{E}}(\tilde{x}, \tilde{y}) \operatorname{Leb}_2(\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}) \right) \rho(\,\mathrm{d}\theta) \\
&= \lambda \iint\limits_{\mathbb{R}^2} \mathbb{1}_{\tilde{E}}(\tilde{x}, \tilde{y}) \operatorname{Leb}_2(\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}) = \lambda \operatorname{Leb}_2(\tilde{E})
\end{aligned}$$

Hence,

$$\mathbb{P}[\tilde{\Phi} \cap \tilde{E} = 0] = \mathbb{P}[\Phi^* \cap E = 0] = \exp\left(-\lambda(\operatorname{Leb}_2 \otimes \rho)(E)\right) = \exp\left(-\lambda \operatorname{Leb}_2(\tilde{E})\right)$$

as required. □

Now we relate the length intensity at (3) to the lengths of the segments h. From the above construction we only need to consider the case where all segments are horizontal, because if K is a disk centered at the origin we know that $\mathbb{E}[L(\Xi \cap K)] = \mathbb{E}[L(R\Xi \cap K)]$. Hence, as $R\Xi$ can be represented by $\Phi \oplus [0, h]$, where Φ is a planar Poisson point process of intensity λ. Therefore in (3) we get

$$\nu_\lambda^h = \frac{\mathbb{E}[\mathrm{Leb}_1([K])]}{\mathrm{Leb}_2(K)} = \frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\Phi \oplus [0, h]) \cap K\right)\big]}{\mathrm{Leb}_2(K)}$$

However if $h_1 \in (0, h)$, it is clear that we can express $\Phi \oplus [0, h]$ as the following disjoint union $(\Phi \oplus [0, h_1]) \cup (\Phi \oplus [h_1, h])$. So ν_λ^h is linear in h

$$\begin{aligned}\nu_\lambda^h = \frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\Phi \oplus [0, h]) \cap K\right)\big]}{\mathrm{Leb}_2(K)} &= \frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\Phi \oplus [0, h_1]) \cap K\right)\big]}{\mathrm{Leb}_2(K)} + \frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\Phi \oplus [h_1, h]) \cap K\right)\big]}{\mathrm{Leb}_2(K)} \\ &= \nu_\lambda^{h_1}(K) + \frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\{\Phi + h_1\} \oplus [0, h - h_1]) \cap K\right)\big]}{\mathrm{Leb}_2(K)} = \nu_\lambda^{h_1} + \nu_\lambda^{h-h_1}\end{aligned}$$

Here $\Phi + h_1$ has the same distribution as Φ, since Φ is a stationary Poisson point process. As a consequence we may deduce $\nu_\lambda^h = h\nu_\lambda^1$.

But ν_λ^h is also linear in λ: apply the superposition theorem to decompose the original Poisson point process Φ of intensity λ into two independent Poisson point process Φ_1 and Φ_2 with respective intensities λ_1 and λ_2, such that $\lambda = \lambda_1 + \lambda_2$. Therefore

$$\nu_\lambda^h = \nu_{\lambda_1}^h + \nu_{\lambda_2}^h$$

In sum, we conclude that ν_λ^h should be of the form $c\lambda h$. We determine the value of $c = \nu_1^1$ by considering the case when K is the unit square $[0, 1]^2$. If we denote by $N(K)$ the amount of points from Φ (a Poisson point process with intensity λ) that falls in K we get (recall $N(K) \sim Po(\lambda\,\mathrm{Leb}_2(K)))$

$$\begin{aligned}\nu_\lambda^h = h\nu_\lambda^1 = h\frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\Phi \oplus [0, 1]) \cap [0, 1]^2\right)\big]}{\mathrm{Leb}_2([0, 1]^2)} &= h\mathbb{E}\left[\sum_{\xi \Uparrow [0,1]^2} \mathrm{Leb}_1(\xi \cap [0, 1]^2)\right] \\ &= h\mathbb{E}\left[\sum_{k=1}^{N([0,1]^2)} U(0, 1) + \sum_{k=1}^{N([-1,0]\times[0,1])} U(0, 1)\right] = 2h\mathbb{E}[N([0, 1]^2)] \times \mathbb{E}[U(0, 1)] = h\lambda\end{aligned}$$

Here we use the fact that if $\xi \Uparrow [0, 1]^2$ then its marker point $\boldsymbol{x}$ falls either in $[0, 1]^2$ or in $[-1, 0] \times [0, 1]$. Either way, $\mathrm{Leb}_1(\xi \cap [0, 1]^2)$ is distributed according to a $U[0, 1]$ random variable (recall the construction of the Poisson point process where we draw $N([0, 1]^2)$ points scattered as independent and identically distributed uniformly on $[0, 1]^2$).

