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Besov Spaces and Parabolic Systems

by

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Thesis

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Contents

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Declarations

I declare that, to the best of my knowledge, the material contained in this dissertation is original and my own work except where otherwise indicated, cited, or commonly known.

The material in this dissertation is submitted to the University of Warwick for the degree of Master of Science, and has not been submitted to any other university or for any other degree.

Abstract

In this dissertation, we study equivalent definitions of Besov spaces, a class of function space which generalise many standard spaces used to study PDEs. The most important definitions involve the Littlewood-Paley decomposition, the heat kernel and real interpolation spaces.

For each definition we study both the non-homogeneous and homogeneous type, the related properties, equivalences between the definitions and examples of when these definitions are used to prove interesting results.

Of further interest, where we can, we define and study these spaces on bounded domains as there are interesting problems solved on the whole space but not on bounded domains.

Chapter 1

Introduction

1.1 Motivation

In the study of existence, uniqueness and regularity of solutions to PDEs, functions spaces are of key importance. Sobolev spaces are well known for use in PDEs as they provide the necessary framework to define weak solutions properly. One may wish to attain higher regularity and there are other important function spaces to look for solutions, we consider for instance, Besov spaces. An example is given in Chapter 4.

The study of Besov spaces is an active and growing area of research. Besov space denoted $B^s_{p,q}$ and Triebel-Lizorkin space denoted $F^s_{p,q}$ for $s \in \mathbb{R}$ and $1 \leq p, q, \leq \infty$ are useful as they are a generalisation of many standard spaces used to study PDEs. For instance in Cannone et al. [2004] $L_q = F^0_{2,q}$, $W^s_q = F^s_{2,q}$ and $C^s = B^s_{\infty,\infty}$ where W^s_q is the Sobolev-s space in L_q and C^s is the Hölder-s space. With these generalisations we can further prove properties like embedding theorems in higher generality than before. These embedding theorems can be applied to more problems.

Of further interest is the difficulties of generalising the definition of Besov spaces to sub domains. This is due to the main definition of the space that is widely used, the Littlewood-Paley definition. This is one of the most explicit definitions and due to the decomposition of the function, we have powerful tools that can be used with this definition for solving calculations. However there is no easy generalisation of this definition to sub domains and thus further study into other definitions that can be generalised is vital for the study of problems involving sub domains.

1.2 Outline

We introduce first the classical definition of a Sobolev norm on a domain Ω and observe the different spaces that can be generated from this norm. Further we study generalisations to Sobolev spaces with differentiability in $\mathbb{R}$. Finally we introduce homogeneous Sobolev norms and the intrinsics of modifying the space so that we obtain a norm. This chapter serves as a foundation so that future work in Besov spaces is understandable.

In Chapter 3 we introduce the Littlewood-Paley decomposition of a function and use this to define a Besov space on the whole space. We then study lemmas and properties related to the Littlewood-Paley decomposition to see how useful this definition is for calculations. Finally we apply all this to the Onsager conjecture, a conjecture related to the incompressible Euler equations. It details the relation between the space of initial conditions and the conservation of energy from the Euler equations. We note that there is no easy way to generalise this definition to sub domains.

In Chapter 4 we study the definition of a Besov space using the heat kernel. Here we go over special bounds the heat kernel provides and prove the equivalence of the heat kernel definition of a Besov space to the Littlewood-Paley definition. Then we look at an example of applying our new theory to an example involving the heat equation, and compare our bounds to those generated by conventional Sobolev methods. Finally we generalise this definition to sub domains so in the future we can study properties and problems on these sub domains.

In Chapter 5 we introduce interpolation theory and use this general theory to present another definition of a Besov space that is easily generalisable to the sub domains. From this definition we use the interpolation theory to develop properties of Besov spaces.

In chapter 6 we introduce two more definitions of Besov spaces and see another generalisation to sub domains. Here we investigate the links between these definitions, the Littlewood-Paley definition and the interpolation definitions and use this to give better understanding of how the different ways of defining a Besov space interact.

Chapter 2

Sobolev Spaces

We start with an introduction of the classical Sobolev spaces and their properties. Looking at these simpler, well known spaces, we shall introduce important properties and concepts of Sobolev spaces. This will then give us inspiration about which properties would be useful to study for the more general Besov spaces.

2.1 Sobolev Spaces with Integer Derivatives

The Sobolev norm, on a domain $\Omega \subseteq \mathbb{R}^d$, is defined as follows and is given in Chapter 7 of Adams and Fournier [2003].

For m positive integer and $1 \leq p \leq \infty$,

$$\|u\|_{X_p^m} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{\frac{1}{p}}, \tag{2.1.1}$$

and for $p = \infty$

$$\|u\|_{X_\infty^m} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L_\infty(\Omega)}.$$

This norm considers positive integer derivatives only.

From this norm we can state three natural definitions of a Sobolev space, which are:

1. $H_p^m(\Omega) :=$ the completion of $\{u \in C^m(\Omega) : \|u\|_{X_p^m(\Omega)} < \infty\}$ with respect to the norm $\|\cdot\|_{X_p^m(\Omega)}$
2. $W_p^m(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L_p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}$ D^α weak derivative.
3. $W_{p,0}^m(\Omega) :=$ the closure of $C_0^\infty(\Omega)$ in the space $W_p^m(\Omega)$

When considering the entire space $\Omega = \mathbb{R}^d$ we find that these definitions are all equivalent. However, interestingly, this does not hold when Ω is a strict subset of the entire space. Therefore when we consider Besov spaces on strict subsets of $\mathbb{R}^d$, this complication will have to be considered.

In fact, as explained in full detail in Chapter 3 of Adams and Fournier [2003], we discover that $H_p^m(\Omega) \subseteq W_p^m(\Omega)$ and that $H_p^m(\Omega)$ is equivalent to $W_p^m(\Omega)$ for any domain as long as $p \neq \infty$. Further we discover for $p \neq 1, \infty$ for $m \geq 1$ and $p \geq 2$ then $W_p^m(\Omega)$ is equivalent to $W_{p,0}^m(\Omega)$ if and only if Ω^c is (m, p')-polar (a condition described in Chapter 3 of Adams and Fournier [2003]). To help understand this condition, if Ω^c has positive measure it cannot be (m, p')-polar. This is a strong condition on Ω and in most cases will not hold.

From this example we see that we must be careful when defining function spaces on bounded domains, since these extra complications can occur.

2.2 Sobolev Spaces with Real Derivatives

In the previous section we discussed spaces with only integer differentiability. For further generalisation of these spaces we would like to define Sobolev spaces for any real valued differentiability.

There are two methods for this generalisation. The first, discussed in Peetre [1976], involves using Fourier analysis to define the Sobolev space and so we can only take $\Omega = \mathbb{R}^d$.

This generalisation involves defining the operator $J = (1-\Delta)^{\frac{1}{2}}$. This can be defined with use of the Fourier transform as $Jf(\xi) = (1+|\xi|^2)^{\frac{1}{2}}\hat{f}(\xi)$. Then fractional powers can be easily defined by $J^s f(\xi) = (1 + |\xi|^2)^{\frac{s}{2}}\hat{f}(\xi)$ for $f \in \mathcal{S}'$, the space of tempered distributions. Once this is done we define the Sobolev space for $s \in \mathbb{R}$ and $p \in [1, \infty]$ by

$$W_p^s(\mathbb{R}^d) = \{f \in \mathcal{S}' : \|J^s f\|_{L_p} < \infty\}, \tag{2.2.1}$$

with the norm $\|f\|_{W_p^s} = \|J^s f\|_{L^p}$.

The second generalisation involves interpolation spaces which will be defined later and is explained in Peetre [1976] and Chapter 7 of Adams and Fournier [2003]. The fractional $W_p^s(\Omega)$, for any $\Omega \subseteq \mathbb{R}^d$, is defined by interpolating between an L_p space and a W_p^a space for a an integer larger than s. This method of interpolation can be used to define a Besov space for any $\Omega \subseteq \mathbb{R}^d$ and is a useful definition for looking at properties of these spaces.

2.3 Homogeneous Sobolev Spaces

A further variant of Sobolev spaces, and later Besov spaces, comes from defining these spaces so that the norm is homogeneous. An example of a homogeneous space is the $L_p(\Omega)$ spaces since $\|f(\lambda\cdot)\|_{L_p} = \lambda^{-\frac{n}{p}}\|f\|_{L_p}$. In general, for a homogeneous space, if one scales the variable then the norm scales as well.

For scaling to occur for the integer derivative Sobolev space norm, as defined in (2.1.1), we see that each different level of differentiation will give a different scaling order. This is due to the chain rule as each differential gives an extra order of the constant, $D(f(\lambda x)) = \lambda D(f(x))$. Thus a norm with a sum of mixed orders of differentials will have a mixed order of scaling for each term and no overall scaling for the sum. To correct this we shall remove the lower order differentials and just leave the highest differentials remaining. Thus we modify the norm into,

$$\|u\|_{\dot{X}_p^m} = \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty, \tag{2.3.1}$$

and

$$\|u\|_{\dot{X}_\infty^m} = \max_{|\alpha|=m} \|D^\alpha u\|_{L_\infty(\Omega)}.$$

We notice that this now becomes a semi-norm. This is because any polynomial of degree less than m has norm zero and in order to create a normed space we must quotient out by the space of polynomial of degree $m-1$ and less. This is a general problem which we will correct for the Besov cases later with a similar method.

Considering again the generalised Sobolev norm defined earlier by (2.2.1), we can define a homogeneous version of the norm by replacing the operator J by I, where $I^s = (-\Delta)^{\frac{s}{2}}$. Again to define the homogeneous Sobolev space $\dot{W}_p^s$ we must quotient out by polynomials to define a norm. Thus we define for $N > s - \frac{n}{p}$ the space as,

$$\dot{W}_p^s = \dot{W}_p^s(\mathbb{R}^d) = \{f : f \in \mathcal{S}'/\mathcal{P}_N \text{ and } \|I^s f\|_{L_p} < \infty\}$$

with semi-norm

$$\|f\|_{\dot{P}_p^s} = \|I^s f\|_{L_p}$$

To show that this is the homogeneous norm we see that in the non-homogeneous case we can expand the operator $J^s = (1-\Delta)^{\frac{s}{2}}$ in a series expansion of Δ and obtain terms involving derivatives of all orders. Thus removing these terms to leave the highest derivative gives us the operator I.

Chapter 3

Littlewood-Paley Theory

The relation between Besov spaces and Sobolev spaces is an equivalence as follows: $B^s_{2,2} = W^s_2$. In general, Besov spaces are a generalisation of many spaces including Sobolev spaces with the embedding

$$B^s_{p,1} \hookrightarrow W^s_p \hookrightarrow B^s_{p,\infty} \tag{3.0.1}$$

shown by the comparison theorem for interpolation spaces in the book Peetre [1976]. The equivalences between Besov spaces and other function spaces are in general non-trivial to prove.

Besov spaces have many separate equivalent definitions on $\mathbb{R}^d$. These have varying forms and different definitions can be more useful in certain circumstances than others. In this section we will look at the Littlewood-Paley definition and its properties. We will investigate how useful this definition is, in particular for ease of calculations.

3.1 Littlewood-Paley Definition

The first definition of a Besov space is the Littlewood-Paley definition as described in Cannone et al. [2004], Cannone [1995],Peetre [1976] and Bahouri et al. [2011]. This is the most useful definition for calculations as it forms a systematic breakdown of the space into dyadic blocks of the Fourier modes of the function. Once split into these dyadic blocks, the decomposed function now satisfies nicer properties and operators acting on the function are easier to bound. We then just deal with a countable sum of these modes rather than dealing with one complicated norm. This being said, the definition heavily relies on Fourier analysis tools and thus it is not applicable if the domain is not $\mathbb{R}^d$ or if the domain is not periodic. Thus, we will need to use different definitions to look at problems in bounded domains.

To understand how this decomposition of the function is achieved let us look at a specific example in $\mathbb{R}^3$.

