PAC Report

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1 Table of contents

The contents of the Thesis will be as follows:

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2 Sketch of contents

1. Chapter 1: Basically is the motivation of this research, that is to understand in more depth the traffic behaviour over the Poissonian city. The interest in this model come from the fact that it represents an efficient transportation network [AK08]. This efficiency is measure in the sense that the network is cheap to construct, i.e. small total length (only order $o(n)$ bigger than the Steiner Tree); and it still provides short routes between any two points in averages (distances over the network differs from the Euclidean distance only by order $\mathcal{O}(\log n)$). Previous work (e.g. [AK08], [AS10] [Ken11], [Ken14b], [Ken14a]) will be briefly mentioned here. Along with the required notation for following chapters. Finally, it will state the aim of this research which are basically two:

   - Analyse the traffic behaviour over the Poissonian city and compare this against real transportation networks, e.g. British railway (development in Chapter 2).
   - Different approaches to provide a more realistic framework for the Poissonain city (development in Chapter 3).

2. Chapter 2: Along this chapter I will analyse the traffic behaviour over the Poissonian city. To do this, we assume that routes between any two points are given by the semi-perimeter algorithm and the traffic flow is divided equally over these two routes. With this, we are able to compute the mean asymptotic amount of traffic flow through any point on the network and see how this changes as the point moves away form the centre of the city. Afterwards, we compare this behaviour against the one observe in the British railway on the Beeching report [Boa63]. As we observe a similar performance, we think this fit could be improved, instead of consider the Poissonian city over a disk, we use an ellipse, which is a better approximation for the shape of Great Britain. This chapter is already in progress and I hope to submitted as a paper for a major journal by the end of this academic year.

3. Chapter 3: The aim of this chapter is to analyse the Poissonian city by changing some assumptions in order to make it a more realistic model. In the first section instead of working with line processes (infinite long lines) we restrict to the case of segment processes. Here the length for all segments is fixed, say h. To start, one had to obtain a similar expression for the length intensity of the segment processes, in such a way that the total length of the processes is preserved as we change the value of h. As expected, the the amount of segments should be proportional to $1/h$. However, with this approach the semi-perimeter algorithm will require some modifications. The second approach use fibre processes, with weak curvature instead of line processes. This proposal could be separated in two cases (if time allows it): one with infinitely long fibres, and the second with fibres of a fixed length h.

4. Chapter 4: Conclusions of this research and mention what else can be done in the future.

3 Timeline

So far the main progress have been made on Chapter 2. However, there is also a small work done on Chapter 3 (here I sketch the main ideas to work with Segment processes, material already presented on first PAC report). With this in mind, the timeline for completion will be as follows: By September 2016 have a final draft of Chapter 2, this should be submitted to major journals as a paper on Traffic behaviour over the Poissonian city.
Along term 1 and term 2 (Academic year 2016 - 2017) work towards completion of ideas for Chapter 3 (if time becomes a factor, maybe forget about the model for fibre processes with a fixed length h, hopefully this won't be the case).
From Easter 2017 to Summer 2017 polish the final version of the thesis and put all Chapters together. Also improve the contents of Chapter 1 (Introduction and motivation of this research), to be written up at the end so it is aligned with the achieved results. And finally write up the last Chapter regarding conclusions and future research.

To be notice, specially in the Draft of Chapter 2 there are a lot of comments which I include in color (red or gray). These kind of comments refer to what should be the content of that section (gray) or things to be done yet (red).

4 Draft of Chapter 1

4.1 Motivation

The aim of this research is to develop further on the idea of a "Poissonian city", that is we will try to establish a more realistic framework for this idealized model and at the same time compare this ideal city with the configuration of real transportation networks (railways and/or road networks). The Poissonian city was first introduced by Kendall on a follow up work [Ken11] to the ideas previously presented by Aldous and Kendall at [AK08].
Essentially the original idea was to construct a network to join n points in such a way that it balanced on the following criteria:

1. Short total length of the whole network. That is, considering railways for example, we are hoping to get an economic network, as the length of the network can be associated with costs (construction and/or maintenance). Remember that the shortest network is given by the Steiner Tree [PS02].

2. Small excess regarding the Euclidean distance in average. That is, however we want an economic network that still gave us, in average, short routes between either two points considered in the network. In contrast to the previous criterion, this would be achieved by the complete network (that which joins all the possible pair of points, therefore it gives the shortest route between every pair of points).

As a consequence, we can think of this problem as a trade-off between the Steiner tree network (smallest network use to join n points) and the complete network. See figure 1

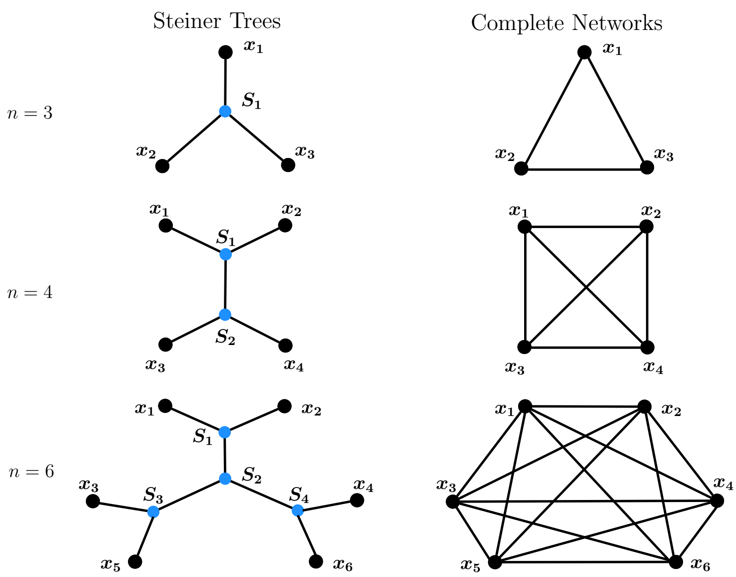

Figure 1: Examples of Steiner Trees and Complete Networks for 3, 4 and 6 points. Steiner Trees figures are based on pictures from [PS02] and [Ski09].

The "Poissonian City" provides an idealized model that emerge from the answer to the above problem presented in [AK08]. Therefore, the main aim of this research is to set down this model to be more close to reality or practical applications. This could be achieved by different means, for example: consider segment processes instead of (whole infinite) lines processes, allow weak curvature of the roads by using fibre processes. Another important approach to be considered is to compare the roadways configuration of real cities/states (only consider principal avenues/main roads, not the whole streets network) against the one proposed by the Poissonian City. However we first need to be clear on the criterion that will be used to compare these networks.

4.2 Literature Review

We have already mentioned that in the design of a network to join n points, $\boldsymbol{x^n} = \{\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}\}$, one can consider two criteria. First, the total length of the network should not be much greater than the shortest network which connects all the points (Steiner tree, $ST(\boldsymbol{x^n})$). Second, the average route length should not be much greater than the average Euclidean distance (straight-lines between source and destination points), i.e. in average the distance between two points should stay close to the one in the complete network.
The main problem addressed by Aldous and Kendall in [AK08] is regarding the order of this excesses, that is, how small can we make them? To answer this question the authors constructed a random network such as the above excesses are of order $o(n)$ and $\mathcal{O}(\log n)$, respectively.
First, it is important to be clear on how to measure the mentioned excesses in a given network, $G(\boldsymbol{x^n})$. The following notation is required to define these measures: $ST(\boldsymbol{x^n})$ denote the Steiner tree corresponding to the configuration of n points given by $\boldsymbol{x^n}$, $\ell(\boldsymbol{x_i}, \boldsymbol{x_j})$ is the shortest route length between points $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$ in the given network $G(\boldsymbol{x^n})$, $d(\boldsymbol{x_i}, \boldsymbol{x_j})$ is the Euclidean distance between the points $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$. Hence, the aim is to control the order of the following quantities:

- excess length $(G(\boldsymbol{x^n})) = \text{length}(G(\boldsymbol{x^n})) - \text{length}(ST(\boldsymbol{x^n}))$.
- excess distance $(G(\boldsymbol{x^n})) = \frac{1}{n(n-1)} \sum_{i \neq j} \big(\ell(\boldsymbol{x_i}, \boldsymbol{x_j}) - d(\boldsymbol{x_i}, \boldsymbol{x_j})\big)$