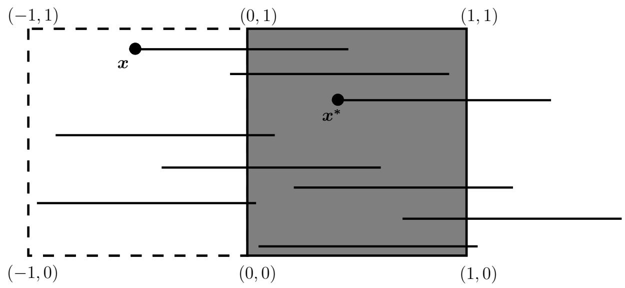

Figure 4: Explanation for the fact that $\mathrm{Leb}_1(\xi \cap [0,1]^2)$ has distribution $2U(0,1)$. If there is a left-end point $\boldsymbol{x} = (x_1, y_1)$ in the square $[-1,0] \times [0,1]$ then the length of the intersection of that segment with the square $[0,1]^2$ is given by $1 + x_1$ which is Uniform on $(0,1)$. Similarly, if there is a left-end point $\boldsymbol{x^*} = (x_2, y_2)$ in the square $[0,1]^2$ then the length of that segment with the square $[0,1]^2$ is given by $1 - x_2$ which is again Uniform $(0,1)$.

Now, recalling that the unit Poisson line process Π (where $\lambda = 1/2$, since we are considering undirected lines) has length intensity equal to $\pi/2$. To obtain this result one need to notice that the length of a chord in $B_1(\boldsymbol{o})$ of a line at a distance r from the origin $\boldsymbol{o}$ is given by $2\sqrt{1-r^2}$, see figure 5. Therefore the calculations for the length intensity leads to:

$$\nu_\lambda \quad = \quad \frac{\mathbb{E}\big[L\big(\Pi \cap B_1(\boldsymbol{o})\big)\big]}{\mathrm{Leb}_2(B_1(\boldsymbol{o}))} = \frac{\lambda}{\pi}\int_0^\pi \int_{-1}^1 2\sqrt{1-r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \frac{2\lambda\pi}{\pi}\int_{-1}^1 \sqrt{1-r^2}\,\mathrm{d}r = \lambda\pi = \frac{\pi}{2}$$

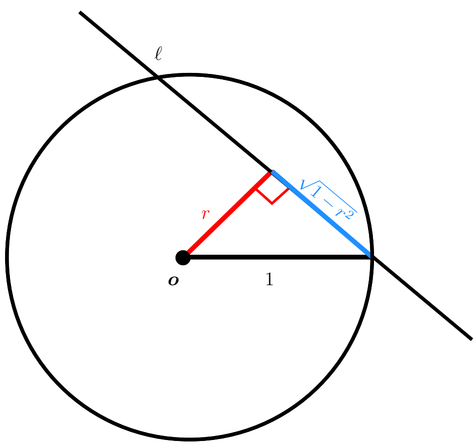

Figure 5: Illustration of the length of the chord in $B(\boldsymbol{o}, 1)$.

Therefore $c = \pi/2$ gives the similar desired result for segment processes. Our particular interest is with the case where ρ is a uniform distribution. Notice that a Poisson segment process of length h with intensity $\lambda = 1/h$ has the same length intensity as an unitary Poisson line process.