Firstly we take an arbitrary function $\Phi \in \mathcal{S}$ such that;

$$0 \le \hat{\Phi}(z) \le 1, \quad \hat{\Phi}(z) = 1 \quad \text{if } |z| \le \frac{3}{4}, \quad \hat{\Phi}(z) = 0 \quad \text{if } |z| \ge \frac{3}{2},$$

and then for $j \in \mathbb{Z}$ let

$$\phi(x) = 8\Phi(2x) - \Phi(x), \quad \Phi_j(x) = 2^{3j}\Phi(2^j x), \quad \phi_j(x) = 2^{3j}\phi(2^j x).$$

Then we further define $*$ to be the convolution operator and then define $S_j = \Phi_j *$ and $\Delta_j = \phi_j *$. Since the Fourier transform of a convolution of functions becomes the multiplication of the Fourier transforms. Thus by performing a convolution here we perform multiplication in Fourier space. Because of the properties of $\{\phi_j\}$ and $\{\Phi_j\}$, we perform a nice cut off in Fourier modes that we wanted for our decomposition. Finally the set $\{S_j, \Delta_j\}_{j\in\mathbb{Z}}$ as the Littlewood-Paley decomposition, so that

$$\mathrm{Id} = S_0 + \sum_{j\ge 0} \Delta_j.$$

The physical interpretation of supp$\hat{f}$ consists of those frequencies which build up f from linear combinations of $e^{ix\xi}$. Thus with a nice support and the smooth cut off that the decomposition gives us, we have a good control of the function frequencies when trying to bound the function. With only certain wave frequencies to worry about, the calculations become simplified and we have more tools at our disposal.

We now define the general properties that are necessary to produce the desired Littlewood-Paley decomposition of our function and throughout the rest of this report we will hopefully see why these properties are chosen.

Let $\{\phi_j\}_{j\in\mathbb{Z}}$ sequence of test functions have the following properties as described in Peetre [1976];

1 $\phi_j \in \mathcal{S}$

2 $\hat{\phi}(\xi) \ne 0$ iff $\xi \in R_j^\circ$ where $R_j := \{2^{j-1} \le |\xi| \le 2^{j+1}\}$

3 $|\hat{\phi}_j(\xi)| \ge C_\varepsilon > 0$ if $\xi \in R_{j,\varepsilon}$ where $R_{j,\varepsilon} := \{(2-\varepsilon)^{-1}2^j \le |\xi| \le (2-\varepsilon)2^j\}$

4 $|D^\beta \hat{\phi}_j(\xi)| \le C_\beta 2^{-j|\beta|}$ for every β

5 (Sometimes require) $\sum_{j=-\infty}^{\infty} \hat{\phi}_j(\xi) = 1$ or $\sum_{j=-\infty}^{\infty} \phi_j(\xi) = \delta(x)$.

Let Φ have the properties;

1. $\Phi \in \mathcal{S}$
2. $\hat{\Phi}(\xi) \neq 0$ iff $\xi \in K^\circ$ where $K := \{|\xi| \leq 1\}$
3. $|\hat{\Phi}(\xi)| \geq C_\varepsilon \neq 0$ if $\xi \in K_\varepsilon^\circ$ where $K_\varepsilon = \{|\xi| \leq 1 - \varepsilon\}$.

Then with any such set of functions $\{\{\phi_j\}_{j\in\mathbb{Z}}, \Phi\}$ we can define the Littlewood-Paley decomposition $\{S_j, \Delta_j\}_{j\in\mathbb{Z}}$ analogously to before.

There are a few simplifications that can be made on this set of eight properties. Firstly, we see that if we have ϕ_0 with the properties 1 to 3 then we can define ϕ_j by $\hat{\phi}_j(\xi) = \hat{\phi}(\xi/2^j)$. Further assuming that $\hat{\phi}(\xi) \geq 0$ then we get 5 by replacing $\hat{\phi}_j(\xi)$ by $\hat{\phi}_j(\xi)/\sum_{j=-\infty}^{\infty} \hat{\phi}_j(\xi)$. Then in this case we can set $\hat{\Phi}(\xi) = \sum_{j=-\infty}^{-1} \hat{\phi}_j(\xi)$. With this cut off using elements of $\mathcal{S}$, we can define the action on any element of $\mathcal{S}'$ and look at L_p norms of these decomposed functions. We can now define a Besov space.

Definition 3.1.1 (Littlewood-Paley Besov space) *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$, $0 < q \leq \infty$. Then we set*

$$B_{p,q}^s = B_{p,q}^s(\mathbb{R}^d) = \left\{ f : f \in \mathcal{S}' \text{ and } \|S_0 f\|_{L_p} + \left(\sum_{j=0}^{\infty} \left(2^{js}\|\Delta_j f\|_{L_p}\right)^q \right)^{\frac{1}{q}} < \infty \right\}. \tag{3.1.1}$$

With the norm of,

$$\|f\|_{B_{p,q}^s} = \|S_0 f\|_{L_p} + \left(\sum_{j=0}^{\infty} \left(2^{js}\|\Delta_j f\|_{L_p}\right)^q \right)^{\frac{1}{q}}.$$

The definition can be shown to be independent of the choice $\{\phi_j\}_{-\infty}^{\infty}$ by showing that this definition is equivalent to the interpolation space definition (5.1.3) given later.

We now wish to define a homogeneous definition of a Besov space. To achieve a homogeneous norm we must only care about the highest derivative and not any fractions beneath as these will stop the scaling from working. Thus in the definition, to keep it a normed space and not a semi-normed space, we remove lower degree polynomials that only give an overall function shape structure and not the local structure.

Definition 3.1.2 (Littlewood-Paley Homogeneous Besov space) *Let $s \in \mathbb{R}$, $1 \le p \le \infty$, $0 < q \le \infty$, $N(s,d,p) > s - \frac{d}{p}$ or with $q \le 1$ $N(s,d,p) \ge s - \frac{d}{p}$, $\mathcal{P}_N$ polynomials of degree N. Then we set*

$$\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^d) = \left\{ f : f \in \mathcal{S}/\mathcal{P}'_N \text{ and } \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \|\Delta_j f\|_{L_p}\right)^q \right)^{\frac{1}{q}} < \infty \right\}. \quad (3.1.2)$$

With the semi-norm of,

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \|\Delta_j f\|_{L_p}\right)^q \right)^{\frac{1}{q}}$$

modulo polynomials of degree $\mathcal{P}_N$.

For the Homogeneous case, instead of taking $f \in \mathcal{S}/\mathcal{P}'_N$ we can define distributions vanishing at infinity. The space $\mathcal{S}'_0(\mathbb{R}^n)$ is the space of distributions such that $\lim_{m \to -\infty} S_m f = 0$ in $\mathcal{S}'$. This definition removes the polynomials as supp $\hat{p} = \{0\}$ for p a polynomial, and also because $\hat{p}$ is a constant of the delta distribution.

3.2 Littlewood-Paley Properties

The first and possibly the most useful lemmas are the Bernstein Lemmas.

Lemma 3.2.1 (Bernstein) *For A an annulus and B a ball there exists a constant C such that for $k \in \mathbb{Z}$ and $1 \le p, q \le \infty$ with $p \le q$ and for any function $u \in L_p$, we have*

$$\text{Supp } \hat{u} \subset \lambda B \implies \|D^k u\|_{L_q} \le C^{k+1} \lambda^{k + d(\frac{1}{p} - \frac{1}{q})} \|u\|_{L_p}, \quad (3.2.1)$$

$$\text{Supp } \hat{u} \subset \lambda A \implies C^{-k-1} \lambda^k \|u\|_{L_p} \le \|D^k u\|_{L_p} \le C^{k+1} \lambda^k \|u\|_{L_p}. \quad (3.2.2)$$

As we see these give bounds, dependent on the support in Fourier space, for the derivatives of a function. Therefore we see that the Littlewood-Paley decomposition creates these nicely supported functions where we can apply the Bernstein lemmas. These are useful to understand further properties and also used in showing the equivalences of different definitions of Besov spaces.

Proof λ is just dilation so can assume $\lambda = 1$. For the first implication, let $\phi \in \mathcal{D}$ with value 1 over B. As we can write $\hat{u}(\xi) = \phi(\xi)\hat{u}(\xi)$ so we have $u(x) = \check{\phi} \star u(x)$ and thus $\partial^\alpha u(x) = \partial^\alpha \check{\phi} \star u(x)$. We can then apply Young's inequality to obtain

$$\|\partial^\alpha \check{\phi} \star u\|_{L_q} \le \|\partial^\alpha \check{\phi}\|_{L_r} \|u\|_{L_p},$$

where $\frac{1}{r} = -\frac{1}{p} + \frac{1}{q} + 1$. To continue we must find a bound on the $\|\partial^\alpha \check{\phi}\|_{L_r}$ term and then we have the inequality we want. Notice that the use of Young's inequality is valid since we have $p \leq q$. First we can state that

$$\|\partial^\alpha \check{\phi}\|_{L_r} \leq \|\partial^\alpha \check{\phi}\|_{L_\infty} + \|\partial^\alpha \check{\phi}\|_{L_1}.$$

The sup-norm bounds the function over any compact set. The L_1 bound is a tail bound as functions with tails in L_1 must decay faster than in any other L_p space. We then want to collect the terms under one L_∞ bound. This is done by multiplying the function under the L_1 bound by $\frac{|x|^{2d}}{|x|^{2d}}$. Then we take the sup of $\partial^\alpha |x|^{2d}$ out for a bound. This gives

$$\leq C\| \left(1 + |\cdot|^2\right)^d \partial^\alpha \check{\phi}\|_{L_\infty}.$$

We then use that the Fourier transform maps L_∞ to L_1 and thus,

$$\leq C\| \left(Id - \Delta\right)^d \left((\cdot)^\alpha \phi\right) \|_{L_1} \leq C^{k+1}.$$

We then use the condition that $\phi \in \mathcal{D}$ to get the bound we want.

Again we need to take a $\psi \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$ with value 1 in neighborhood of $\mathcal{A}$. This is similar to before but now as the point $\{0\}$ is removed this works for the annulus case and we get the upper bound in a similar argument to above.

For the lower bound we again write $\hat{u}(\xi) = \psi(\xi)\hat{u}(\xi)$. We can then, for some constants C_α, write this term as.

$$\hat{u}(\xi) = \psi \sum_{|\alpha|=k} (i\xi)^\alpha C_\alpha \frac{(-i\xi)^\alpha}{|\xi|^{2d}} \hat{u}(\xi) = \sum_{|\alpha|=k} (i\xi)^\alpha \hat{u}(\xi) C_\alpha \frac{(-i\xi)^\alpha}{|\xi|^{2d}} \psi$$

Then we can apply the inverse Fourier transform to both sides and we obtain

$$u(x) = \sum_{|\alpha|=k} (\partial)^\alpha u(\xi) \star \check{\psi}_\alpha \text{ where } \check{\psi}_\alpha = C_\alpha \mathcal{F}^{-1}\left(\frac{(-i\xi)^\alpha}{|\xi|^{2d}} \psi\right).$$

We can then apply Young's inequality to both sides. To complete the proof we need to bound $\|\check{\psi}_\alpha\|_{L_1}$. This is done similarly to before and so we get the lower bound we want. ☐

One problem that needs to be considered is what to do when dealing with a product of two functions when we are taking the Littlewood-Paley decomposition. The method to deal with this is discussed in Bae [2009] and Bahouri et al. [2011].

For this explanation we shall absorb the S_0 term into the Δ_0 term and thus can write the Littlewood-Paley decomposition of the function as $\sum_{j \geq 0} \Delta_j u$. For a

u and v in $\mathcal{S}'$ we then have that their product can be written as a Littlewood-Paley decomposition by

$$uv = \sum_{j,k} \Delta_j u \Delta_k v$$

where here $j, k \in \mathbb{N}$.

This formula is a double sum over the product of all the combinations of the decomposition. As this is complicated to deal with let us consider splitting the sum up into three parts considering an arbitrary k and N_0 dependent on the functions used for the decomposition,

$$uv = \sum_{j,k} \Delta_j u \Delta_k v = \sum_{k \leq j - N_0 - 1} \Delta_j u \Delta_k v + \sum_{k \geq j + N_0 + 1} \Delta_j u \Delta_k v + \sum_{|k-j| \leq N_0} \Delta_j u \Delta_k v.$$

As the support of the Fourier transform of each element is finite and the Littlewood-Paley decomposition was chosen so that each support only intersects with a finite number of the other terms in the decomposition, we can simplify these three separate terms.

Looking at the first two terms, N_0 is chosen so that the supports in Fourier space no longer intersect. To make life easy with the specific example chosen earlier we can choose N_0 to be one. We can now simplify the first term by noticing that as k is always less than j with independent support in Fourier space we can collect the sum of these smaller terms into $S_{j-N_0} v$. Similarly we can do this for the second term.