The desired network arises from considering a hierarchical construction. At small scales the idea is to use the underlying Steiner tree to connect points. However, for long distances the network will depend on the sparse collection of randomly oriented lines, that is one realisation of a stationary and isotropic Poisson line process, which turns out to be the key characteristic to achieve an excess of $\mathcal{O}(\log n)$. Finally, to simplify calculations an intermediate scale consisting of grid lines is required. With this in mind, the rest of the paper [AK08] focus mainly on the following result:

Theorem 1 (Upper bound on minimum excess network length)**.** *[AK08, Theorem 1]*
For any configuration $\boldsymbol{x^n}$ in square side $\sqrt{n}$ and for any $\epsilon > 0$ there are connecting networks $G(\boldsymbol{x^n})$ such that

$$\textit{excess length } (G(\boldsymbol{x^n})) \leq \epsilon n$$
$$\textit{excess distance } (G(\boldsymbol{x^n})) \leq \mathcal{O}(\log n)$$

Also, this paper analyze a similar result to get lower bounds on the average excess distance as long as one had an extra condition on the configuration of the n points, $\boldsymbol{x^n}$, which are joint by the network. Basically, one desires to avoid non-random configurations on these points, that is we want *equidistributed* points.

Definition 1 (L_n-equidistributed). *[AK08, Definition 3]*
Let $\boldsymbol{x^n}$, for varying n, form a sequence of configurations in the plane and let $L_n > 0$. We said that $\boldsymbol{x^n}$ is L_n-equidistributed *if a random choice $\boldsymbol{x_n}$ of a city point can be coupled to a uniformly random point $\boldsymbol{y_n}$ so that*

$$\mathbb{E}\left[\min\left\{1, \frac{|\boldsymbol{x_n} - \boldsymbol{y_n}|}{L_n}\right\}\right] \to 0 \quad \text{as } n \to \infty$$

It turns out that a wide range of possible point patterns can be seen to be L_n-equidistributed. For this cases, we have the following lower-bound

Theorem 2 (Lower bound on minimum excess network length). *[AK08, Theorem 2]*
Given an L_n-equidistributed configurations of cities, $\boldsymbol{x^n}$, in $[0, \sqrt{n}]^2$ with $L_n = o\big(\sqrt{\log n}\big)$, then any connecting network $G(\boldsymbol{x^n})$ with length bounded above by a multiple of n connects the city points with average connection length exceeding average Euclidean connection length by at least $\Omega(\sqrt{\log n})$, that is

$$\textit{excess distance } (G(\boldsymbol{x^n})) = \Omega(\sqrt{\log n})$$

Thus, $\liminf_{n\to\infty}$ *excess distance* $(G(\boldsymbol{x^n}))/\sqrt{\log n} > 0$.

At the same time, it is important to notice the main two branches that emerge from the network proposed by Aldous and Kendall [AK08]:

- How to define a good statistic that captures the short-routes property in a given network? As shown by the hierarchical network the ratio statistic or the excess average doesn't seem to work out completely, because this gives an example of a non-realistic transportation network. Some further thoughts on this direction are presented by Aldous and Shun [AS10] where they introduced an intermediate statistic between the average distance excess and the maximum distance excess to achieve this goal.

- What else can be said about a network consisting of one realization of an stationary and isotropic Poisson line process? Specifically what are its geometric properties and how the semi-perimeter route (near-geodesic) compare to actual geodesic in this network. This and other questions are solved in [Ken11], which will be the approach to be followed along this research.

The concept of a *Poissonian city* was first introduced in [Ken11], that is a random network generated by a unit intensity stationary and isotropic Poisson line process, Π. The idea is to use such a network to connect n points in a disk of radius n through the semi-perimeter algorithm that was already introduced in [AK08] to establish the upper bound result, see figure 2. The results presented in this work explores the following ideas:

- What is the shape of near-geodesics in this network?
- Given a good trade-off between the excess length ($G(\boldsymbol{x}^{\boldsymbol{n}})$) and excess distance ($G(\boldsymbol{x}^{\boldsymbol{n}})$), how might the variance of these near-geodesics behave?
- The upper-bound is obtained by controlling these near-geodesic paths. How might true geodesics behave in comparison to the semi-perimeter route?
- Consider we attach certain amount of traffic to this network. What can be said about flows of traffic in this network? In particular, how does the traffic behave at the centre of the Poissonian city?

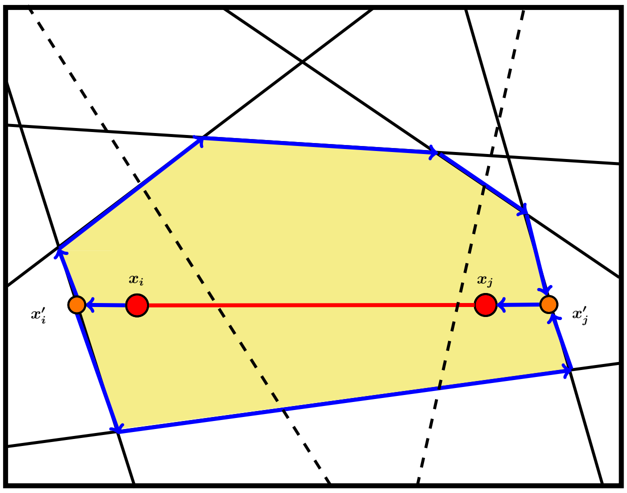

Figure 2: Construction of the semi-perimeter paths.

Specifically we will focus on the properties concerning the last two questions.

The main result regarding the shape of true geodesics is actually an improvement to the lower bound presented by Aldous and Kendall [AK08].

Theorem 3 (Lower Bound from the Poissonian city)**.** *[Ken11, Theorem 4]*
In the Poissonian city network, consider any path from $\boldsymbol{p}^-$ to $\boldsymbol{p}^+$ that is contained in the cell $\mathcal{C}(\boldsymbol{p}^-, \boldsymbol{p}^+)$ (determined by the semi-perimeter route). If $d(\boldsymbol{p}^-, \boldsymbol{p}^+) = n$, then the path must have mean excess exceeding

$$2\left(\log 4 - \frac{5}{4}\right)\log n + o(\log n) = 0.27258872\ldots \log n + o(\log n)$$

One of the specific aims of my research will be to generalize this result to the case where the connections are made by a Poisson segment process of fixed length h and intensity λ in such a way that the *length intensity* of this segment process is equivalent to the unitary Poisson line process. Alternative approaches could be to consider fibre processes with weak curvature.

On the other hand, when we consider the flow of traffic through the center of the city, the main result tell us its mean asymptotic behaviour

Theorem 4 (Mean flow through the centre). *[Ken11, Theorem 5]*
The mean flow through a line at the center of a Poissonian city that connects n points is given by the expression

$$\mathbb{E}[F_n] \quad = \quad \int_0^\pi \int_0^n \int_0^n \exp\left\{-\frac{1}{2}(r+s-\rho)\right\} r \,\mathrm{d}rs \,\mathrm{d}s\theta \,\mathrm{d}\theta \tag{1}$$

where $\rho = \sqrt{r^2+s^2+2rs\cos\theta}$. *Asymptotically, as* $n \to \infty$,

$$\mathbb{E}[F_n] \sim 2n^3 \tag{2}$$

Now, another specific aim for my research is to analyze how the amount of traffic fluctuates around the city, depending mainly on the Euclidean distance from the analyzed point to the centre of the city. For example, we can consider the proportion of traffic that is contained in a disk centered at the centre of the city with varying radius $r \in [0, n]$ and see if this behaviour is similar in some way to real transportation network, e.g. the british railway system, see [Boa63].

5 Draft of Chapter 2 / Paper

Abstract

5.1 Introduction

General ideas to be discussed:

- *Poissonian City (approaches already taken on [AK08], [Ken11], [Ken14b], [Ken14a]).*
- *Semi-perimeter algorithm, Convex hull [AK08].*
- *Conditioning on one line being part of the line process [Ken11].*
- *Palm theory[CSKM13] (maybe look for the original reference).*
- *Notation*
- *Directory of results*
- *So far...*

Important to work on the organization and structure of each section. Put the maths in concrete sentences (Theorems, Lemmas, etc.)