3.2 Question 2

Alternative explanation of the Markov property for the Markov path in the *Poisson-Delaunay graph* as constructed by Baccelli, F., Tchoumatchenko, K., & Zuyev, S. (2000) [BTZ00]

First, we explain the construction of this Markov path on the Poisson-Delaunay graph (triangulation). The idea is to join two random, but fixed points, say $\boldsymbol{s}$ and $\boldsymbol{t}$, through the edges of the Delaunay graph constructed with respect to the vertex set Φ, given by a planar stationary Poisson point process. The construction of the Delaunay graph can be expressed as follows: connect all the pair of points $\{\boldsymbol{Z_i}, \boldsymbol{Z_j}\}$ from Φ such that there exists a disk having $\boldsymbol{Z_i}$ and $\boldsymbol{Z_j}$ on its boundary and no points of Φ in its interior. Notice, that whenever the points of Φ are distributed in a general position, that is no three points are co-linear and no four points are co-circular, then the Delaunay graph constitutes a triangulation of the plane.

The *Voronoi tessellation* with respect to the set Φ is another geometrical object related to the *Delaunay triangulation*. The construction of the Voronoi tessellation consists of all the Voronoi cells $V_{\boldsymbol{Z_i}}$ with nucleus $\boldsymbol{Z_i}$ (a point from Φ). Here a Voronoi cell $V_{\boldsymbol{Z_i}}$ is the convex set of points in the plane that are closer to $\boldsymbol{Z_i}$ than to any other $\boldsymbol{Z_j} \in \Phi$, that is $V_{\boldsymbol{Z_i}} = \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x} - \boldsymbol{Z_i}\| = \text{dist}(\boldsymbol{x}, \Phi)\}$. The Voronoi tessellation and the Delaunay triangulation are dual: there is an edge between $\boldsymbol{Z_i}$ and $\boldsymbol{Z_j}$ in the Delaunay triangulation if and only if their respective Voronoi cells $V_{\boldsymbol{Z_i}}$ and $V_{\boldsymbol{Z_j}}$ share an edge.

Since the Poisson point process is invariant under translations and rotations then without loss of generality we can focus on the case where $\boldsymbol{s} = (0,0)$ and $\boldsymbol{t} = (t,0)$. Therefore, the line ℓ that joins $\boldsymbol{s}$ and $\boldsymbol{t}$ is given by the x-axis. The path considered by Baccelli et al [BTZ00] consists of those segments $\{[\boldsymbol{Z_{k-1}}, \boldsymbol{Z_k}] : k \in \mathbb{N}\}$ such that the sequence of vertices $\{\boldsymbol{Z_k} : k \in \mathbb{N}\}$ corresponds to the nuclei of the Voronoi cells $V_{\boldsymbol{Z_k}}$ that are crossed by the line that joins $\boldsymbol{s}$ and $\boldsymbol{t}$, in our case, the x-axis. Notice $\boldsymbol{Z_0}$ corresponds to the nearest point of Φ to $\boldsymbol{s}$, that is $\boldsymbol{s} \in V_{\boldsymbol{Z_0}}$.

An alternative construction of this Markov path is given by the following recursive relations:

1. Define the nearest point of Φ to $\boldsymbol{s}$ as $\boldsymbol{Z_0}$.

2. Consider the line ℓ which connects $\boldsymbol{s}$ and $\boldsymbol{t}$, in our particular case the x-axis. Move from $\boldsymbol{s}$ towards $\boldsymbol{t}$ until one hits the first point in ℓ for which $\boldsymbol{Z_0}$ is not the only nearest point from Φ. Denote this point by $\boldsymbol{T_1} = (T_1, 0)$. Notice, since $\boldsymbol{s} \in V_{\boldsymbol{Z_0}}$, and the Voronoi cells are convex, then the whole line segment from $\boldsymbol{s} = (0,0)$ to $\boldsymbol{T_1}$ belongs to $V_{\boldsymbol{Z_0}}$, i.e. $[0, T_1] \subset V_{\boldsymbol{Z_0}}$. So we can define T_1 as follows:

   $$T_1 := \sup\{u \in \mathbb{R} : [0,u] \subset V_{\boldsymbol{Z_0}}\} \qquad \text{equivalently} \qquad T_1 := \inf\left\{v \in \mathbb{R} : (v, \infty) \subset V^c_{\boldsymbol{Z_0}}\right\}$$

   Remark: If $T_1 > t$, as $\boldsymbol{t} = (t, 0)$, then join $\boldsymbol{Z_0}$ with $\boldsymbol{t}$ and finish.