We are now ready to define the paraproduct of v and u with the remainder term as well as described in page nine of Danchin [2012].

Definition 3.2.2 (Paraproduct) *The paraproduct of v by u is defined as.*

$$T_u v := \sum_j S_{j-N_0} u \Delta_j v$$

The remainder of u and v is defined as.

$$R(u, v) := \sum_{|j-\nu| \leq N_0} \Delta_j u \Delta_\nu v$$

Thus with the paraproduct defined we can write the Bony decomposition of the product of two functions uv .

Definition 3.2.3 (Bony decomposition) *For the product of two functions uv*

with $u, v \in \mathcal{S}'$ we can write the Bony decomposition as.

$$uv := T_u v + T_v u + R(u, v)$$

This gives a way to decompose products of functions. All this above can be done for the homogeneous case as discussed in Bahouri et al. [2011] where the terms are all the same yet the homogeneous Littlewood-Paley decomposition is used instead.

This decomposition will be used later in the proof of Onsager's conjecture where we have to deal with a term of products of functions to find a bound. Further this is useful to look at embeddings of products of functions and there properties.

3.3 Littlewood-Paley Applications

Here we shall try to give some examples of applications where the Littlewood-Paley definiton of a Besov space can be used to prove results for Besov spaces that other spaces, such as L_p spaces or W_p^s spaces, do not necessarily satisfies.

3.4 Embeddings

One case of this is the embedding of $H^1(\mathbb{R}^2)$ into BMO but not into L_∞ which would be useful in many theorems. However what can be shown is the embedding of $B^1_{2,1}$ into L_∞ which works as $B^1_{2,1}$ is a slightly smaller space than H^1 "yet is very similar in size as $H^1 = B^1_{2,2}$".

Example 3.4.1 $B^1_{2,1}(\mathbb{R}^2)$ *embeds into* $L_\infty(\mathbb{R}^2)$

Proof We know that from equation (3.0.1) $B^0_{\infty,1}$ continuously embeds into L_∞ so want to show the embedding into $B^0_{\infty,1}$ then we are done. Thus for a $u \in B^0_{\infty,1}$ it has norm $\|S_0 u\|_{L_\infty} + \sum_{j\in\mathbb{N}}(\|\Delta_j u\|_{L_\infty})$. Then we can multiply by $\frac{2^j}{2^j}$ and this gives.

$$\|S_0 u\|_{L_\infty} + \sum_{j\in\mathbb{N}} \frac{2^j}{2^j}(\|\Delta_j u\|_{L_\infty}) = \|S_0 u\|_{L_\infty} + \sum_{j\in\mathbb{N}} 2^j(\frac{1}{2^j}\|\Delta_j u\|_{L_\infty})$$

Thus all that is left to prove is that is that $(\frac{1}{2^j}\|\Delta_j u\|_{L_\infty}) \leq \|\Delta_j u\|_{L_2})$. This is done by the scaling part of the Bernstein lemma (3.2.1) which gives $\|\Delta_j\|_{L_q} \leq \lambda^{d\left(\frac{1}{p}-\frac{1}{q}\right)}\|\Delta_j\|_{L_p}$. Thus with $\lambda = 2^j$, $d = 2$, $p = 2$ and $q = \infty$ we get the term 2^j out which cancels with the other 2^j to give $\sum_{j\in\mathbb{N}} 2^j\|\Delta_j u\|_{L_\infty}$ which is bounded by assumption.

To finish the proof we notice the same scaling applies to the term $\|S_0 u\|_{L_\infty}$ yet here $\lambda = 1$ and thus we have $\|S_0 u\|_{L_\infty} \leq \|S_0 u\|_{L_2}$ trivially. $\square$

This important example is a specific case of an embedding theorem, Proposition 2.20 in Bahouri et al. [2011], which states:

Proposition 3.4.1 *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then for $s \in \mathbb{R}$ $B^s_{p_1,r_1}$ is continuously embedded in $B^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_1,r_1}$.*

Proof This proof is similar to the proof above using $\|\Delta_j u\|_{L_q} \leq \lambda^{d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|\Delta_j\|_{L_p}$ with $\lambda = 2^j$. Then we just use the fact that l_{r_1} is continuously embedded in l_{r_2}, and we are done. $\square$

Further we find in Proposition 2.39 that in a homogeneous setting the previous embedding in fact embeds into the space C_0 (continuous functions that decay to 0 at infinity). We can show this by noting that $\mathcal{S}_0$ is dense in $\dot{B}^{\frac{d}{p}}_{p,1}$ where in our case is $\dot{B}^1_{2,1}$ as $d = p = 2$.

We see that this gives an interesting relation of the interplay between the regularity and the integrability of functions in Besov spaces with more generality than other embedding theorems.

3.5 Onsager's Conjecture

3.5.1 Original Idea

Onsager's conjecture considers the incompressible Euler equations given by the system described in Shvydkoy [2010]

$$\partial_t u = (\nabla u)u - \nabla p \qquad \text{Momentum balance} \tag{3.5.1}$$

$$\nabla \cdot u = 0 \qquad \text{Incompressibility condition.} \tag{3.5.2}$$

Here u is a divergence-free velocity field, p is the internal pressure and consider the entire space say $\mathbb{R}^2$ or $\mathbb{R}^3$ so we can use Fourier tools.

Here we care about the energy conservation of the system above. For an initial condition u_0 we obtain

$$\int_\Omega |u(t)|^2 dx = \int_\Omega |u_0|^2 dx \text{ for all } t \geq 0.$$

We notice above that this condition holds if, once multiplying by a test function u and integrating over time, the RHS of the equation (3.5.1) becomes

0. Now for smooth solutions with regular enough domains the RHS is zero and conservation of energy holds. Assuming that we can perform integration by parts with $u|_{\partial\Omega} = 0$ and use the incompressibility condition then this term becomes 0. The question is what is the minimal regularity needed for this conservation of energy to hold.

Consider for instance that u is a distribution where we take u weakly continuous in time and L^2 in space $u \in L_\infty(0, t, L^2)$ then this gives zero for the second term on the RHS of (3.5.1). Thus to answer the question of minimal regularity, our attention falls to the first term on the RHS of (3.5.1). Thus we care about the total energy flux Π,

$$\Pi := \int_\Omega (\nabla u)u \cdot u dx.$$

Then using the identity $\nabla \cdot (u \otimes u) = (\nabla u)u + (\nabla \cdot u)u$ from Gonzalez and Stuart [2008],

$$\Pi := \int_\Omega \nabla \cdot (u \otimes u) \cdot u - (\nabla \cdot u)u \cdot u dx.$$

The condition $u \in L_\infty(0, t, L^2)$ does not allow us to integrate by parts here and obtain that $\Pi = 0$. Here $u \otimes u$ is defined componentwise as the elements $u_i u_j$ for $i, j = 1, \ldots, d$. We have seen that after performing a Littlewood-Paley decomposition on a function, due to Bernstein lemmas, the derivative seems to act like multiplication by constants and thus we could deduce that

$$\Pi \cong \int_\Omega (|\nabla|^{\frac{1}{3}} u)^3 dx.$$

So naïvely if u has Hölder continuity $\frac{1}{3}$ then Π would make sense and any better regularity would be sufficient for integration by parts and give $\Pi = 0$. This is Onsager's conjecture that:

- Every weak solution to the incompressible Euler equations with smoothness $h > \frac{1}{3}$ conserves energy.

- Conversely there exists a weak solution to the incompressible Euler equations of smoothness exactly $\frac{1}{3}$ that does not conserve energy.

Onsager's original heuristic justification was based on laws of turbulence and not the Littlewood-Paley decomposition.

3.5.2 Critical Spaces and $B^{\frac{1}{3}}_{3,c_0}$

To explain what we mean by critical spaces let $h \in C_0$ be a scalar mollifier with dilation $h_\delta(x) = \frac{1}{\delta^n} h(\frac{x}{\delta})$. Then $u_\delta(t,x) = u(t,\cdot) \star h_\delta(x)$ and for $u \in L_\infty(0,t,L^2)$ this is well defined. Further, for $\delta > 0$ u_δ, is a smooth approximation of u and as $\delta \to 0$ we return u.

We can thus multiply (3.5.1) by $(u_\delta)_\delta$ and integrate over space and time to obtain

$$\int_0^t \int_{\mathbb{R}^d} \partial_t u \cdot (u_\delta)_\delta \; dx \; ds + \int_0^t \int_{\mathbb{R}^d} (\nabla u) u \cdot (u_\delta)_\delta \; dx \; ds = \int_0^t \int_{\mathbb{R}^d} -\nabla p \cdot (u_\delta)_\delta \; dx \; ds.$$

We can push the second mollification onto the other terms and with this smoothness integrate by parts so that using incompressibility the RHS term vanishes. For the far left term, we can use the identity $\partial u \cdot u = \frac{1}{2}\partial_t(u^2)$ and integrate over time. Finally for the remaining term we can use the identity $\nabla \cdot (u \otimes u) = (\nabla u)u + (\nabla \cdot u)u$ and integrate by parts to obtain

$$\frac{1}{2}(\|u_\delta(t)\|^2_{L_2} - \|u_\delta(0)\|^2_{L_2}) = \int_0^t \int_{\mathbb{R}^n} (u \otimes u)_\delta : \nabla u_\delta \; dx \; ds \tag{3.5.3}$$

where $A : B = \text{Trace}(AB)$. The RHS is the energy flux through different orders scales δ. We notice that we have three u terms on the RHS. Thus for the optimal function space we would want some bound in terms of $\|u\|^3_X$ in some function space X involving time and space. Now with scale analysis we have these three outcomes:

- We have a one dimensional integral over time (T), this will scale at rate T.
- We have three velocity terms, if we fix an average velocity (U) this will scale at rate U^3.
- We have one d-dimensional integral and one differential which together scale with length(L) at a rate L^{d-1}.

Overall, this gives the formula to define an Onsager critical space as having the scaling

$$(\text{dim}\|\cdot\|_X)^3 = TU^3L^{d-1}.$$

Some of these spaces in three dimensions are $L_3\left(0,t,L_{\frac{9}{2}}\right)$ and $L_3\left(0,t,H^{\frac{5}{6}}\right)$ when in two dimensions we have $L_3(0,t,L_6)$. Yet of particular interest are $L_3\left(0,t,B^{\frac{d(3-p)+p}{3p}}_{p,r}\right)$ and thus $L_3\left(0,t,B^{\frac{1}{3}}_{3,l}\right)$ for $l \in [1,\infty]$ are critical for any dimension. In fact, the proof will be done in the space $L_3\left(0, T, B^{\frac{1}{3}}_{3,c_0}\right)$ where this is defined as follows

$$B^s_{p,c_0} := \left\{ u \in B^s_{p,\infty} : \lim_{j\to\infty} 2^{js}\|\Delta_j u\|_{L_p} = 0 \right\}.$$

3.5.3 Proving Onsager's Conjecture in $B^{\frac{1}{3}}_{3,c_0}$

To prove this conjecture in the Besov space $B^{\frac{1}{3}}_{3,c_0}$ we will need the definition of the weak solution to the incompressible Euler equations. We will also need a lemma from Shvydkoy [2010] that relates this definition to Littlewood-Paley theory.

Definition 3.5.1 (Weak solution to Euler equations) *A weakly continuous vector field u from $[0,T]$ to L_2 $u \in C_w(0,T,L_2)$, is a weak solution to the Euler equations with initial data in $u_0 \in L_2$ if for every compactly supported test function $\phi \in C_0^\infty([0,T]\times\mathbb{R}^d)$ with $\nabla\cdot\phi = 0$ and for every $t \in [0,T]$ we have*

$$\int_{\mathbb{R}^d\times t} u\cdot\phi\ dx - \int_{\mathbb{R}^d\times 0} u\cdot\phi\ dx - \int_0^t\int_{\mathbb{R}^d} u\cdot\partial_s\phi\ dx\ ds = \int_0^t\int_{\mathbb{R}^d}(u\otimes u) : \nabla\phi\ dx\ ds \tag{3.5.4}$$

and $\nabla u(t) = 0$ in the sense of distributions.

With the next lemma one can pass from the weak formulation of the equation to the mollified equation or to the integral equations. Then with the mollified equation we can write this as a partial sum of the Littlewood-Paley decomposition of the function u and then check the sum converges to 0.