5.2 Mean traffic at a particular point

In 2011, Kendall showed that the mean traffic flow at the centre of the Poissonian city, conditioned on the presence of one line that goes through the centre, behaves asymptotically as $2n^3$ [Ken11, Theorem 5]. This section generalize the previous result to any point $\boldsymbol{q}$ of the Poissonian city, now conditioning on a particular line passing through $\boldsymbol{q}$.

First, consider the mean flow through a point $\boldsymbol{q}$ at a distance tn ($t \in (0,1)$) from the centre of the city, $\boldsymbol{o}$, conditioning on the existence of a line that goes through both points $\boldsymbol{q}$ and $\boldsymbol{o}$. The flow through $\boldsymbol{q}$ results if every $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathcal{B}_n(\boldsymbol{o})$ generates an infinitesimal flow of amount $\mathrm{d}\boldsymbol{x}\ \mathrm{d}\boldsymbol{y}$ divided equally between the two possible routes given by the semi-perimeter algorithm.

To achieve this goal it is required to have an expression for the flow through the point $\boldsymbol{q}$, which is given by the following 4-volume

$$\mathcal{D}_n^q \quad = \quad \left\{(\boldsymbol{p}^-, \boldsymbol{p}^+) \in \mathcal{B}_n(\boldsymbol{o})^2 : p_1^- < p_1^+, \boldsymbol{q} \in \partial\mathcal{C}(\boldsymbol{p}^-, \boldsymbol{p}^+)\right\},$$

where $\mathcal{C}(\boldsymbol{p}^-, \boldsymbol{p}^+)$ stands for the convex cell containing $\boldsymbol{p}^-$ and $\boldsymbol{p}^+$ which arises by deleting all the Poisson lines which separates $\boldsymbol{p}^-$ from $\boldsymbol{p}^+$ as explained in [AK08]. In consequence, $\partial\mathcal{C}(\boldsymbol{p}^-, \boldsymbol{p}^+)$

represents the *near geodesics* used to connect $\boldsymbol{p}^-$ and $\boldsymbol{p}^+$ through the semi-perimeter algorithm. Then the distribution of the total traffic through $\boldsymbol{q}$ is given by

$$T_n^q \quad = \quad \frac{1}{2}\iint \mathbb{1}_{\{(\boldsymbol{p}^-,\boldsymbol{p}^+)\in\mathcal{D}_n^q\}}\,\mathrm{d}\boldsymbol{p}^-\ \mathrm{d}\boldsymbol{p}^+ = \frac{1}{2}\iint\limits_{\mathcal{B}_n(\boldsymbol{o})^2} \mathbb{1}_{\{p_1^-<p_1^+,\boldsymbol{q}\in\partial\mathcal{C}(\boldsymbol{p}^-,\boldsymbol{p}^+)\}}\,\mathrm{d}\boldsymbol{p}^-\ \mathrm{d}\boldsymbol{p}^+ .$$

Due to the disk symmetries, one can only consider the case where both points lies within the upper semicircle, i.e. $p_2^-, p_2^+ > 0$, to obtain

$$\begin{aligned}\mathbb{E}[T_n^q] \quad &= \quad 2\left(\frac{1}{2}\iint\limits_{\{\mathcal{B}_n(\boldsymbol{o})^2:p_2^-,p_2^+>0\}} \mathbb{E}[\mathbb{1}_{\{p_1^-<p_1^+,\boldsymbol{q}\in\partial\mathcal{C}(\boldsymbol{p}^-,\boldsymbol{p}^+)\}}]\,\mathrm{d}\boldsymbol{p}^-\ \mathrm{d}\boldsymbol{p}^+\right)\\
&= \iiiint \exp\left\{-\frac{1}{2}(r+s-\rho)\right\} r\,\mathrm{d}r\, s\,\mathrm{d}s\ \mathrm{d}\varphi_1\ \mathrm{d}\varphi_2 = \iiint \exp\left\{-\frac{1}{2}(r+s-\rho)\right\} r\,\mathrm{d}r\, s\,\mathrm{d}s\,\theta\,\mathrm{d}\theta\end{aligned}$$

Here $\boldsymbol{p}^- = (r,\varphi_1)$ and $\boldsymbol{p}^+ = (s,\varphi_2)$ in polar coordinates, and $\rho = \sqrt{r^2+s^2+2rs\cos\theta}$ (by cosine law, with $\theta = \varphi_1+\varphi_2$). For the last equality we used the auxiliary variable $\psi = \varphi_1 - \varphi_2$ to obtain

$$\int_0^\pi\int_0^\pi \mathrm{d}\varphi_1\ \mathrm{d}\varphi_2 = \int_0^{2\pi}\int_{-\theta}^{\theta}\frac{1}{2}\,\mathrm{d}\psi\ \mathrm{d}\theta = \int_0^{2\pi}\theta\,\mathrm{d}\theta.$$

The difference regarding the mean flow through the point $\boldsymbol{q}$ instead of the centre of the city $\boldsymbol{o}$ is given by the region of integration. The integral that represents the mean flow through $\boldsymbol{q}$ can be split up in a similar manner to the way in which [Ken11] splits up the integral for the mean traffic through the centre. So the region of integration can be splitted in terms of the angle θ. Taking $\theta_n \downarrow 0$ the range of integration for θ can be separated into $[0,\theta_n]$ and $(\theta_n,\pi]$. Notice that the region $(\pi,2\pi]$ should not be considered to avoid counting twice the same flow (from r to s and from s to r). The second region is bounded above by

$$\int_{\theta_n}^{\pi}\int_0^{(1+t)n}\int_0^{(1+t)n}\exp\left\{-\frac{1}{2}(r+s-\rho)\right\} r\,\mathrm{d}r\, s\,\mathrm{d}s\,\theta\,\mathrm{d}\theta\,.$$

This is illustrated in figure 3. Here the polar coordinates (r,φ_1) and (s,φ_2) (recall $\theta=\varphi_1+\varphi_2$) corresponds now to the polar coordinates of $\boldsymbol{p}^-$ and $\boldsymbol{p}^+$ regarding the point $\boldsymbol{q} = (tn,0)$ as its reference centre. To get the asymptotics, scale the coordinates r and s by n, and use the symmetry between both variables to obtain

$$2n^4\int_{\theta_n}^{\pi}\int_0^{1+t}\int_0^{s}\exp\left\{-\frac{n}{2}(r+s-\sqrt{r^2+s^2+2rs\cos\theta})\right\} r\,\mathrm{d}r\, s\,\mathrm{d}s\,\theta\,\mathrm{d}\theta \tag{3}$$

Scaling now the variable r by s to get

$$2n^4\int_{\theta_n}^{\pi}\int_0^{1+t}\int_0^{1}\exp\left\{-\frac{ns}{2}(r+1-\sqrt{r^2+1+2r\cos\theta})\right\} r\,\mathrm{d}r\, s^3\,\mathrm{d}s\,\theta\,\mathrm{d}\theta$$

Rewrite $\sqrt{r^2+1+2r\cos\theta}$ as $\sqrt{(r+1)^2-2r(1-\cos\theta)}$. Then use the inequality $\sqrt{1-2z}\leq$

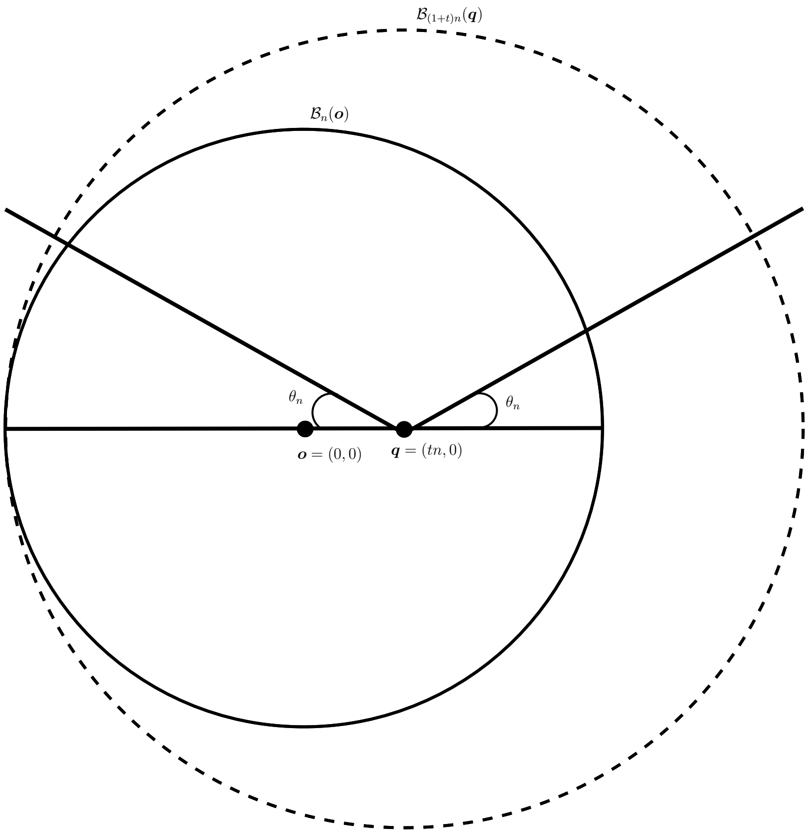

Figure 3: How to bound the area corresponding to $[\theta_n, \pi]$.