3. Recall that the points Φ are in general position, therefore almost surely there exists only one other point from Φ that is as close to $\boldsymbol{T_1}$ as $\boldsymbol{Z_0}$, we denote this point by $\boldsymbol{Z_1}$. Notice that by construction the disk centred at $\boldsymbol{T_1}$ with radius $\|\boldsymbol{T_1} - \boldsymbol{Z_0}\| = \|\boldsymbol{T_1} - \boldsymbol{Z_1}\|$ contains both points: $\boldsymbol{Z_0}$ and $\boldsymbol{Z_1}$ on its boundary and contains no points from Φ in its interior, otherwise if $\boldsymbol{Z_k} \in B^0_{\|\boldsymbol{T_1}-\boldsymbol{Z_0}\|}(\boldsymbol{T_1})$ we will have $\|\boldsymbol{T_1} - \boldsymbol{Z_k}\| < \|\boldsymbol{T_1} - \boldsymbol{Z_0}\|$, a contradiction to the fact that $\boldsymbol{T_1} \in V_{\boldsymbol{Z_0}}$. Therefore, there exist an edge on the Delaunay graph which connects $\boldsymbol{Z_0}$ and $\boldsymbol{Z_1}$.

4. Again, move from $\boldsymbol{T_1}$ along the line ℓ towards $\boldsymbol{t}$ until one hits the first point in ℓ for which $\boldsymbol{Z_1}$ is not the only nearest point from Φ. So we can define the point $\boldsymbol{T_2} = (T_2, 0)$ as follows:

$$T_2 := \sup\{u \in \mathbb{R} : [T_1, u] \subset V_{\boldsymbol{Z_1}}\} \qquad \text{equivalently} \qquad T_2 := \inf\left\{v \in \mathbb{R} : (v, \infty) \subset V^c_{\boldsymbol{Z_1}}\right\}$$

   If $T_2 > t$ then join $\boldsymbol{Z_1}$ with $\boldsymbol{t}$ and finish, else iterate step 3 with $\boldsymbol{T_2}$ instead of $\boldsymbol{T_1}$.

Thus, in general, if we consider $\boldsymbol{T_0} := \boldsymbol{s}$ and first define $\boldsymbol{Z_0}$ as the nearest point from Φ to $\boldsymbol{T_0}$, we can construct the sequences $\{\boldsymbol{T_k} = (T_k, 0) : k \in \mathbb{N}\}$ and $\{\boldsymbol{Z_k} \in \Phi : k \in \mathbb{N}\}$ with the following recursion: While $T_k < t$, define

$$T_{k+1} \quad := \quad \sup\{u \in \mathbb{R} : [T_k, u] \subset V_{\boldsymbol{Z_k}}\} = \inf\{v \in \mathbb{R} : (v, \infty) \subset V^c_{\boldsymbol{Z_k}}\} \tag{4}$$

$$\boldsymbol{Z_{k+1}} \quad := \quad \text{(a.s.) unique point of } \Phi \setminus \{\boldsymbol{Z_k}\} \text{ such that } \|\boldsymbol{T_{k+1}} - \boldsymbol{Z_{k+1}}\| = \|\boldsymbol{T_{k+1}} - \boldsymbol{Z_k}\| \tag{5}$$

Here, once we stop the recursion, say at n^*, we join $\boldsymbol{Z_{n^*}}$ to $\boldsymbol{t} = (t, 0)$ to end the Markov path $\hat{p}(\boldsymbol{s}, \boldsymbol{t}, \Phi) = \{[\boldsymbol{Z_{k-1}}, \boldsymbol{Z_k}]\}_{k=1}^{n^*}$, plus the extra segments $[\boldsymbol{s}, \boldsymbol{Z_0}]$ and $[\boldsymbol{Z_{n^*}}, \boldsymbol{t}]$. Notice that the end points are essentially not relevant in this construction (they only determined the beginning and ending of the recursion), to create the recursion we only required the line ℓ that joints them in order to create such a path, the end points only provide a criterion where to start and when should we end this recursion. Actually, it is clear that the sequence of vertices $\{\boldsymbol{Z_k} : k = 0, 1, \ldots, n^*\}$ of the path $p(\boldsymbol{s}, \boldsymbol{t}, \Phi)$ is a finite subsequence of the sequence $\{\boldsymbol{Z_k} : k \in \mathbb{N}\}$ of the nuclei of the Voronoi cells crossed by the line ℓ starting at $\boldsymbol{s} \in \ell$. We denote by $p(0, \infty, \Phi)$ the infinite path created by the segments $\{[\boldsymbol{Z_{k-1}}, \boldsymbol{Z_k}]\}_{k\in\mathbb{N}}$.