Lemma 3.5.2 *Let u be a weak solution to the incompressible Euler equations. Then for each fixed $\delta > 0$, $u_\delta\colon [0,T] \to W^s_q$ is absolutely continuous for all $s > 0$ and $q \geq 2$ and moreover*

$$\partial_t u_\delta = -\nabla\cdot(u\otimes u)_\delta - \nabla p_\delta. \tag{3.5.5}$$

Furthermore, (3.5.4) *is equivalent to the integral equation,*

$$u(t) = u_0 - \int_0^t \left[\nabla\cdot(u\otimes u) + \nabla p\right]\ ds$$

in the sense of distributions for all $t \in [0,T]$.

We will have to introduce some more notation. We know that for a Littlewood-Paley decomposition $\{S_j, \Delta_j\}_{j\in\mathbb{Z}}$ we get the identity operator by $\mathrm{Id} = S_0 + \sum_{j\geq 0}\Delta_j$. Let us instead introduce the partial sum and define $u_{\leq q} := (S_0 + \sum_{j\geq 0}^q \Delta_j)(u)$. So this is only a partial Littlewood-Paley decomposition missing out the parts of u with Fourier modes of the order 2^k for $k \in \mathbb{Z}, k > q$. We see that from the definition of a mollifier that the process of mollification smooths out the modes of order $\frac{1}{\delta}$ and greater. Thus as $\delta \to 0$ it smooths out only the higher modes until $\delta = 0$ and we get the identity. Thus we see the relation, for δ there is a q where the partial sum up to q is the same as the mollified u_δ. Further we can denote $\tilde{u}_q = (\Delta_{q-1} + \Delta_q + \Delta_{q+1})u$.

Finally we can define the Littlewood-Paley energy flux through wave number 2^q by

$$\Pi_{\leq q} = -\int_{\mathbb{R}^d} (u \otimes u)_{\leq q} : \nabla u_{\leq q} \; dx. \tag{3.5.6}$$

From the lemma above we know that if $u(t)$ is a weak solution to the incompressible Euler equations then it satisfies (3.5.5). Thus from the discussion above we can then write this mollifier equation in terms of partial Littlewood-Paley decompositions as follows

$$\partial_t u_{\leq q} = -\nabla \cdot (u \otimes u)_{\leq q} - \nabla p_{\leq q}.$$

To this equation, we can multiply by $u_{\leq q}$ and integrate over time and space. Due to the absolute continuity of $u_{\leq q}$, as it is only a partial sum, we can follow the same procedures as for equation (3.5.3) and obtain the equation.

$$\frac{1}{2}(\|u_{\leq q}(t)\|_{L_2}^2 - \|u_{\leq q}(0)\|_{L_2}^2) = -\int_0^t \Pi_{\leq q}(s) \; ds \tag{3.5.7}$$

Now we can define the following localization kernel $K = \lim_{q\to\infty} K_q$ which we will use later to give an important bound on the flux for $u \in B^{\frac{1}{3}}_{3,\infty}$. As described in Shvydkoy [2010], the important feature of this bound is that it features a strongly decreasing tail which greatly penalises far off interactions giving a quasi-local bound on $\Pi_{\leq q}$.

$$K_q = \begin{cases} 2^{q\frac{2}{3}} & \text{if } q \leq 0 \\ 2^{-q\frac{4}{3}} & \text{if } q > 0. \end{cases}$$

With the following lemma we can easily prove the result. This interesting bound to prove in Besov spaces for the operator $\Pi_{\leq q}$ will finish off this proof.

Lemma 3.5.3 *The energy flux of a divergence free vector field $u \in B^{\frac{1}{3}}_{3,\infty}$ satisfies the following estimate.*

$$|\Pi_{\leq q}| \leq C \sum_{j \geq 1} K_{q-j} 2^j \|\Delta_j u\|_{L_3}^3 \tag{3.5.8}$$

where $C > 0$ is an absolute constant.

Theorem 3.5.4 (Onsager in $B^{\frac{1}{3}}_{3,\infty}$) *Every weak solution u to the incompressible Euler equations on a time interval $[0,T]$ that satisfies*

$$\lim_{q\to\infty}\int_0^T 2^q\|u_q(t)\|^3_{L_3}\ dt = 0,$$

conserves energy on the entire interval $[0,T]$. In particular, energy is conserved for every solution in the class $L_3\left(0,T,B^{\frac{1}{3}}_{3,c_0}\right)$.

Proof Taking the modulus of both sides of (3.5.7) we get the following

$$\frac{1}{2}(\|u_{\leq q}(t)\|^2_{L_2} - \|u_{\leq q}(0)\|^2_{L_2}) \leq \int_0^t |\Pi_{\leq q}(s)|ds \leq \limsup_{q\to\infty}\int_0^t |\Pi_{\leq q}(s)|ds.$$

Then we can apply the above lemma to obtain the inequality

$$\leq \limsup_{q\to\infty}\int_0^t C\sum_{j\geq 1} K_{q-j}2^j\|\Delta_j u\|^3_{L_3}ds \leq C\limsup_{q\to\infty}\int_0^t 2^q\|\Delta_q u\|^3_{L_3}ds = 0.$$

The last inequality holds as the sum of K_q is exponentially decreasing for all j except $j = q$ then the limit as $q \to \infty$ is bounded except for $j = q$ and so this term remains.

□

Proof (Proof of 3.5.3) To prove this we need to split up $(u \otimes u)_{\leq q}$ which we can see as a matrix of all the multiplications of the components of u and thus to split up this double sum we need to use paradifferential calculus and obtain the Bony decomposition of the product of the functions Bahouri et al. [2011].

This gives the following from Constantin et al. [1994] and Shvydkoy [2010]

$$(u\otimes u)_{\leq q} = r_{\leq q}(u,u) - u_{>q}\otimes u_{>q} + u_{\leq q}\otimes u_{\leq q}$$

where,

$$r_{\leq q}(u,u)(x) = \int_{\mathbb{R}^d}\check{\Phi}_q(y)(u(x-u)-u(x))\otimes(u(x-u)-u(x))\ dy.$$

Thus substituting into the definition of $\Pi_{\leq q}$ gives the equation

$$|\Pi_{\leq q}| \leq \int_{\mathbb{R}^d} |r_{\leq q}(u,u):\nabla u_{\leq q}| + |u_{>q}\otimes u_{>q}:\nabla u_{\leq q}| + |u_{\leq q}\otimes u_{\leq q}:\nabla u_{\leq q}|\ dx. \quad (3.5.9)$$

We now need to bound each term separately so for the first (1) term on the RHS of (3.5.9) we use Hölders inequality to give,

$$(1)\text{RHS} \leq \|r_{\leq q}(u,u)\|_{L_{\frac{2}{3}}} \|\nabla u_{\leq q}\|_{L_3}$$

Where further as $0 \leq \check{\Phi}_q \leq 1$ we obtain,

$$\|r_{\leq q}(u,u)\|_{L_{\frac{2}{3}}} \leq \int_{\mathbb{R}^d} |\check{\Phi}_q(y)| \|u(\cdot - y) - u(\cdot)\|^2_{L_3} \, dy.$$

Now we want to bound $\|u(\cdot - y) - u(\cdot)\|^2_{L_3}$. First we multiply by $\frac{|y|}{|y|}$ inside the modulus and see that we have

$$|y|^2 \left\| \frac{u(\cdot - y) - u(\cdot)}{|y|} \right\|^2_{L_3}.$$

The inside term looks like a differential. Due to the multiplication by the function $|\check{\Phi}_q(y)|$ we see that this differential term only gives weight to Fourier modes of order less than or equal to q. We can split it into two sums one over $\leq q$ and one over $> q$ and using Bernstein's lemma for the differential term we obtain

$$\|u(\cdot - y) - u(\cdot)\|^2_{L_3} \leq \sum_{p \leq q} |y|^2 2^{2p} \|\Delta_p u\|^2_{L_3} + \sum_{p > q} \|\Delta_p u\|^2_{L_3}.$$

We want a bound with the term $\left(2^{\frac{p}{3}} \|\Delta_p u\|_{L_3}\right)^3$ thus we multiply, inside the sum, the first term by $\frac{2^{\frac{4}{3}(q-p)}}{2^{\frac{4}{3}(q-p)}}$ and the second term by $\frac{2^{\frac{2}{3}(q-p)}}{2^{\frac{2}{3}(q-p)}}$ and collect terms to give,

$$= |y|^2 2^{q\frac{4}{3}} \sum_{p \leq q} 2^{-\frac{4}{3}(q-p)} (2^{\frac{p}{3}} \|\Delta_p u\|_{L_3})^2 + 2^{-\frac{2}{3}q} \sum_{p > q} 2^{\frac{2}{3}(q-p)} (2^{\frac{p}{3}} \|\Delta_p u\|)^2_{L_3}.$$

We notice that these sums form a pattern of the K kernel introduced before and thus we can simplify to,

$$\left(|y|^2 2^{q\frac{4}{3}} + 2^{-\frac{2}{3}q}\right) \sum_p K_{q-p} (2^{\frac{p}{3}} \|\Delta_p u\|_{L_3})^2.$$

Substituting this back and using Bernstein inequality on $\|\nabla u_{\le q}\|_{L_3}$ we get that

$$
\begin{aligned}
(1)\mathrm{RHS} &\le \int_{\mathbb{R}^d} |\check{\Phi}_q(y)| \left(|y|^2 2^{q\frac{4}{3}} + 2^{-\frac{2}{3}q}\right) \sum_p K_{q-p} (2^{\frac{p}{3}} \|\Delta_p u\|_{L_3})^2 \, dy \left(\sum_{p \le q} 2^{2p} \|\Delta_p u\|_{L_3}^2\right)^{\frac{1}{2}} \\
&= \sum_p K_{q-p} (2^{\frac{p}{3}} \|\Delta_p u\|_{L_3})^2 \left(\int_{\mathbb{R}^d} |\check{\Phi}_q(y)| |y|^2 2^{q\frac{4}{3}} \, dy + 2^{-\frac{2}{3}q}\right) \left(\sum_{p \le q} 2^{p\frac{4}{3}} \left(2^{\frac{p}{3}} \|\Delta_p u\|_{L_3}\right)^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Now we need to calculate the integral. As $|\check{\Phi}_q(y)|$ is supported only for $|y| \le 2^{-q}$ we can pull the supremum out of the integral to obtain

$$
\int_{\mathbb{R}^d} |\check{\Phi}_q(y)| |y|^2 2^{q\frac{4}{3}} \, dy \le 2^{-2q} 2^{q\frac{4}{3}} \int_{\mathbb{R}^d} |\check{\Phi}_q(y)| \, dy \le C 2^{-q\frac{2}{3}}.
$$

Substituting this back in we obtain

$$
\begin{aligned}
(1)\mathrm{RHS} &\le \sum_p K_{q-p} (2^{\frac{p}{3}} \|\Delta_p u\|_{L_3})^2 C 2^{-q\frac{2}{3}} \left(\sum_{p \le q} 2^{p\frac{4}{3}} \left(2^{\frac{p}{3}} \|\Delta_p u\|_{L_3}\right)^2\right)^{\frac{1}{2}} \\
&= C \sum_p K_{q-p} (2^{\frac{p}{3}} \|\Delta_p u\|_{L_3})^2 \left(\sum_{p \le q} 2^{-(q-p)\frac{4}{3}} \left(2^{\frac{p}{3}} \|\Delta_p u\|_{L_3}\right)^2\right)^{\frac{1}{2}} \\
&\le C \left(\sum_p K_{q-p} (2^{\frac{p}{3}} \|\Delta_p u\|_{L_3})^2\right)^{\frac{3}{2}}.
\end{aligned}
$$

Then bringing this power inside the sum as greater than one we get the final result

$$
(1)\mathrm{RHS} \le C \sum_p K_{q-p} (2^{\frac{p}{3}} \|\Delta_p u\|_{L_3})^3.
$$

We now have to deal with the other two terms, these are simpler. For instance the (2)RHS of (3.5.9), with use of Hölder's inequality, similarly to before, we obtain $\|u_{>q}\|_{L_3}^2 \|\nabla u_{\le q}\|_{L_3}$. Then we can bound these terms again using the same methods as above. Thus we are done. $\square$

Chapter 4

The Heat Kernel

Further equivalent definitions for a Besov space can be defined with use of the heat kernel. We will see that the heat kernel definition, in general, has a rather complicated structure. Yet there are occasions when this definition is useful for example considering problems when the domain is not the whole space. Further, it is useful when dealing with a problem which one can write a solution in terms of the heat kernel. For instance, this is found in Cannone et al. [2004].