$1 - z$, for $z \in [0, 1/2]$, with $z = r(1-\cos\theta)/(r+1)^2$ to bound the integral (3) by

$$\begin{aligned}
& 2n^4 \int_{\theta_n}^{\pi} \int_0^{1+t} \int_0^1 \exp\left\{ -\frac{ns(r+1)}{2} \left(1 - \sqrt{1 - 2\frac{r(1-\cos\theta)}{(r+1)^2}} \right) \right\} r \, \mathrm{d}r \, s^3 \, \mathrm{d}s \, \theta \, \mathrm{d}\theta \\
\leq \quad & 2n^4 \int_{\theta_n}^{\pi} \int_0^{1+t} \int_0^1 \exp\left\{ -\frac{nsr}{2(r+1)} (1 - \cos\theta) \right\} r \, \mathrm{d}r \, s^3 \, \mathrm{d}s \, \theta \, \mathrm{d}\theta \\
\leq \quad & 2n^4 \frac{\pi^2 - \theta_n^2}{2} \int_0^{1+t} \int_0^1 \exp\left\{ -\frac{nsr}{2(r+1)} (1 - \cos\theta_n) \right\} r \, \mathrm{d}r \, s^3 \, \mathrm{d}s \\
\leq \quad & n^4 (\pi^2 - \theta_n^2) \int_0^{1+t} \int_0^{\infty} \exp\left\{ -\frac{nsr}{4} (1 - \cos\theta_n) \right\} r \, \mathrm{d}r \, s^3 \, \mathrm{d}s \\
= \quad & (\pi^2 - \theta_n^2) \left(\frac{4}{1 - \cos\theta_n} \right)^2 n^2 \int_0^{1+t} s \, \mathrm{d}s = 8(\pi^2 - \theta_n^2) \left(\frac{1+t}{1 - \cos\theta_n} \right)^2 n^2
\end{aligned}$$

On the other hand, the integral of the region corresponding to $[0, \theta_n]$ can be approximated by the rotational invariant regions given by

$$(r, \varphi_1) \in [0, (1+t)n] \times [0, \theta_n] \qquad (s, \varphi_2) \in [0, (1-t)n] \times [0, \theta_n]$$

Thus, figure 4, the mean traffic through $\boldsymbol{q} = (tn, 0)$ is asymptotically given by the expression

$$
\begin{aligned}
\mathbb{E}[T_n^q] &\sim \int_0^{\theta_n} \int_0^{(1+t)n} \int_0^{(1-t)n} \exp\left\{-\frac{1}{2}(r+s-\rho)\right\} r\,\mathrm{d}r\, s\,\mathrm{d}s\, \theta\,\mathrm{d}\theta \\
&= n^4 \int_0^{\theta_n} \int_0^{1+t} \int_0^{1-t} \exp\left\{-\frac{n}{2}(r+s-\sqrt{r^2+s^2+2rs\cos\theta})\right\} r\,\mathrm{d}r\, s\,\mathrm{d}s\, \theta\,\mathrm{d}\theta \\
&= n^4 \left[\int_0^{\theta_n} \int_0^{1+t} \int_0^{\frac{1-t}{1+t}s} \exp\left\{-\frac{n}{2}(r+s-\sqrt{r^2+s^2+2rs\cos\theta})\right\} r\,\mathrm{d}r\, s\,\mathrm{d}s\, \theta\,\mathrm{d}\theta \right. \qquad (4) \\
&\qquad \left. + \int_0^{\theta_n} \int_0^{1+t} \int_{\frac{1-t}{1+t}s}^{1-t} \exp\left\{-\frac{n}{2}(r+s-\sqrt{r^2+s^2+2rs\cos\theta})\right\} r\,\mathrm{d}r\, s\,\mathrm{d}s\, \theta\,\mathrm{d}\theta \right]
\end{aligned}
$$

Now analyse the final two integrals. First rewrite ρ in the same way as before. Then notice that $\sqrt{1-2z} \sim 1-z$ with same z as above (since $\sqrt{1-2z} \leq 1-z$ for $z \in [0, 1/2]$ and $1 - z(1+\theta) \leq \sqrt{1-2z}$ for $z \in [0, 2\theta/(1+\theta)^2]$). After a change of variable, we apply a Taylor series expansion to get that $\theta/\sin\theta \sim 1$ as $\theta \downarrow 0$ (to obtain this recall that $\sin\theta_n = \theta_n - \theta_n^3/3! + \varphi^5/5!$ for some $\varphi \in [0, \theta_n]$ by Taylor expansion with residue, therefore $1 \leq \theta_n/\sin\theta_n \leq \theta_n/(\theta_n - \theta_n/3!) = 1 + \theta_n^2/(6-\theta_n^2)$).

$$
\begin{aligned}
&\int_0^{\theta_n} \int_0^{1+t} \int_0^{\frac{1-t}{1+t}} \exp\left\{-\frac{ns}{2}(r+1-\sqrt{r^2+1+2r\cos\theta})\right\} r\,\mathrm{d}r\, s^3\,\mathrm{d}s\, \theta\,\mathrm{d}\theta \\
&= \int_0^{\theta_n} \int_0^{1+t} \int_0^{\frac{1-t}{1+t}} \exp\left\{-\frac{ns(r+1)}{2}\left(1-\sqrt{1-2\frac{r(1-\cos\theta)}{(r+1)^2}}\right)\right\} r\,\mathrm{d}r\, s^3\,\mathrm{d}s\, \theta\,\mathrm{d}\theta \\
&\sim \int_0^{\theta_n} \int_0^{1+t} \int_0^{\frac{1-t}{1+t}} \exp\left\{-\frac{nsr}{2(r+1)}(1-\cos\theta)\right\} r\,\mathrm{d}r\, s^3\,\mathrm{d}s\, \theta\,\mathrm{d}\theta \\
&= \int_0^{1-\cos\theta_n} \int_0^{1+t} \int_0^{\frac{1-t}{1+t}} \exp\left\{-\frac{nsr}{2(r+1)}u\right\} r\,\mathrm{d}r\, s^3\,\mathrm{d}s\, \frac{\theta}{\sin\theta}\,\mathrm{d}u \qquad [u = 1-\cos\theta] \\
&\sim \int_0^{\frac{1-t}{1+t}} \int_0^{1+t} \int_0^{\infty} \exp\left\{-\frac{nsr}{2(r+1)}u\right\} \mathrm{d}u\, s^3\,\mathrm{d}s\, r\,\mathrm{d}r \\
&= \int_0^{\frac{1-t}{1+t}} \int_0^{1+t} \frac{2(r+1)}{nsr} s^3\,\mathrm{d}s\, r\,\mathrm{d}r = \frac{1}{n}\int_0^{\frac{1-t}{1+t}} 2(r+1)\,\mathrm{d}r \int_0^{1+t} s^2\,\mathrm{d}s \\
&= \frac{1}{n}\left[\left(\frac{1-t}{1+t}+1\right)^2 - 1\right]\frac{(1+t)^3}{3} = \frac{1}{n}\frac{(3+t)(1-t)}{(1+t)^2}\frac{(1+t)^3}{3} = \frac{(3+t)(1-t^2)}{3n}
\end{aligned}
$$