Therefore, with the above construction it can be deduced that the sequence of segments given by $\{[\boldsymbol{Z_{k-1}}, \boldsymbol{Z_k}]\}_{k\in\mathbb{N}}$ form a Markov chain. To get an idea of this fact, fix the history of the process up to the n-th step, that is up to the segment $[\boldsymbol{Z_{n-1}}, \boldsymbol{Z_n}]$, so we have the truncated path given by $\{[\boldsymbol{Z_{k-1}}, \boldsymbol{Z_k}] : k = 1, 2, \ldots n\}$. Since we already know the direction in which we are moving along, given by the line ℓ, we only required the last segment $[\boldsymbol{Z_{n-1}}, \boldsymbol{Z_n}]$ to determine the point $\boldsymbol{T_n}$ over the line ℓ, such that $\boldsymbol{Z_{n-1}}$ and $\boldsymbol{Z_n}$ are both on the boundary of the disk $B_n(\boldsymbol{Z_{n-1}}, \boldsymbol{Z_n}) = B_{\|\boldsymbol{T_n}-\boldsymbol{Z_n}\|}(\boldsymbol{T_n})$ and there is no points from Φ on its interior. Then notice, that with this disk we can partition the plane in three regions:

$$
\begin{aligned}
i) & D_{n+1} := \{\boldsymbol{x} = (x, y) \in \mathbb{R}^2 : x > x_n\} \setminus B_n \\
ii) & D_{n+1}^c \setminus B_n \\
iii) & B_n
\end{aligned}
$$

where $\boldsymbol{Z_n} = (x_n, y_n)$. Notice, that we can forget about the disk B_n because by construction we already know there are no points of Φ in its interior. Finally, due to the strong Markov property of the Poisson point process, the points distributed on D_{n+1} are independent from the ones on $D_{n+1}^c \setminus B_n$ and due to its construction it is clear that all past segments from the chain, that is $\{[\boldsymbol{Z_{k-1}}, \boldsymbol{Z_k}] : k = 1, 2, \dots n-1\}$, belongs to $D_{n+1}^c \setminus B_n$, while the next point $\boldsymbol{Z_{n+1}}$ will belong to D_{n+1}.

Finally, taken advantage of this fact the authors establish the convergence of this path to the stationary regime and find its stationary distribution. With this, it can be seen that the class of Markov paths described above $4/\pi$-approximates in mean and asymptotically the Euclidean distance, that is to say respectively

$$
\ell(\boldsymbol{Z_i}, \boldsymbol{Z_j}) = \frac{4}{\pi} d(\boldsymbol{Z_i}, \boldsymbol{Z_j}) \quad \text{and} \quad \limsup_{d(\boldsymbol{Z_i} - \boldsymbol{Z_j}) \to \infty} \ell(\boldsymbol{Z_i}, \boldsymbol{Z_j}) = \frac{4}{\pi} d(\boldsymbol{Z_i}, \boldsymbol{Z_j})
$$

4 Future Research

Along the following months I'll be dealing with new problems that emerge from considering segment processes instead of lines processes. Fundamentally, there are two aspects to be consider:

- Percolation of paths: If we made connections through a Poisson segment process, we will need to consider the fact that there might not exist a continuos path from one point to another. Some papers that can provide some insight in this matter are [Roy93] and [RT03].

- The presence of a convex hull, as the one given by the cell $\mathcal{C}(\boldsymbol{p}^-, \boldsymbol{p}^+)$ in the semi-perimeter algorithm, made the computations on [Ken11, Theorem 4] feasible. However, what can be done in the absence of this convex hull? as is the case for segment processes.

Also, regarding the comparison between the Poissonian city and real transportation networks still need to be done and compare the shape of the plots (% network against % traffic) as in the next figure shown part of the analysis realized by the British Railways Board on [Boa63].