4.1 Heat Kernel Definition

We have two definitions from Lemarié-Rieusset [2010] for Besov spaces associated to the heat kernel, for the non-homogeneous and the homogeneous cases.

Definition 4.1.1 (Non-Homogeneous Heat Kernel) *For $s \in \mathbb{R}$, $1 \leq q \leq \infty$, $t_0 > 0$, $\alpha \geq 0$ so that $\alpha > s$ and for $f \in \mathcal{S}'(\mathbb{R}^d)$. Then for all $t > 0$*

$$B^s_{p,q} = \left\{ f : e^{t\Delta} f \in L_p \text{ and } \left(\int_0^{t_0} \left(\| t^{-\frac{s}{2}} (-t\Delta)^{\frac{\alpha}{2}} e^{t\Delta} f \|_{L_p} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\} \tag{4.1.1}$$

With the equivalent norms,

$$\|f\|_{B^s_{p,q}} = \|e^{t_0\Delta} f\|_{L_p} + \left(\int_0^{t_0} \left(\| t^{-\frac{s}{2}} (-t\Delta)^{\frac{\alpha}{2}} e^{t\Delta} f \|_{L_p} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Note this norm is similar to norms defining Lorentz/interpolation spaces which we will look at later.

We notice that the parameter $j \in \mathbb{Z}$ seems to have been replaced by the conditions parameter $0 < t < \infty$ as the countable sum is replaced by the integral weighted by $\frac{1}{t}$. This weighted integral and sum perform the same measurement on the function. This will be more obvious when we prove the equivalence of the Littlewood-Paley definition and the heat kernel definition later.

The definition of the homogeneous case which is defined in Lemarié-Rieusset [2010] and Bahouri et al. [2011] is introduced below.

Definition 4.1.2 (Homogeneous Heat Kernel) *Let $s < 0$, $1 \leq q \leq \infty$, $f \in \mathcal{S}'(\mathbb{R}^d)$. Then for all $t > 0$*

$$\dot{B}^s_{p,q} \cap \mathcal{S}'_0 = \left\{ f : e^{t\Delta} f \in L_p \text{ and } \left(\int_0^\infty \left(\|t^{-\frac{s}{2}} e^{t\Delta} f\|_{L_p} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\} \tag{4.1.2}$$

With the equivalent norms,

$$\|f\|_{B^s_{p,q}} = \left(\int_0^\infty \left(\|t^{-\frac{s}{2}} e^{t\Delta} f\|_{L_p} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

We work in $\mathcal{S}'_0(\mathbb{R}^n)$ as we again want to work modulo polynomials to allow us to have a norm and not a semi-norm. As we know supp $\hat{p} = \{0\}$ for p a polynomial we have to work in a space with these elements are removed.

We will show the equivalences between the Littlewood-Paley definitions and the heat kernel definitions later after seeing bounds on the heat kernel for functions f with supp$\hat{f}$ in some annulus.

4.2 Heat Kernel Properties

A useful bound given below describes the action of the heat kernel on functions whose support in Fourier space is on an annulus. This is very useful for applications to look at solutions to the heat equation. Further for showing properties of Besov spaces and the equivalences in definitions using Littlewood-Paley and the heat kernel. The next lemma is stated in Bahouri et al. [2011] and though it is stated on an annulus it can further be shown to hold on a ball around 0 though c a constant in the lemma is 0.

Lemma 4.2.1 (Heat Kernel Bound) *For any annulus A there exists positive constants c and C such that for any $p \in [1, \infty]$ and any pair of positive real numbers (t, λ) we have*

$$\text{Supp } \hat{u} \subset \lambda A \implies \|e^{t\Delta} u\|_{L_p} \leq C e^{-ct\lambda^2} \|u\|_{L_p}. \tag{4.2.1}$$

Proof As we are dealing with a function supported on an annulus A we again need a function $\phi \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$ which is identically 1 near A to use in the proof. We will force all the calculations on to it and with its nicer properties calculations will be easier. Again we can let $\lambda = 1$, as this is just a dilation.

We multiply by ϕ which does not change the action of the heat kernel on A as it has value one. As the Fourier transform of the heat kernel is $e^{-t|\xi|^2}$ we obtain

$$e^{t\Delta}u = \mathcal{F}^{-1}\left(\phi(\xi)e^{-t|\xi|^2}\hat{u}(\xi)\right) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\phi(\xi)e^{-t|\xi|^2}\ d\xi \star u.$$

Applying Young's inequality to both sides gives the desired result provided we show for c, C real positive constants and for all $t > 0$

$$\left\| \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\phi(\xi)e^{-t|\xi|^2}\ d\xi \right\|_{L_1} := \|f(t,\cdot)\|_{L_1} \leq Ce^{-ct}.$$

To bound f in L_1, the idea is to pull out a $\frac{1}{(1+|x|^2)^d}$ as this is bounded in L_1 and then try to eliminate the $\left(1+|x|^2\right)^d$ we created. So first we multiply by 1 in the correct manner

$$f(t,x) = \frac{1}{(1+|x|^2)^d}\int_{\mathbb{R}^d}\left(1+|x|^2\right)^d e^{ix\cdot\xi}\phi(\xi)e^{-t|\xi|^2}\ d\xi.$$

Now we use the properties of the Fourier transform and integration by parts as $\phi(\xi)$ compactly supported and thus we obtain

$$\frac{1}{(1+|x|^2)^d}\int_{\mathbb{R}^d}\left((Id-\Delta_\xi)^d e^{ix\cdot\xi}\right)\phi(\xi)e^{-t|\xi|^2}\ d\xi =$$
$$= \frac{1}{(1+|x|^2)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\left(Id-\Delta_\xi\right)^d\phi(\xi)e^{-t|\xi|^2}\ d\xi.$$

We then have to use Faa di Bruno's formula, an algebraic identity, which gives in our case for some constants C_β^α

$$\left(Id-\Delta_\xi\right)^d\phi(\xi)e^{-t|\xi|^2} = \sum_{\beta\leq\alpha, |\alpha|\leq 2d} C_\beta^\alpha\left(\partial^{(\alpha-\beta)}\phi(\xi)\right)\left(\partial^\beta e^{-t|\xi|^2}\right).$$

Now as the support of ϕ is in an annulus we can find a couple of bounds. With c the square of the lower bound of the support of ϕ and C being a function of the upper bound of the support of ϕ. Also we notice that as $\phi \in \mathcal{D}$, $\left|\partial^{(\alpha-\beta)}\phi\right|$ is bounded. Further each $\partial e^{-t|\xi|^2}$ will drop down a constant of t and ξ yet the ξ can be bounded by C so we have

$$\left|\left(\partial^{(\alpha-\beta)}\phi(\xi)\right)\left(\partial^{\beta}e^{-t|\xi|^2}\right)\right| \leq C(1+t)^{|\beta|}e^{-t|\xi|^2}.$$

Then using the lower bound c we obtain

$$\leq C(1+t)^{|\beta|}e^{-ct}.$$

Thus

$$\begin{aligned}
\|f(t,\cdot)\|_{L_1} &\leq \int_{\mathbb{R}^d}\left|\frac{1}{(1+|x|^2)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\,(Id-\Delta_\xi)^d\,\phi(\xi)e^{-t|\xi|^2}\,d\xi\right|\,dx \\
&\leq \int_{\mathbb{R}^d}\frac{1}{(1+|x|^2)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\left|(Id-\Delta_\xi)^d\,\phi(\xi)e^{-t|\xi|^2}\right|\,d\xi\,dx \\
&\leq \sum_{\beta\leq\alpha,|\alpha|\leq 2d} C^{\alpha}_{\beta}C(1+t)^{|\beta|}e^{-ct}\int_{\mathbb{R}^d}\frac{1}{(1+|x|^2)^d}\,dx\int_{\mathbb{R}^d} e^{ix\cdot\xi}\,d\xi \\
&\leq Ce^{-ct}\int_{\mathbb{R}^d}\frac{1}{(1+|x|^2)^d}\,dx \\
&\leq Ce^{-ct}.
\end{aligned}$$

This gives us the bound we needed. If we are dealing with a ball rather than the annulus then the lower bound $c = 0$. □

There is a corollary in Bahouri et al. [2011] that describes the behaviour of the solution to the heat equation for initial conditions whose Fourier transform is supported in an annulus. This behaviour relates the integrability over time to the initial condition.

Corollary 4.2.2 *Let A be an annulus and λ a positive real number. Let initial condition u_0 and forcing $f(t,x)$ have its Fourier transform supported in λA for all $t\in[0,T]$. Consider the solution $u(t)$ to the heat equation*

$$\partial_t u - \nu\Delta u = 0 \text{ and } u(0,\cdot) = u_0(\cdot).$$

Consider v a solution of forced heat equation

$$\partial_t v - \nu\Delta v = f \text{ and } v(0,\cdot) = 0.$$

Then there exist positive constant C depending only on A such that for $1\leq a\leq b\leq$

∞ *and* $1 \leq p \leq q \leq \infty$ *we have*

$$\|u\|_{L_q(0,T,L_b)} \leq C(\nu\lambda^2)^{-\frac{1}{q}}\lambda^{n\left(\frac{1}{a}-\frac{1}{b}\right)}\|u_0\|_{L_a},$$

$$\|v\|_{L_q(0,T,L_b)} \leq C(\nu\lambda^2)^{-1+\left(\frac{1}{p}-\frac{1}{q}\right)}\lambda^{n\left(\frac{1}{a}-\frac{1}{b}\right)}\|f\|_{L_p(0,T,L_a)}.$$

Proof For a $u(t)$ satisfying the heat equation we can use the heat kernel to to write $u(t) = e^{\nu t\Delta}u_0$. Then we start with the definition of $\|u\|_{L_q(0,T,L_b)}$ and use the lemma above (4.2.1) this gives

$$\left(\int_0^T \left(\left\|e^{\nu t\Delta}u_0\right\|_{L_b}\right)^q \, dt\right)^{\frac{1}{q}} \leq \left(\int_0^T \left(Ce^{-ct\nu\lambda^2}\left\|u_0\right\|_{L_b}\right)^q \, dt\right)^{\frac{1}{q}}.$$

We can then use Bernstein's Lemma to get a bound from L_a to L_b and rearrange

$$\left(\int_0^T \left(Ce^{-ct\nu\lambda^2}\lambda^{d\left(\frac{1}{a}-\frac{1}{b}\right)}\left\|u_0\right\|_{L_a}\right)^q \, dt\right)^{\frac{1}{q}} \leq C\lambda^{d\left(\frac{1}{a}-\frac{1}{b}\right)}\left\|u_0\right\|_{L_a}\left(\int_0^T e^{-ct\nu\lambda^2 q} \, dt\right)^{\frac{1}{q}}.$$

We can then integrate out to obtain

$$C\left\|u_0\right\|_{L_a}\lambda^{d\left(\frac{1}{a}-\frac{1}{b}\right)}\left(\frac{1}{c\nu\lambda^2 q}\right)^{\frac{1}{q}}\left(1-e^{-cT\nu\lambda^2 q}\right)^{\frac{1}{q}}.$$

Thus as $cT\nu\lambda^2 q > 0$ we have $e^{-cT\nu\lambda^2 q} \in (0,1)$ and we can bound $\left(1-e^{-cT\nu\lambda^2 q}\right)^{\frac{1}{q}} \leq 1$. Then absorbing constants gives the desired result.

For the second part we similarly start with $v(t) = \int_0^t e^{\nu(t-\tau)\Delta}f(\tau) \, d\tau$ and the proof follows in a similar fashion. $\square$

As mentioned earlier we want to show the equivalence between the Littlewood-Paley definition of a Besov space and the heat kernel definition. So far with the heat kernel bound and the Bernstein's lemmas we now have enough to prove the equivalence of these two definitions. The proof will have inspiration from Bahouri et al. [2011] yet here the proof is only for the homogeneous case where $s < 0$ and we look at the non-homogeneous case for all $s \in \mathbb{R}$. This is further mentioned in Lemarié-Rieusset [2010].