On the other hand, notice that the second integral from (4) can be rewritten by exchanging the order of integration of r and s as follows

$$
\int_0^{\theta_n} \int_0^{1-t} \int_0^{\frac{1+t}{1-t}r} \exp\left\{-\frac{n}{2}(r+s-\sqrt{r^2+s^2+2rs\cos\theta})\right\} s\,\mathrm{d}s\, r\,\mathrm{d}r\, \theta\,\mathrm{d}\theta
$$

Rescaling s by r and following similar computations as for the first integral in (4), one obtains

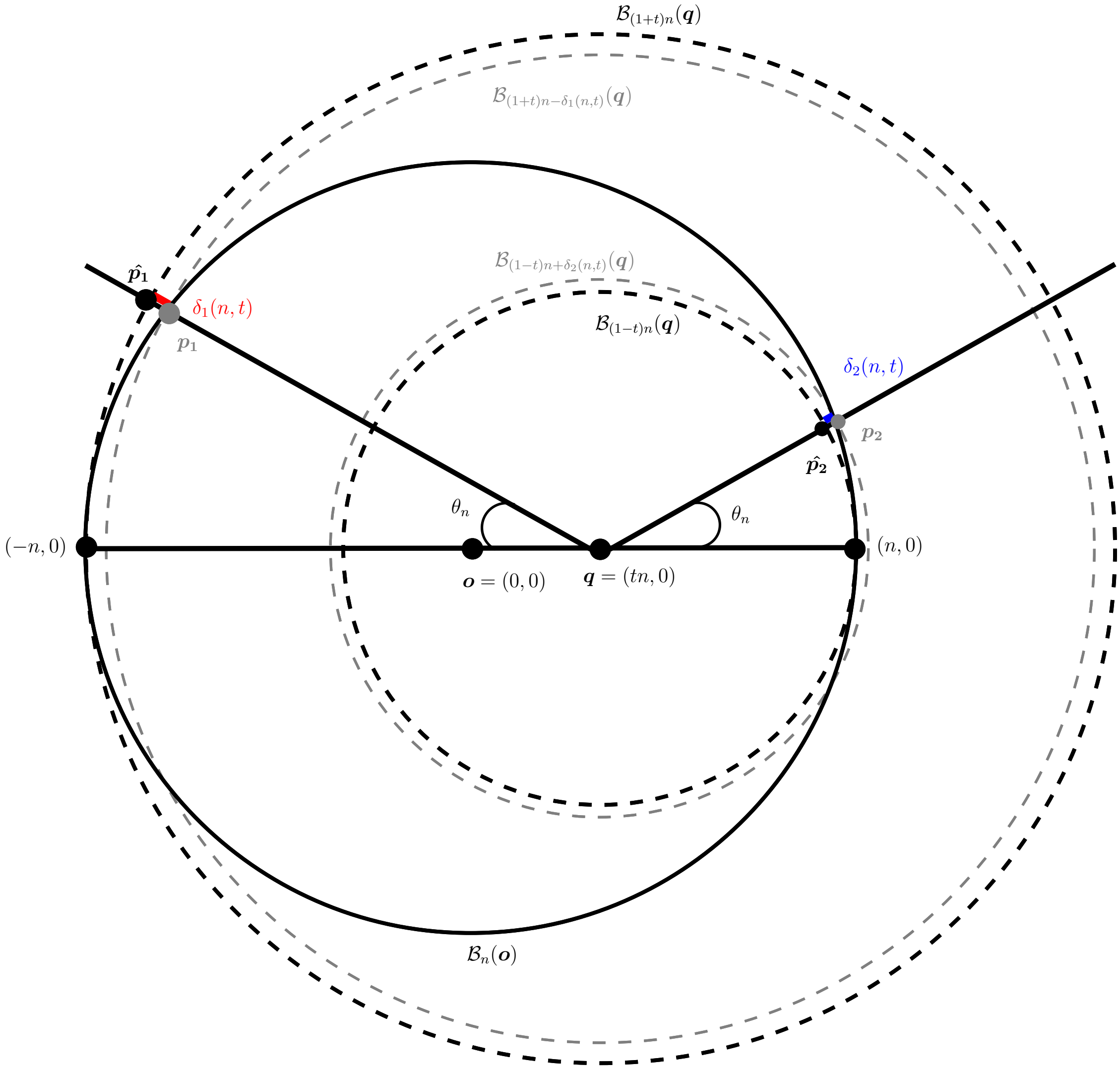

Figure 4: How to approximate the area corresponding to $[0,\theta_n]$.

$$
\begin{aligned}
&\int_0^{\theta_n}\int_0^{1-t}\int_0^{\frac{1+t}{1-t}} \exp\left\{-\frac{nr}{2}(s+1-\sqrt{s^2+1+2s\cos\theta})\right\} s\,\mathrm{d}s\, r^3\,\mathrm{d}r\,\theta\,\mathrm{d}\theta \\
&= \int_0^{\theta_n}\int_0^{1-t}\int_0^{\frac{1+t}{1-t}} \exp\left\{-\frac{nr(s+1)}{2}\left(1-\sqrt{1-2\frac{s(1-\cos\theta)}{(s+1)^2}}\right)\right\} s\,\mathrm{d}s\, r^3\,\mathrm{d}r\,\theta\,\mathrm{d}\theta \\
&\sim \int_0^{\theta_n}\int_0^{1-t}\int_0^{\frac{1+t}{1-t}} \exp\left\{-\frac{nrs}{2(s+1)}(1-\cos\theta)\right\} s\,\mathrm{d}s\, r^3\,\mathrm{d}r\,\theta\,\mathrm{d}\theta \\
&= \int_0^{1-\cos\theta_n}\int_0^{1-t}\int_0^{\frac{1+t}{1-t}} \exp\left\{-\frac{nrs}{2(s+1)}u\right\} s\,\mathrm{d}s\, r^3\,\mathrm{d}r\,\frac{\theta}{\sin\theta}\,\mathrm{d}u \qquad [u=1-\cos\theta] \\
&\sim \int_0^{\frac{1+t}{1-t}}\int_0^{1-t}\int_0^{\infty} \exp\left\{-\frac{nrs}{2(s+1)}u\right\} \mathrm{d}u\, r^3\,\mathrm{d}r\, s\,\mathrm{d}s \\
&= \int_0^{\frac{1+t}{1-t}}\int_0^{1-t} \frac{2(s+1)}{nrs} r^3\,\mathrm{d}r\, s\,\mathrm{d}s = \frac{1}{n}\int_0^{\frac{1+t}{1-t}} 2(s+1)\,\mathrm{d}s \int_0^{1-t} r^2\,\mathrm{d}r \\
&= \frac{1}{n}\left[\left(\frac{1+t}{1-t}+1\right)^2-1\right]\frac{(1-t)^3}{3} = \frac{1}{n}\frac{(3-t)(1+t)}{(1-t)^2}\frac{(1-t)^3}{3} = \frac{(3-t)(1-t^2)}{3n}
\end{aligned}
$$

In conclusion, asymptotically as $n \to \infty$, the mean traffic flow at the point $\boldsymbol{q} = (tn, 0)$ ($t \in (0,1)$) conditioning on there being a line that goes through $\boldsymbol{o}$ and $\boldsymbol{q}$ is given by

$$\mathbb{E}[T_n^q] \sim n^4 \left[\frac{(3-t)(1-t^2)}{3n} + \frac{(3+t)(1-t^2)}{3n} \right] = 2n^3(1-t^2) = 2n\big[(1+t)n\big]\big[(1-t)n\big] \quad (5)$$

Notice that when using the rotational invariance area there are two errors to be analysed yet illustrated on figure 4. Nonetheless, one can show that the contribution from these regions is negligible. To show this consider the following lemma

Add correct version of this lemma, since previous one have a mistake on it. With this in mind correct the following text.