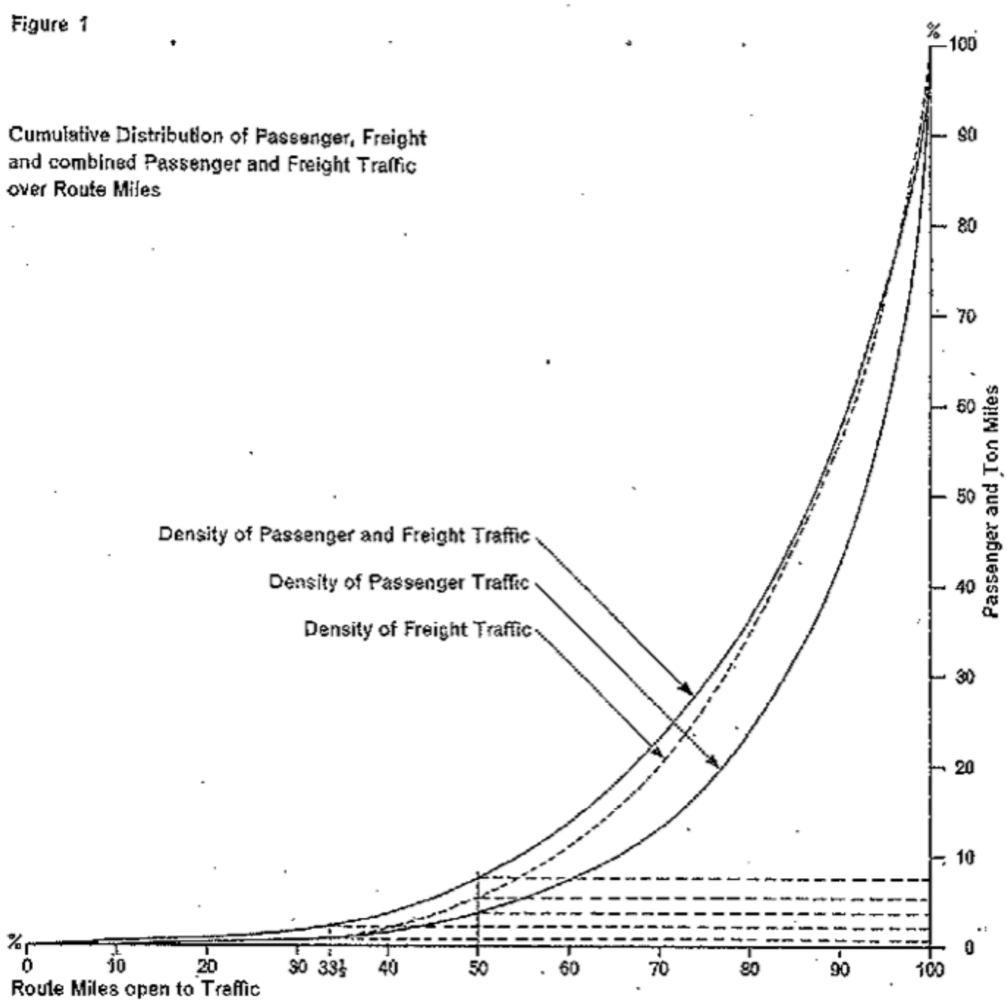

Figure 6: Cumulative Distribution of different types of Traffic over Route Miles in the British Railways system, figure taken from the Appendix 1 of [Boa63, Figure 1].

From figure 6 one can notice the concentration of traffic in a small amount of the network, say around 40% of the total traffic take place only on 10% of the route miles in the railways system. Intuitively, one will guess this occurs at the centre of the network, in case for the British railways system that is London. We will analyze if a similar phenomenon take place in the Poissonian city, for this case the centre of the network will correspond to the centre of the disk. In order to achieve this goal one approach is to generalized the result referring to the mean flow of traffic through the centre [Ken11, Theorem 5] to a general point. After that, we will be able to establish the proportion of traffic that goes through a contained disk at the centre of the city for any radius $r \in [0, n]$ and conclude if its behavior is similar to the one observed in the British railways system at 1963 (see figure 6) as r increase. Nevertheless, as computations for this could be troublesome, an alternative would be to only analyze the asymptotic behaviour of a scaled version of this flow, which gives rise to the improper anisotropic Poisson line processes studied in [Ken14].

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