Theorem 4.2.3 (Littlewood-Paley and heat kernel equivalence) *The Littlewood-Paley definition of a Besov space and the heat kernel definition are equivalent.*

Proof First we want to look at the heat kernel integrand and using the heat kernel bound above and the Bernstein lemma for the operator $I^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ we have

$$\|t^{-\frac{s}{2}}\Delta_j(-t\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p} \leq Ct^{\frac{\alpha-s}{2}}2^{j\alpha}e^{-ct2^{2j}}\|\Delta_j u\|_{L_p}.$$

Then by multiplication by $\frac{2^{js}}{2^{js}}$ we can pull the lower term with the $2^{j\alpha}$ term and take the remaining term to create a Littlewood-Paley Besov norm after being summed like so

$$\begin{aligned}\|t^{-\frac{s}{2}}(-t\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p} &= \sum_{j\in\mathbb{Z}}\|t^{-\frac{s}{2}}\Delta_j(-t\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p} \leq \\ &\leq C\sum_{j\in\mathbb{Z}} t^{\frac{\alpha-s}{2}}2^{j(\alpha-s)}e^{-ct2^{2j}}2^{js}\|\Delta_j u\|_{L_p} \leq C\|u\|_{B^s_{p,q}}\sum_{j\in\mathbb{Z}} t^{\frac{\alpha-s}{2}}2^{j(\alpha-s)}e^{-ct2^{2j}}c_{q,j}.\end{aligned}$$

Where $c_{q,j}$ is a generic element of the unit ball in $l_q(\mathbb{Z})$. This last inequality comes from using Hölder's inequality to bound the last two terms on the left hand side in l_q and get the Littlewood-Paley Besov norm from the $2^{js}\|\Delta_j u\|_{L_p}$ term. We get left with a term in $l_{q'}$, for q' the conjugate of q. Then we can use the duality definition for an element in an l_r space. (For an element in l_r we know $\sum_j x_r y_{r'} < \infty$ for all $x_r \in l_r$ and $y_{r'}$ any element in the unit ball of $l_{r'}$.) We use this definition, of an element in $l_{q'}$, for the rest.

For the rest of this proof we need something to deal with the spare term and we need a lemma in Bahouri et al. [2011] which helps bound this term as we chose $\alpha - s > 0$.

Lemma 4.2.4 *For any positive m, we have the bound*

$$\sup_{t>0}\sum_{j\in\mathbb{Z}} t^{\frac{m}{2}}2^{mj}e^{-ct2^{2j}} < \infty. \tag{4.2.2}$$

From this lemma we first notice that for the case with $q = \infty$ we are done and have the first inequality after taking the supremum over $t > 0$ of both sides of our previous bound.

We now consider the case of $q < \infty$. So we want to bound

$$\int_0^{t_0}\|t^{-\frac{s}{2}}(-t\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|^q_{L_p}\,\frac{dt}{t} \leq C\|u\|^q_{B^s_{p,q}}\int_0^{t_0}\left(\sum_{j\in\mathbb{Z}} t^{\frac{\alpha-s}{2}}2^{j(\alpha-s)}e^{-ct2^{2j}}c_{q,j}\right)^q\frac{dt}{t}. \tag{4.2.3}$$

We can split the term inside the integral in preparation to use Hölder's inequality

$$\sum_{j\in\mathbb{Z}} t^{\frac{\alpha-s}{2}} 2^{j(\alpha-s)} e^{-ct2^{2j}} c_{q,j} = \sum_{j\in\mathbb{Z}} \left(\left(t^{\frac{\alpha-s}{2}} 2^{j(\alpha-s)} e^{-ct2^{2j}} \right)^{\frac{1}{q}} c_{q,j} \right) \left(t^{\frac{\alpha-s}{2}} 2^{j(\alpha-s)} e^{-ct2^{2j}} \right)^{\frac{q-1}{q}}.$$

We then use Hölder's inequality with the first bracket term to the power of q and the second $q' = \frac{q}{q-1}$. This then, after we cancel the powers gives,

$$\leq C\|u\|^q_{B^s_{p,q}} \int_0^{t_0} \left(\sum_{j\in\mathbb{Z}} t^{\frac{\alpha-s}{2}} 2^{j(\alpha-s)} e^{-ct2^{2j}} \right)^{q-1} \left(\sum_{j\in\mathbb{Z}} t^{\frac{\alpha-s}{2}} 2^{j(\alpha-s)} e^{-ct2^{2j}} c^q_{q,j} \right) \frac{dt}{t}.$$

We take the supremum over time of the first term out and can bound by the previous lemma. We only have to worry about the second term in the integral and we can use Fubini's theorem for this. To get the bound, we need to calculate the following integral and show it is finite. To simplify, let $k = \frac{\alpha-s}{2}$

$$\sum_{j\in\mathbb{Z}} c^q_{q,j} \int_0^{t_0} t^{k-1} 2^{j2k} e^{-ct2^{2j}} \, dt \leq \sum_{j\in\mathbb{Z}} c^q_{q,j} \int_0^{\infty} t^{k-1} 2^{j2k} e^{-ct2^{2j}} \, dt.$$

Then we observe by calculation that $\int_0^\infty t^{k-1} e^{-ct2^{2j}} \, dt = \frac{C}{2^{j2k}}\Gamma(k)$. Thus we find that the whole term becomes

$$\leq \sum_{j\in\mathbb{Z}} C c^q_{q,j} \Gamma(k) \leq C\Gamma(k).$$

We have the first inequality.

To prove the other direction, we use the fact that the last bound is of the form of a gamma distribution to derive a useful identity

$$\widehat{\Delta_j u} = 1 \star \widehat{\Delta_j u} = \frac{1}{\Gamma(k+1)} \int_0^\infty x^k e^{-x} \, dx \star \widehat{\Delta_j u}.$$

Then use the substitution $x = t|\xi|^2$ and the Fubini's theorem, we obtain,

$$\frac{1}{\Gamma(k+1)} \int_0^\infty t^k |\xi|^{2(k+1)} e^{-x|\xi|^2} \, dx \star \widehat{\Delta_j u} = \frac{1}{\Gamma(k+1)} \int_0^\infty t^k |\xi|^{2(k+1)} e^{-t|\xi|^2} \star \widehat{\Delta_j u} \, dt.$$

Finally we can apply the inverse Fourier transform to both sides and we have,

$$\Delta_j u = \frac{1}{\Gamma(k+1)} \int_0^\infty t^k (-\Delta)^{k+1} e^{t\Delta} \Delta_j u \, dt. \tag{4.2.4}$$

Now let $k = \frac{\alpha-s}{2}$ and split up $e^{t\Delta}u = e^{\frac{t}{2}\Delta}e^{\frac{t}{2}\Delta}u$. This lets us use the heat kernel bound without losing the heat kernel properties for bounding the integral. After performing these two substitutions we can bound the L_p norm with the usual use of the Bernstein's Lemmas and heat kernel bound

$$\begin{aligned}\|\Delta_j u\|_{L_p} &\leq C\int_0^\infty t^{\frac{\alpha-s}{2}}2^{j(-s+2)}e^{-ct2^{2j}}\|\Delta_j(-\Delta)^{\frac{\alpha}{2}}e^{\frac{t}{2}\Delta}u\|_{L_p}\ dt\\ &\leq C\int_0^\infty t^{\frac{\alpha-s}{2}}2^{j(-s+2)}e^{-ct2^{2j}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}\ dt.\end{aligned}$$

Now we want to look at the two cases again $q=\infty$ and $q<\infty$. For the first case we want to bound $\sup_{j\in\mathbb{Z}}\left(2^{js}\|\Delta_j u\|_{L_p}\right)$. From the above we obtain,

$$\sup_{j\in\mathbb{Z}}\left(2^{js}\|\Delta_j u\|_{L_p}\right) \leq C\sup_{j\in\mathbb{Z}}\left(2^{js}\int_0^\infty t^{\frac{\alpha-s}{2}}2^{j(-s+2)}e^{-ct2^{2j}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}\ dt\right).$$

Then we want to take the supremum over t out of the integral so that we obtain the heat kernel definition and want the rest of the terms to be bounded

$$C\sup_{j\in\mathbb{Z}}\left(2^{js}\sup_{t>0}\left(t^{\frac{\alpha-s}{2}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}\right)\int_0^\infty 2^{j(-s+2)}e^{-ct2^{2j}}\ dt\right).$$

We now just have to calculate the integral and rearrange We discover that everything cancels nicely,

$$\begin{aligned}&C\sup_{j\in\mathbb{Z}}\left(2^{js}2^{-js}\sup_{t>0}\left(t^{\frac{\alpha-s}{2}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}\right)2^{j2}\left(-c2^{-2j}e^{-ct2^{2j}}\Big|_0^\infty\right)\right)=\\ &\qquad\qquad = C\sup_{t>0}\left(t^{\frac{\alpha-s}{2}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}\right).\end{aligned}$$

We now have to look at the case of $q<\infty$. This is again going to use similar methods to the $q<\infty$ case before. We have to deal with the inequality below

$$\sum_{j\in\mathbb{Z}}\left(2^{js}\|\Delta_j u\|_{L_p}\right)^q \leq C\sum_{j\in\mathbb{Z}}\left(2^{js}\int_0^\infty t^{\frac{\alpha-s}{2}}2^{j(-s+2)}e^{-ct2^{2j}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}\ dt\right)^q.$$

We can split this up to prepare for Hölder's inequality, used in the same way as before,

$$C\sum_{j\in\mathbb{Z}}2^{jsq}\left(\int_0^\infty e^{-ct2^{2j}\frac{1}{q}}t^{\frac{\alpha-s}{2}}2^{j(-s+2)}e^{-ct2^{2j}\frac{q-1}{q}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}\ dt\right)^q.$$

Then we can apply Hölder's inequality,

$$\leq C\sum_{j\in\mathbb{Z}} 2^{jsq}\left(\int_0^\infty e^{-ct2^{2j}}dt\right)^{q-1}\left(\int_0^\infty t^{q\frac{\alpha-s}{2}}2^{qj(-s+2)}e^{-ct2^{2j}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}^q\ dt\right).$$

Calculation of the first integral gives

$$C\sum_{j\in\mathbb{Z}} 2^{jsq}2^{-2j(q-1)}\left(\int_0^\infty t^{q\frac{\alpha-s}{2}}2^{qj(-s+2)}e^{-ct2^{2j}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}^q\ dt\right).$$

Now we can use Fubini's theorem to switch the sum and integral and then we can simplify, this gives,

$$= C\int_0^\infty \sum_{j\in\mathbb{Z}} 2^{j2}t^{q\frac{\alpha-s}{2}}e^{-ct2^{2j}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}^q\ dt.$$

Multiplying by $\frac{t}{t}$ and rearranging gives us the measure we need for the integral in the heat kernel definition of Besov space,

$$= C\int_0^\infty t^{q\frac{\alpha-s}{2}}\|(-\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}^q\sum_{j\in\mathbb{Z}}\left(t2^{j2}e^{-ct2^{2j}}\right)\ \frac{dt}{t}.$$

Then taking the supremum of the sum out of the integral and bounding by the previous lemma, this rearranges to

$$\leq C\int_0^\infty \|t^{-\frac{s}{2}}(-t\Delta)^{\frac{\alpha}{2}}e^{t\Delta}u\|_{L_p}^q\ \frac{dt}{t}.$$

This is the bound we wanted and together we have both the lower and upper bounds so the equivalence is proved. □

We have shown that the Littlewood-Paley definition and the heat kernel definition are equivalent. This proves that the Littlewood-Paley definition must be independent of the choice of functions used to create the decomposition. This is because the heat kernel definition and the equivalence to the Littlewood-Paley definiton is independent of the choice of functions used.

4.3 Heat Equation Applications

This application of Besov spaces relates the heat equation on $\mathbb{R}^2$ and the relations between the initial conditions and the solution some time t later.

For the Sobolev case we discover that for $u_0 \in L_2$ we can achieve a bound to show that at time $t \in (0,T]$, $u(t)$ is in $L_1(0,T,H^1)$ for any T. This calculation will be given below and other simpler more general forms can be found in Evans [1998]. We take the equation, multiply by u, and integrate over space and time then perform integration by parts on the gradient term and use the identity $u_t u = \frac{1}{2}\frac{d}{dt}|u|^2$. Switching integral over space and derivative gives

$$\|u(T)\|_{L_2}^2 + 2\int_0^T \|\nabla u\|_{L_2}^2 \; dt = \|u(0)\|_{L_2}^2.$$

This gives the desired bound for controling the gradient with the L_2 norm of the initial condition.

We were able to increase the regularity by a derivative for the Sobolev case above yet now we will show that with a slightly smaller Besov space than L_2 we can achieve extra regularity of two derivatives.