Lemma 1. *The area of any circular sector formed by the fixed angle θ and the line that joins two points, $\boldsymbol{x}$ (central point) and $\boldsymbol{y}$ (point over the arc segment of the circular sector), that lies on a disk of radius r is bounded above by $2r^2\theta$*

Thence, the first error can be bounded by the area of the circular sector formed by the central point $\boldsymbol{q}$, the angle θ_n and the point $\hat{\boldsymbol{p}}_1$. Notice, we can use just the area of this region as a bound, as the exact error will be an integral of a probability quantity (less or equal than 1) over a smaller region, which is contained on this circular sector. The smaller region that we are referring to is detemined by the points $\hat{\boldsymbol{p}}_1$, $\boldsymbol{p}_1$ and $(-n, 0)$ and their connecting curves, see figure 4. Therefore

$$\text{Error}_1 \leq A_{\text{Circular Sector}_1} = \frac{1}{2}[(1+t)n]^2\theta_n.$$

Similarly, the second error is bounded above by the circular sector formed by the central point $\boldsymbol{q}$, the angle θ_n and the point $\boldsymbol{p}_2$. So, as both points lies on the disk of radius n centre at the origin, applying Lemma 1 one gets the next upper bound

$$\text{Error}_2 \leq A_{\text{Circular Sector}_2} \leq 2n^2\theta_n.$$

Returning to the asymptotic behaviour given by equation (5) one can infer that the mean asymptotic traffic flow through a point $\boldsymbol{q}$ in a line ℓ with endpoints $\boldsymbol{a}$ and $\boldsymbol{b}$ ($\vec{\boldsymbol{ab}}$) is given by the product of the distances between $\vec{\boldsymbol{ab}}$, $\vec{\boldsymbol{aq}}$ and $\vec{\boldsymbol{qb}}$.

To prove the above claim consider figure 5. As long as the two overestimating areas (delimited by the points: $(\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2})$, $(\tilde{\boldsymbol{x_0}}, \tilde{\boldsymbol{x_1}}, \tilde{\boldsymbol{x_2}})$ and their connecting curves); and the two underestimating areas (bounded by the points: $(\boldsymbol{x_0}, \boldsymbol{x_3}, \boldsymbol{x_4})$, $(\tilde{\boldsymbol{x_0}}, \tilde{\boldsymbol{x_3}}, \tilde{\boldsymbol{x_4}})$ and their connecting curves) have an expression of order $o(n^3)$ then the result will follow from the previous argument. To achieve this goal, notice that Lemma 1 can be applied to four circular sectors, where each one contains one of the triangular sectors in question.

Notice that the triangular area between $(\boldsymbol{x_0}, \boldsymbol{x_1}, \boldsymbol{x_2})$ is contained in the circular sector formed by the central point $\boldsymbol{q}$, the angle θ_n and $\boldsymbol{x_2}$, see figure 5. In a similar manner, the triangular area between $(\tilde{\boldsymbol{x_0}}, \tilde{\boldsymbol{x_1}}, \tilde{\boldsymbol{x_2}})$ is contained in the circular sector delimited by the central point $\boldsymbol{q}$, the angle θ_n and $\tilde{\boldsymbol{x_2}}$. Now, by construction the points $\boldsymbol{x_2}, \tilde{\boldsymbol{x_2}}$ and $\boldsymbol{q}$ lies on the disk $\mathcal{B}_{n\sqrt{1-u^2}}(0, un)$ (since $\boldsymbol{x_2}$ ($\tilde{\boldsymbol{x_2}}$) is the intersection of the circle $x^2 + (y - un)^2 = n^2(1 - u^2)$ with the line $y = -\tan\theta_n(x - tn)$ ($y = \tan\theta_n(x - tn)$, respectively)). Therefore, each of these error is bounded above by $2(n\sqrt{1-u^2})^2\theta_n \leq 2n^2\theta_n$ for any $u \in [0, 1]$.

Similarly, as $\boldsymbol{x_4}, \tilde{\boldsymbol{x_4}}$ and $\boldsymbol{q}$ lies on the disk $\mathcal{B}_n(\boldsymbol{o})$, as a consequence of Lemma 1 each one of the underestimating areas is bounded above by $2n^2\theta_n$.

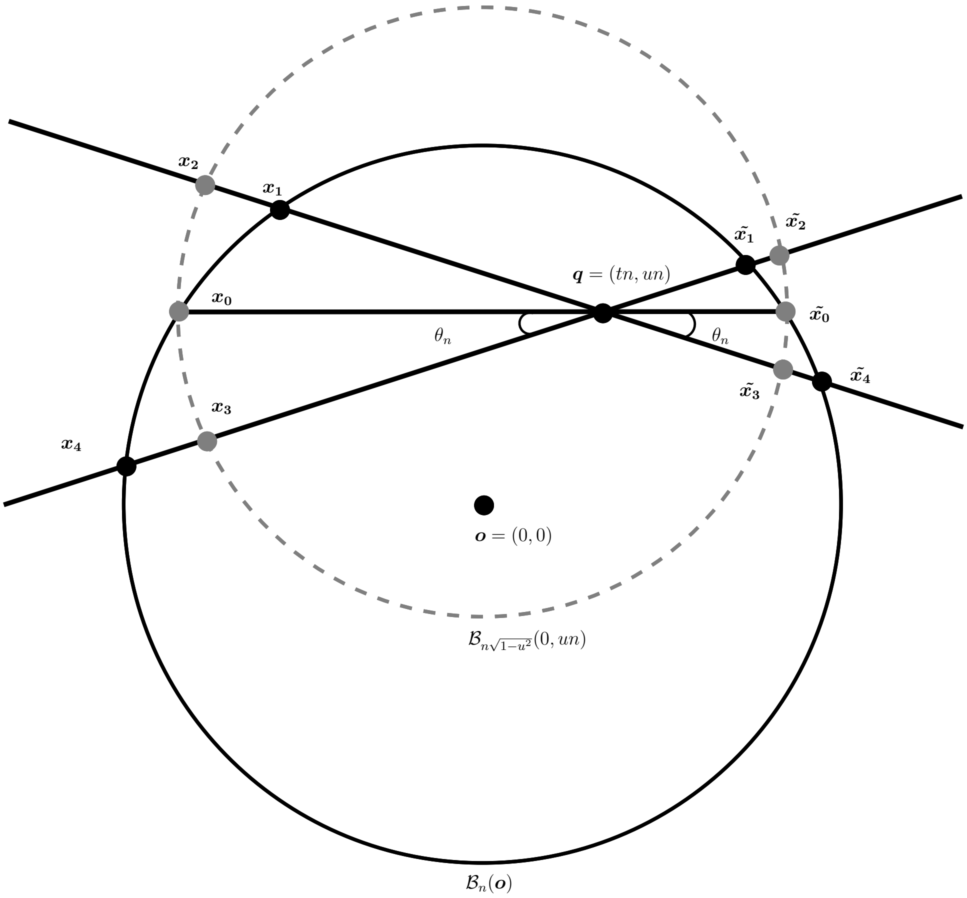

Figure 5: How to generalize the result to any point $\boldsymbol{q} = (tn, un)$.

Add link with next section.

5.3 Exact Asymptotics

Work on exact asymptotics from previous calculations and add link with the next sections.

5.4 Palm Distribution

Now that there is a formula for the traffic through a particular point $\boldsymbol{q}$ at the Poissonian city conditioning on the presence of a particular line ℓ that passes through $\boldsymbol{q}$. *Palm theory* provides the required insight to compute the mean traffic over the whole disk (or sub-disks) in terms of this deterministic formula; instead of working with random points over the Poisson line process (that represents the origin and destination of the traffic flow) one can work with the whole disk of radius n. To achieve this exchange consider the following relation:

$$\begin{aligned}
\mathbb{E}\left[\sum_{x\in\Phi}\sum_{y\in\Phi}\int_{\Pi} f_{x,y}\,\mathcal{H}^1_{\Pi}(\,\mathrm{d}t\,)\right] &= \mathbb{E}\left[\sum_{x\in\Phi}\sum_{y\in\Phi}\sum_{\ell\in\Pi}\int_{\ell} f_{x,y}\,\mathcal{H}^1_{\ell}\,(\,\mathrm{d}t\,)\right]\\
&= \lambda^2\mathbb{E}\left[\int_{\Pi}\int_{\Pi}\left(\sum_{\ell\in\Pi}\int_{\ell} f_{x,y}\,\mathcal{H}^1_{\ell}(\,\mathrm{d}t\,)\right)\mathcal{H}^1_{\Pi}(\,\mathrm{d}y\,)\,\mathcal{H}^1_{\Pi}(\,\mathrm{d}x\,)\right]\\
&= \lambda^2\iint_{\mathcal{B}_n(\boldsymbol{o})}\mathbb{E}\left[\int_{\Pi\cup\ell}\int_{\Pi\cup\ell}\left(\int_{\ell(r,\theta)} f_{x,y}\,\mathcal{H}^1_{\ell}(\,\mathrm{d}t\,)\right)\mathcal{H}^1_{\Pi}(\,\mathrm{d}y\,)\,\mathcal{H}^1_{\Pi}(\,\mathrm{d}x\,)\right]\Lambda(\,\mathrm{d}r,\,\mathrm{d}\theta\,)
\end{aligned}$$

REMARK: Include computations that justify using the deterministic integral that I will use on the next section to compute how the mean amount of traffic is decreasing as one consider bigger sub disks centred at the origin.
Also, add more context for the requirement of Palm Theory and its role on the next section. Compare $\tilde{T}_n$ against T_n, see Wilfrid notes regarding this subject.

5.5 Comparison against actual transportation networks

Comparison against the Beeching report, figure 1 of the first appendix.

5.6 Poissonian city over an ellipse

Generalize previous ideas for the case where the Poisson process lines is analysed over ellipses instead of disks.

5.7 Final remarks

6 Draft for Chapter 3 / Segment Processes

Consider a stationary Poisson segment process of fixed length h. This subsection addresses the following question: what intensity, ν_λ^h (for the marked point process of intensity λ defining the segments) leads to a length intensity which is the same as the unit intensity Poisson line process?

We denote by Ξ the segment process of fixed length h and intensity λ with rose of directions given by ρ (we measure the angles from the horizontal axis in the anti-clockwise direction). We seek an expression for its *length intensity*, given by

$$\nu_\lambda^h \quad = \quad \frac{\mathbb{E}\big[L\big(\Xi \cap K\big)\big]}{A(K)} \quad = \quad \frac{\mathbb{E}[\mathrm{Leb}_1([K])]}{\mathrm{Leb}_2(K)} \tag{6}$$

for $K \subseteq \mathbb{R}^2$, any compact convex set on $\mathbb{R}^2$. Here we denote by $\xi \Uparrow K$ the event "ξ hits K", meaning $\xi \cap K \neq \emptyset$, then $[K]$ stands for the hitting set, which is defined as the set of segments of Ξ that hit K, i.e. $[K] = \{\xi \in \Xi : \xi \Uparrow K\}$.
We can identify the segment process Ξ with a marked point process Φ^* in $\mathbb{R}^2 \times (0, \pi]$. Here each point $\boldsymbol{x}$ of the process represents the lower-end point of the segment ξ (in case of horizontal segments we chose the left-end point). Thus $\boldsymbol{x} = (x^*, y^*)$ where:

$$\begin{aligned} y^* \quad &:= \quad \inf\big\{y : (x, y) \in \xi \text{ for some } x\big\} \\ x^* \quad &:= \quad \inf\big\{x : (x, y^*) \in \xi\big\} \end{aligned}$$

We can construct the Poisson segment process by requiring that the points (x^*, y^*) form a Poisson point process of intensity λ and the angular marks θ are independent of each other and of the positions $\boldsymbol{x}$ and identically distributed with distribution given by $\rho(\,\mathrm{d}\theta)$. Viewed as a point process in (x^*, y^*, θ) space, the intensity mean of this point process is given by $\lambda(\mathrm{Leb}_2 \otimes \rho)$.

The first step in addressing this question is to reduce to the case where the mark distribution is nonrandom. To achieve this consider the following rotation

$$R(\xi) = R_\theta(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, 0) =: (\tilde{x}, \tilde{y}, 0)$$

for each segment $\xi = (x, y, \theta)$. The rotated segments will all be horizontal. The rotation $R_\theta(x, y)$ moves the segment ξ in such a way that its reference point, $\boldsymbol{x}$, is still at the same distance, $\|\boldsymbol{x}\|$, from the origin but it has been rotated θ radians in clockwise direction so that the segment is now horizontal, see figure 6. This converts Ξ into a new segment process $R\Xi$ for which all segments are horizontal, but such that the length of intersection with any disk centered at the origin is unchanged.

In particular the new point process is still Poisson.

Theorem 5. *If Φ^* is a marked Poisson point process in $\mathbb{R}^2 \times (0, \pi]$, with intensity λ and rose*

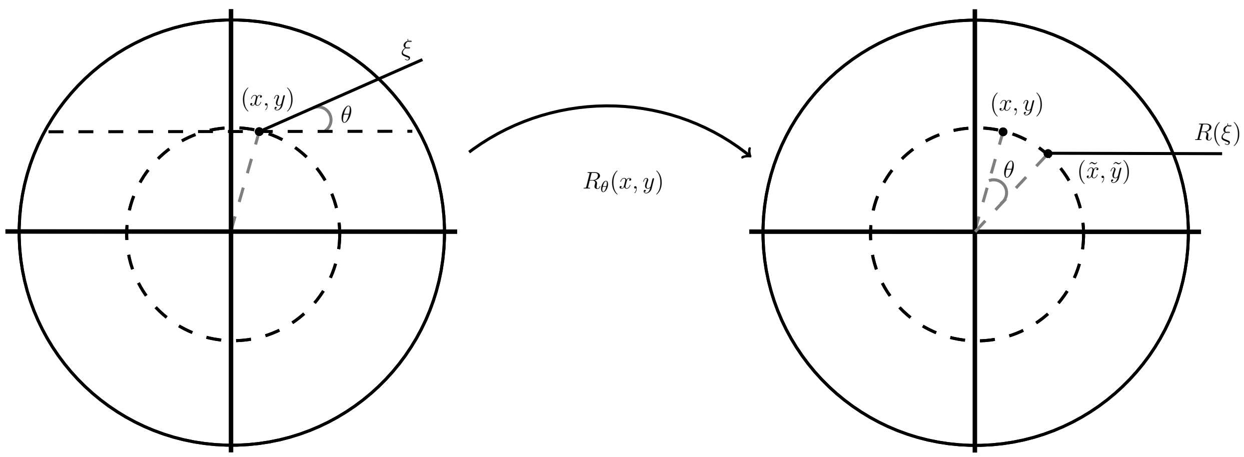

Figure 6: Illustration of the rotation $R_\theta(x, y)$.

of directions ρ then $\tilde{\Phi} = \{(\tilde{x}, \tilde{y}) : (\tilde{x}, \tilde{y}, 0) = R_\theta(x, y)$ with $(x, y, \theta) \in \Phi^*\}$ is a Poisson point process of intensity λ.

Proof. The family of avoidance probabilities determines the distribution of a point process [Kin92], so it is enough to show that:

$$\mathbb{P}[\tilde{\Phi} \cap \tilde{E} = 0] \quad = \quad \exp\left(-\lambda \operatorname{Leb}_2(\tilde{E})\right)$$

for all measurable $\tilde{E} \subseteq \mathbb{R}^2$.
For $\tilde{E}$ any measurable subset in $\mathbb{R}^2$, we define $E \subseteq \mathbb{R}^2 \times (0, \pi]$ as follows:

$$E \quad = \quad \Big\{(x, y, \theta) : (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta) \in \tilde{E}\Big\}$$

Therefore $\{\tilde{\Phi} \cap \tilde{E} = 0\} = \{\Phi^* \cap E = 0\}$. By the rotation invariance of the Leb_2 measure and the fact that ρ is a probability measure we can compute the intensity measure for $\tilde{\Phi}$:

$$\begin{aligned}
\lambda(\operatorname{Leb}_2 \otimes \rho)(E) \quad &= \quad \lambda \int_0^\pi \left(\iint_{\mathbb{R}^2} \mathbb{1}_E(x, y, \theta) \operatorname{Leb}_2(\,\mathrm{d}x\,\mathrm{d}y) \right) \rho(\,\mathrm{d}\theta) \\
= \lambda \int_0^\pi \left(\iint_{\mathbb{R}^2} \mathbb{1}_{\tilde{E}}(R_\theta(x, y)) \operatorname{Leb}_2(\,\mathrm{d}x\,\mathrm{d}y) \right) \rho(\,\mathrm{d}\theta) &= \lambda \int_0^\pi \left(\iint_{\mathbb{R}^2} \mathbb{1}_{\tilde{E}}(\tilde{x}, \tilde{y}) \operatorname{Leb}_2(\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}) \right) \rho(\,\mathrm{d}\theta) \\
&= \lambda \iint_{\mathbb{R}^2} \mathbb{1}_{\tilde{E}}(\tilde{x}, \tilde{y}) \operatorname{Leb}_2(\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}) = \lambda \operatorname{Leb}_2(\tilde{E})
\end{aligned}$$