Example 4.3.1 *Let the initial condition to the heat equation on $\mathbb{R}^d$ be $u(0) = u_0 \in B^0_{2,1}$, then the solution at time $t \in (0,T]$ $u(t)$ is in $L_1(0,T,B^2_{2,1})$ for any T.*

This example shows how easy it is to use Besov spaces and specifically the Littlewood-Paley definition for this calculation.

We see from (3.1.1) that for a function $f \in B^2_{p,q}$ that $f \in \mathcal{S}'$ as well which will be useful as the heat kernel is in $\mathcal{S}$ and thus the integrals will make sense. Further we see that the space norm is the sum of two separate norms and thus we will consider each one separately.

Proof We can write for $u \in \mathcal{S}'$ $u(t) = e^{t\Delta}u_0$ as this is just the action of the heat kernel.

We then want to consider the Littlewood-Paley decomposition of $u(t)$ so first consider the decomposition into the annulus so $\phi_j \star u(t)$ which we can also write $\Delta_j u(t)$ for $j \in \mathbb{N}$. We can write this part of the norm of $u(t)$ by

$$\int_0^T \left| \sum_{j=0}^{\infty} 2^{j2} \left\| e^{t\Delta}\Delta_j u_0 \right\|_{L_2} \right| \; dt.$$

Clearly the modulus can be ignored as the term inside is always positive. We can now get the next inequality by the using the heat kernel estimate for a function whose Fourier transform is supported in an annulus (4.2.1)

$$\int_0^T \sum_{j=0}^{\infty} 2^{j2} \left\| e^{t\Delta}\Delta_j u_0 \right\|_{L_2} \; dt \leq \int_0^T \sum_{j=0}^{\infty} 2^{j2} C e^{-ct2^{2j}} \left\| \Delta_j u_0 \right\|_{L_2} \; dt.$$

We can now use Fubini's theorem to switch sum and integral and obtain

$$\sum_{j=0}^{\infty} 2^{j2} \left\|\Delta_j u_0\right\|_{L_2} \int_0^T Ce^{-ct2^{2j}}\ dt \leq \sum_{j=0}^{\infty} 2^{j2} \left\|\Delta_j u_0\right\|_{L_2} \frac{1}{c2^{2j}} \left(1 - e^{-cT2^{2j}}\right).$$

This simplifies to

$$\sum_{j=0}^{\infty} \left\|\Delta_j u_0\right\|_{L_2} \frac{1}{c} \left(1 - e^{-cT2^{2j}}\right).$$

We see that $cT2^{2j}$ is always positive so $(1 - e^{-cT2^{2j}})$ is always in the interval $(0, 1]$ and so u_0 is in $B^0_{2,1}$ so this is bounded.

The $\Phi_0 \star u(t)$ term can also be written $S_0 u(t)$. We just need to show that $\int_0^T \|e^{t\Delta}\Delta_j u_0\|_{L_2}$ is bounded. We can bound this by $C\int_0^T e^{-ct2^0}\|\Delta_j u_0\|_{L_2}$. As $c = 0$ we just get a bound of $C\int_0^T \|\Delta_j u_0\|_{L_2}$ and we are done. □

We notice here that if instead of $q = 1$ we used $q = 2$ and were using as our spaces $u_0 \in L_2$ and $u(t) \in L_1(0, T, H^2)$ then this proof would fail. This is clear as the cancellation would not be complete and we would be left with a 2^{2j} term in the sum. This extra term comes from the square of 2^{2j} we get from the l_2 norm.

4.4 Heat Kernel on Sub Domains

There exists interesting problems that have been solved on the whole space but not on bounded domains and therefore defining Besov spaces on these bounded domains may help in solving some of these problems.

We have seen from the calculations that when working with problems on the whole domain, the Littlewood-Paley definition seems to be the easiest to work with. However, as seen before there is no analogous definition of the Littlewood-Paley decomposition that can be defined on a bounded domain due to the innate dependence on the Fourier transform.

We need to generalise the heat kernel definition to the domain Ω. For this generalisation, one problem is the choice of boundary data for the Heat equation. The choice of boundary data should depend on the values on the boundary that one wants for elements of the Besov space. Therefore, if we are considering spaces analogous to $W^m_{p,0}$ where we want the functions to go to zero on the boundary we would want to consider the heat equation with zero on the boundary condition.

$$\begin{aligned}\frac{d}{dt}u &= \Delta u \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \\ u_0 = u(0,\cdot) &= f(\cdot).\end{aligned}$$

Then we wish to consider the u that solves this set of equations with the initial condition f.

Definition 4.4.1 (Non-Homogeneous Heat Kernel norm on domain) *For u the solution to the equation above with f the initial data. For $s \in \mathbb{R}$, $1 \leq q \leq \infty$, $t_0 > 0$, $\alpha \geq 0$ so that $\alpha > s$, α even positive integer. Then for all $t > 0$*

$$\|f\|_{B^s_{p,q,0}} = \|u\|_{L_p} + \left(\int_0^{t_0} \left(\|t^{-\frac{s}{2}}(-t\Delta)^{\frac{\alpha}{2}}u\|_{L_p(\Omega)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

Then to define the space it makes sense to have the value of the function at the boundary to be zero. Thus taking inspiration from the Sobolev case we can define, $B^s_{p,q,0}(\Omega) \equiv \overline{\{f \in C_0^\infty(\Omega) : \|f\|_{B^s_{p,q,0}(\Omega)} < \infty\}}^{B^s_{p,q,0}(\Omega)}$.

Chapter 5

Lorentz and Interpolation Spaces

5.1 Background Theory

To define a real interpolation definition of a Besov space and, to give an initial idea of how these definitions are all equivalent, Lorentz and interpolation spaces will be introduced They are discussed in Peetre [1976] and Chapter 7 of Adams and Fournier [2003]. These interpolation spaces are a useful building block for the Besov spaces and useful in proofs of equivalence and properties of the Besov spaces.

These interpolation spaces and the associated Besov space definitions are of importance though again not necessary easy to do calculations with. However this definition is useful as it is more abstract so it can give general properties from interpolation theory and is easier to generalise to other domains.

Definition 5.1.1 (Lorentz Space) *Take a measure space* $(\Omega, \mathscr{B}(\Omega), \mu)$*. If* $0 < p, q \leq \infty$ *we define* $L_{p,q}$ *the Lorentz space the space of* j*-measurable functions such that*

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \, \frac{dt}{t} \right)^{\frac{1}{q}},$$

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} (t^{\frac{1}{p}} f^*(t))$$

and by convention $L_{\infty,\infty} = L_\infty$*, where* f^* *is the decreasing rearrangement of* $|f|$

$$f^*(t) = \inf\{s : j\{|f| > s\} \leq t\}.$$

Also $L_{p,\infty}$ is the weak Lebesgue space. Furthermore $L'_{p,q} \approx L_{p',q'}$ if $1 < p < \infty$,$1 \leq q < \infty$ or $p = q = 1$ here $\frac{1}{p} + \frac{1}{p'} = 1$.

These spaces then lead on to interpolation spaces which can be used to generalise and treat many families of function spaces with the same approach.

Definition 5.1.2 (Banach couple) *For two Banach spaces B_0 and B_1 and a Hausdroff topological vector space $\mathcal{B}$ with B_0 and B_1 continuously embedded in $\mathcal{B}$ then we define $\bar{B} = B_0, B_1$ the Banach couple.*

We now want to define an F that for the Banach couple $\bar{B}$ gives a Banach space $F(\bar{B})$ that is continuously embedded in $\mathcal{B}$. This $F(\bar{B})$ in an interpolation space. It acts on the couple to take certain combinations of the elements of each couple and creates a space out of this.

Real interpolation spaces have two equivalent definitions involving the differing functionals J and K described in Adams and Fournier [2003],Lemarié-Rieusset [2010] and Peetre [1976]. Though, in Adams and Fournier [2003] and Lemarié-Rieusset [2010] the definitions are formed where we first split the function in the interpolation space into a diadic series first and then apply the J or K functional to the series and check it is bounded. Here the definition given will be from Peetre [1976].

Definition 5.1.3 (Lions and Gagliardo Real interpolation spaces) For the Banach situation we introduce two auxiliary functionals K and J. For $0 < t < \infty$, $b \in B_0 + B_1$ we let

$$K(t, b; \bar{B}) = K(t, b; B_0, B_1) = \inf_{b=b_0+b_1} (\|b_0\|_{B_0} + t\|b_1\|_{B_1})$$

If $0 < t < \infty$, $b \in B_0 \cap B_1$ we let

$$J(t, b; \bar{B}) = J(t, b, B_0, B_1) = \max (\|b\|_{B_0}, t\|b\|_{B_1})$$

Then let $0 < \theta < 1$, $0 < q \leq \infty$. Then we define the interpolation space $(\bar{B})_{\theta,q}$ by

$$b \in (\bar{B})_{\theta,q} = (B_0, B_1)_{\theta,q}$$

$$\iff \left(\int_0^\infty \left(\frac{K(t, b)}{t^\theta} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

$\iff$ there exists $u = u(t)$ $(0 < t < \infty)$ such that

$$\left(\int_0^\infty \left(\frac{J(t,u(t))}{t^\theta}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty \text{ and } b = \int_0^\infty u(t) \frac{dt}{t}$$

$\iff$ $u = u_m u$ $j \in \mathbb{Z}$ such that

$$\sum_{j=-\infty}^{\infty} \frac{J(2^j, u_j)}{2^{j\theta}} < \infty \text{ and } b = \sum_{j=-\infty}^{\infty} u_j$$

Here $(\bar{B})_{\theta,q}$ has norm

$$\begin{aligned} \|b\|_{(\bar{B})_{\theta,q}} &= \left(\int_0^\infty \left(\frac{K(t,b)}{t^\theta}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\approx \inf_u \left(\int_0^\infty \left(\frac{J(t,u(t))}{t^\theta}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\approx \inf_u \left(\sum_{j=-\infty}^{\infty} \left(\frac{J(2^j,u_j)}{2^{j\theta}}\right)^q\right)^{\frac{1}{q}} \end{aligned}$$

This dyadic decomposition interpolation definition is like the Littlewood-Paley decomposition of a function mentioned earlier. This gives more understanding of the relations between interpolation and the definitions involving integrals and using the Littlewood-Paley definition where the integral is replaced by this countable sum. We can use this definition to perform interpolation on the Littlewood-Paley definition for calculation purposes.

Theorem 5.1.4 (Equivalence) *In the definition above, the different sub definitions are equivalent.*

Proof (Sketch) First we notice that the first two norms are of the form

$$\left(\int_0^\infty \left(\frac{V}{t^\theta}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

for some V and thus we only have to worry about V for the equivalences to hold.

For $b \in (\bar{B})_{\theta,q}$ with the K functional definition we have $\inf_{b=b_0+b_1}(\|b_0\|_{B_0} + t\|b_1\|_{B_1})$. Thus if we then define $b_{0/1} = \int_0^\infty u_{0/1}(t) \frac{dt}{t}$ we have

$$\inf_{b=\int_0^\infty u_0(t)\frac{dt}{t}+\int_0^\infty u_1(t)\frac{dt}{t}} \left(\left\|\int_0^\infty u_0(t) \frac{dt}{t}\right\|_{B_0} + t\left\|\int_0^\infty u_1(t) \frac{dt}{t}\right\|_{B_1}\right), \tag{5.1.1}$$

and by simple rearrangement of integrals we achieve

$$\inf_{b=\int_0^\infty u_0(t)\frac{dt}{t}+\int_0^\infty u_1(t)\ \frac{dt}{t}} \left(\int_0^\infty \|u_0(t)\|_{B_0} \frac{dt}{t} + t \int_0^\infty \|u_1(t)\|_{B_1} \ \frac{dt}{t} \right). \tag{5.1.2}$$

We know that (5.1.2) is bounded it implies that $\|u_0(t)\|_{B_0}$ and $\|u_1(t)\|_{B_1}$ are bounded so the functional $J = \inf_u(\max(\|u_0(t)\|_{B_0}, t\|u_1(t)\|_{B_1}))$ is bounded by a constant of the K functional. Further, if (5.1.1) is bounded it implies $b = \int_0^\infty u(t)\ \frac{dt}{t}$ for some u.

For the other bound it is easy to see from (5.1.2) that if $b = \int_0^\infty u(t)\ \frac{dt}{t}$ exists and if $J = \inf_u(\max(\|u_0(t)\|_{B_0}, t\|u_1(t)\|_{B_1}))$ is bounded then twice the maximum will bound a linear combination of $u_{0/1}$ and thus the K functional is bounded by a constant of the J functional.