Hence,

$$\mathbb{P}[\tilde{\Phi} \cap \tilde{E} = 0] = \mathbb{P}[\Phi^* \cap E = 0] = \exp\left(-\lambda(\operatorname{Leb}_2 \otimes \rho)(E)\right) = \exp\left(-\lambda \operatorname{Leb}_2(\tilde{E})\right)$$

as required. □

Now we relate the length intensity at (6) to the lengths of the segments h. From the above construction we only need to consider the case where all segments are horizontal, because if K

is a disk centered at the origin we know that $\mathbb{E}[L(\Xi \cap K)] = \mathbb{E}[L(R\Xi \cap K)]$. Hence, as $R\Xi$ can be represented by $\Phi \oplus [0, h]$, where Φ is a planar Poisson point process of intensity λ. Therefore in (6) we get

$$\nu_\lambda^h \quad = \quad \frac{\mathbb{E}[\mathrm{Leb}_1([K])]}{\mathrm{Leb}_2(K)} \quad = \quad \frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\Phi \oplus [0, h]) \cap K\right)\big]}{\mathrm{Leb}_2(K)}$$

However if $h_1 \in (0, h)$, it is clear that we can express $\Phi \oplus [0, h]$ as the following disjoint union $(\Phi \oplus [0, h_1]) \cup (\Phi \oplus [h_1, h])$. So ν_λ^h is linear in h

$$\begin{aligned}\nu_\lambda^h = \frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\Phi \oplus [0, h]) \cap K\right)\big]}{\mathrm{Leb}_2(K)} &= \frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\Phi \oplus [0, h_1]) \cap K\right)\big]}{\mathrm{Leb}_2(K)} + \frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\Phi \oplus [h_1, h]) \cap K\right)\big]}{\mathrm{Leb}_2(K)} \\ &= \nu_\lambda^{h_1}(K) + \frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\{\Phi + h_1\} \oplus [0, h - h_1]) \cap K\right)\big]}{\mathrm{Leb}_2(K)} = \nu_\lambda^{h_1} + \nu_\lambda^{h-h_1}\end{aligned}$$

Here $\Phi + h_1$ has the same distribution as Φ, since Φ is a stationary Poisson point process. As a consequence we may deduce $\nu_\lambda^h = h\nu_\lambda^1$.

But ν_λ^h is also linear in λ: apply the superposition theorem to decompose the original Poisson point process Φ of intensity λ into two independent Poisson point process Φ_1 and Φ_2 with respective intensities λ_1 and λ_2, such that $\lambda = \lambda_1 + \lambda_2$. Therefore

$$\nu_\lambda^h \quad = \quad \nu_{\lambda_1}^h + \nu_{\lambda_2}^h$$

In sum, we conclude that ν_λ^h should be of the form $c\lambda h$. We determine the value of $c = \nu_1^1$ by considering the case when K is the unit square $[0, 1]^2$. If we denote by $N(K)$ the amount of points from Φ (a Poisson point process with intensity λ) that falls in K we get (recall $N(K) \sim Po(\lambda\,\mathrm{Leb}_2(K))$)

$$\begin{aligned}\nu_\lambda^h = h\nu_\lambda^1 = h\frac{\mathbb{E}\big[\mathrm{Leb}_1\left((\Phi \oplus [0, 1]) \cap [0, 1]^2\right)\big]}{\mathrm{Leb}_2([0, 1]^2)} &= h\mathbb{E}\left[\sum_{\xi \Uparrow [0,1]^2} \mathrm{Leb}_1(\xi \cap [0, 1]^2)\right] \\ = h\mathbb{E}\left[\sum_{k=1}^{N([0,1]^2)} U(0, 1) + \sum_{k=1}^{N([-1,0]\times[0,1])} U(0, 1)\right] &= 2h\mathbb{E}[N([0, 1]^2)] \times \mathbb{E}[U(0, 1)] = h\lambda\end{aligned}$$

Here we use the fact that if $\xi \Uparrow [0, 1]^2$ then its marker point $\boldsymbol{x}$ falls either in $[0, 1]^2$ or in $[-1, 0] \times [0, 1]$. Either way, $\mathrm{Leb}_1(\xi \cap [0, 1]^2)$ is distributed according to a $U[0, 1]$ random variable (recall the construction of the Poisson point process where we draw $N([0, 1]^2)$ points scattered as independent and identically distributed uniformly on $[0, 1]^2$).
Now, recalling that the unit Poisson line process Π (where $\lambda = 1/2$, since we are considering undirected lines) has length intensity equal to $\pi/2$. To obtain this result one need to notice that the length of a chord in $B_1(\boldsymbol{o})$ of a line at a distance r from the origin $\boldsymbol{o}$ is given by $2\sqrt{1 - r^2}$,

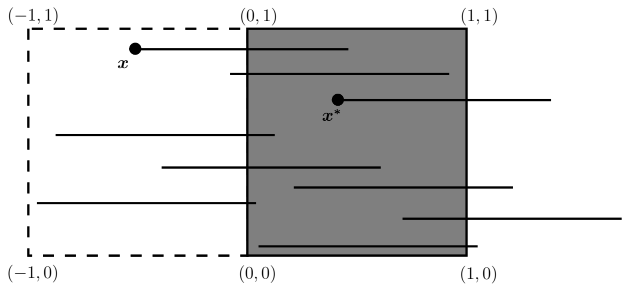

Figure 7: Explanation for the fact that $\text{Leb}_1(\xi \cap [0,1]^2)$ has distribution $2U(0,1)$. If there is a left-end point $\boldsymbol{x} = (x_1, y_1)$ in the square $[-1,0] \times [0,1]$ then the length of the intersection of that segment with the square $[0,1]^2$ is given by $1 + x_1$ which is Uniform on $(0,1)$. Similarly, if there is a left-end point $\boldsymbol{x^*} = (x_2, y_2)$ in the square $[0,1]^2$ then the length of that segment with the square $[0,1]^2$ is given by $1 - x_2$ which is again Uniform $(0,1)$.

see figure 8. Therefore the calculations for the length intensity leads to:

$$\nu_\lambda \quad = \quad \frac{\mathbb{E}\big[L\big(\Pi \cap B_1(\boldsymbol{o})\big)\big]}{\text{Leb}_2(B_1(\boldsymbol{o}))} = \frac{\lambda}{\pi} \int_0^\pi \int_{-1}^1 2\sqrt{1-r^2}\, \mathrm{d}r\, \mathrm{d}\theta = \frac{2\lambda\pi}{\pi} \int_{-1}^1 \sqrt{1-r^2}\, \mathrm{d}r = \lambda\pi = \frac{\pi}{2}$$

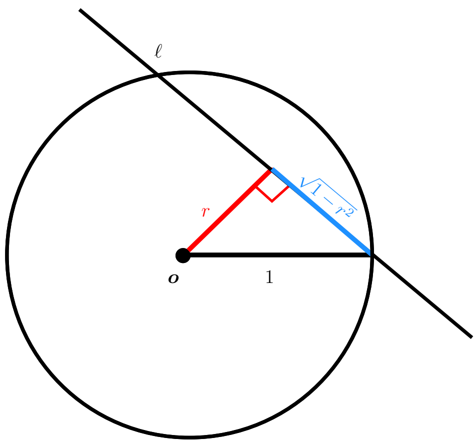

Figure 8: Illustration of the length of the chord in $B(\boldsymbol{o}, 1)$.

Therefore $c = \pi/2$ gives the similar desired result for segment processes. Our particular interest is with the case where ρ is a uniform distribution. Notice that a Poisson segment process of length h with intensity $\lambda = 1/h$ has the same length intensity as an unitary Poisson line process.

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