The proof for discrete K and J functionals is in Lemarié-Rieusset [2010] and Adams and Fournier [2003] . Were we split the intergral into dyadic parts and prove bounds Bernstein like bounds for each dyadic part. $\square$

One reason to look at these interpolation spaces is that the Lorentz spaces are a specific class of interpolation spaces with the correctly chosen functional. Notice $L_{p,q} = [L^1, L^\infty]_{\frac{1}{p},q}$

More importantly the definitions of Besov spaces are defined so that we see the norms are of the form of the interpolation spaces. We can, in fact, define the Besov space via an interpolation of Sobolev spaces and then use theorems on the properties of interpolation spaces to understand the properties of Besov spaces. For instance the duality and reiteration theorems.

From the equivalence between this definition and the others we again show that the Littlewood-Paley definition is independent on the choice of function used in the decomposition. Further using embedding theorems for interpolation spaces we can straight away get embedding theorems for Besov spaces for instance from the comparison theorem in Peetre [1976] tells us that $B^s_{p,1} \hookrightarrow W^s_p \hookrightarrow B^s_{p,\infty}$. Further from the duality theorems we attain $(B^s_{p,q})' \cong B^{-s}_{p',q'}$ as one would expect.

5.1.1 Real Interpolation Definition

Definition 5.1.5 (Real Interpolation Besov space) *From Peetre [1976]. For real interpolation, s real. We have for* $s = (1-\theta)s_0 + \theta s_1$ $(0 < \theta < 1)$

$$\left(W^{s_0}_p, W^{s_1}_p\right)_{\theta,q} = B^s_{p,q}.$$

Further this can be written

$$B^s_{p,q} = \left\{ f : \left(\int_0^\infty \left(\frac{K(t,f)}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}, \tag{5.1.3}$$

With

$$K(t,f) = K(t,f; W_p^{s_0}, W_p^{s_1}) = \inf_{f=f_0+f_1} \left(\|f_0\|_{W_p^{s_0}} + t\|f_1\|_{W_p^{s_1}} \right).$$

Interestingly one sees that from the interpolation definition of a Besov space, we have to interpolate across two Sobolev spaces in the same L_p space but with different derivative exponent. A function in the new space is bounded by the combination $s_0\theta + (1 - \theta s_1)$ with different outer $L_q(\frac{dt}{t})$ norms.

Therefore consider a function in a Besov space. We can bound this function by taking a weighted sum of the norms of Sobolev functions with differentiability either side. By varying θ we get a different weighting of the norms varying convexly with θ.

Further the q determines the integrability we want for this weighting between the norms and the measure $\frac{dt}{t}$ acts as a limit of the sums of these weights as discussed earlier. Overall we see that the combination of these three indices allows for a strong control on what functions we have in a Besov space.

We can define the definition for homogeneous real interpolation Besov spaces similarly to above but using the homogeneous definition of a Sobolev space as defined earlier.

Definition 5.1.6 (Homogeneous Real Interpolation Besov space) *Same as definition above but replace the space $W_p^{s_0}$ and $W_p^{s_1}$ with $\dot{W}_p^{s_0}$ and $\dot{W}_p^{s_1}$.*

5.2 Interpolation on Sub Domains

Finally we see from the definition that we can generalise this definition to bounded domains as we can use Sobolev spaces with integer derivative coefficients which we can define on bounded domains and then interpolate between them. As mentioned in Chapter 2 there are many ways to define a Sobolev space on a bounded domain and this will give many ways to define the bounded domain Besov space.

5.2.1 Interpolation Bounded Domain Definition

This is taken from an analogy to the Sobolev case where we can use interpolation to define Sobolev spaces for any $m \in \mathbb{R}_+$.

Definition 5.2.1 (Real Interpolation Besov space bounded domain) For $X^m_{p,i}$ defined as one of $H^m_p(\Omega), W^m_p(\Omega), W^m_{p,0}(\Omega)$ for i respectively $1, 2, 3$, m positive integer. We have for $s = (1-\theta)m_0 + \theta m_1$ $(0 < \theta < 1)$,

$$\left(X^{m_0}_{p,i}, X^{m_1}_{p,i}\right)_{\theta,q} = B^s_{p,q,i}.$$

Further can be written,

$$B^s_{p,q,i} = \left\{ f : \left(\int_0^\infty \left(\frac{K_i(t,f)}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}, \tag{5.2.1}$$

with

$$K_i(t,f) = K_i(t,f; X^{m_0}_p, X^{m_1}_p) = \inf_{f=f_0+f_1} \left(\|f_0\|_{X^{m_0}_{p,i}} + t\|f_1\|_{X^{m_1}_{p,i}} \right).$$

We see here that we get three different definitions of a Besov space for each different Sobolev space we interpolate over. The most useful for applications will probably be the third where we use $W^m_{p,0}$ as this definition has functions going to zero on the boundary. This is useful to have in most applications.

Also in Chapter 7 of Adams and Fournier [2003], it defines a definition of the bounded domain Besov space as above but with $m_0 = 0$ thus we interpolate between an L_p space and the Sobolev space with derivative order $m > s$ for integer m.

The interpolation can be done with the K or J functionals and possibly of more interest the discrete K or J functionals and thus form a norm with instead of the outer integral we can get a countable sum. This norm would be similar to the Littlewood-Paley definitions and we may be able to use this definition for applications assuming we can prove useful bounds like the Bernstein lemmas.

Chapter 6

Differential Difference and Taibleson Poisson Integral Definition

6.1 Differential Difference Definition

We have seen many different definitions so far for Besov spaces each with their own uses. Here we shall introduce some more different equivalent definitions. These definitions are again useful as there is no explicit use of the Fourier transform, so generalisations to different domains can be done.

These definitions we initially defined for small ranges of $s \in I \subset \mathbb{R}$, where s is the differentiability of the space. Then we modified and generalised to include any $s \in \mathbb{R}$ yet this causes the definitions to get more complicated.

These definitions can be formulated into a form similar to the interpolation space definition yet with a different functional replacing K. Further there are properties similar to the Littlewood-Paley definition when looking at the Fourier transform of the functional.

In Peetre [1976] the Besov spaces can be defined under a generalisation of the operator $\tau_h f(x) = f(x+h) - f(x)$ and $s \in (0,1)$. This is

$$B^s_{p,q} = \left\{ f : f \in L_p \text{ and } \left(\int_{\mathbb{R}^d} \left(\frac{\|\tau_h f\|_{L_p}}{|h|^s} \right)^q \frac{dh}{|h|^d} \right)^{\frac{1}{q}} < \infty \right\}$$

and in the case $q = \infty$, one has

$$\sup \frac{\|\tau_h f\|_{L_p}}{|h|^s} < \infty,$$

with the obvious norm.

We can see that inside the integral we have the form of a differential quotient yet instead of bounding this under a supremum norm to get some Hölder spaces we are looking at a more generalised L_p norm.

We can generalise this for larger s firstly by differentiating out and then checking if the following derivative of the function is in the $0 < s < 1$ case. For example

$$B^s_{p,q} = \left\{ f : f \in W^k_p \text{ and } D_\alpha f \in B^{s-k}_{p,q}(|\alpha| \leq k) \right\}$$

or this can be generlised by defining the kth difference operator

$$\tau^k_h f(x) = \sum_{j=0}^{k} (-1)^k \left(\binom{k}{j} f(x + hj) \right)$$

so have the space for $0 < s < k$ as

$$B^s_{p,q} = \left\{ f : f \in L_p \text{ and } \left(\int_{\mathbb{R}^d} \left(\frac{\|\tau^k_h f\|_{L_p}}{|h|^s} \right)^q \frac{dh}{|h|^d} \right)^{\frac{1}{q}} < \infty \right\}.$$

This can be written in another way that will help us see the connection between this definition and the interpolation spaces defined earlier.

Definition 6.1.1 (Difference Besov space) *Where $e_j = (0, \cdots, 1, \cdot, 0)$ the jth basis vector. $0 < s < k$*

$$B^s_{p,q} = \left\{ f : f \in L_p \textit{ and } \left(\int_0^\infty \left(\frac{\|\tau^k_{te_j} f\|_{L_p}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \quad j = (1, \cdots, n) \right\} \tag{6.1.1}$$

and in the case $q = \infty$ one has the obvious L_∞ modification.

6.2 Differential Difference in Sub Domain

6.2.1 Finite difference Bounded Domains Norm

For this norm on bounded domains, we are going to generalise the finite difference Besov norm that we defined in the previous section.

This generalisation has two quite natural options:

- Firstly we can just make sure the finite difference stays within the space. Thus we give the finite difference a value of 0 if $[x, x+hej \notin \Omega]$ and then just restrict our L_p norms in the definition to Ω.
- Secondly we can keep our finite difference as we have before but integrate in L_p over Ω_h were $\Omega_h \subset \Omega$ and $\Omega_h = \cap_j^{[s]+1}\{x : x+h \in \Omega\}$.

As both generalisations give the same result here we will concentrate on the first method of generalisation since it seems more natural to keep integrating over the same domain when varying h.

We can generalise the kth difference operator by defining,

$$\tau_{h,\Omega}^k f(x) = \begin{cases} \sum_{j=0}^k (-1)^k \left(\binom{k}{j} f(x+hj)\right) & [x, x+kh] \subseteq \Omega \\ 0 & \text{Otherwise} \end{cases}.$$

Thus we have the norm for $0 < s < k$ as

$$\|\cdot\|_{B_{p,q}^s(\Omega)} = \|f\|_{L_p} + \left(\int_{\mathbb{R}^d} \left(\frac{\|\tau_{h,\Omega}^k f\|_{L_p(\Omega)}}{|h|^s}\right)^q \frac{dh}{|h|^d}\right)^{\frac{1}{q}}.$$

This then simplifies to:

Definition 6.2.1 (Difference Besov norm bounded domain) Where $e_j = (0, \cdots, 1, \cdot, 0)$ the jth basis vector. $0 < s < k$, k integer and $\Omega \subset \mathbb{R}^d$.

$$\|\cdot\|_{B_{p,q}^s(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{j=1}^n \left(\int_0^\infty \left(\frac{\|\tau_{te_j,\Omega}^k f\|_{L_p(\Omega)}}{t^s}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{6.2.1}$$

and in the case $q = \infty$ one has the obvious L_∞ modification.

This norm is now the finite difference norm but restricted to the domain Ω. We can now define Besov spaces with use of this norm similar to the Sobolev case before in Chapter 2 and thus get three different associated spaces for this norm.

This definition and similar definitions of finite differences are used in Frehse and Kassmann [2006], Kamiński [2011] and Buch [2006].

6.3 Taibleson Poisson Definition

This definition is interesting as it is derived from considering $u(x, t)$ the (tempered) solution to the PDE problem below. This again links the solution of PDEs to Besov spaces like the heat kernel does. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = -\Delta u \text{ if } t > 0,$$

$$u = f \text{ if } t = 0,$$

with solution given by the Poisson integral of f

$$u(x,t) = \int_{\mathbb{R}^d} \frac{t}{((x-y)^2+t^2)^{\frac{n+1}{2}}} f(y)\ dy.$$

Then similarly to before for $0 < s < 1$ we can define

$$B^s_{p,q} = \left\{ f : f \in L_p \text{ and } \left(\int_0^\infty \left(\frac{\|t\frac{\partial u}{\partial t}\|_{L_p}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}$$

and we can extend this definition like before.

Then for general positive real number s we obtain the following definition.

Definition 6.3.1 (Poisson Besov space) *For* $0 < s < k$

$$B^s_{p,q} = \left\{ f : f \in L_p \textit{ and } \left(\int_0^\infty \left(\frac{\|t^k\frac{\partial^k u}{\partial t^k}\|_{L_p}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\} \tag{6.3.1}$$

and in the case $q = \infty$ *one has the obvious* L_∞ *modification.*

We could do a similar method to above but instead consider $v = v(x,t)$ the solution to the equation

$$\frac{\partial^2 v}{\partial t^2} = (1-\Delta)v \text{ if } t > 0,$$

$$v = f \text{ if } t = 0$$

instead and we get a similar definition of a Besov space to the one above.

We now want to look at the previous two examples and try to determine a pattern in the definitions and see how this could link into earlier definitions. As discussed in Peetre [1976], we notice the outer integrals are of the same form as the interpolation space norm but with different integrands. So we just have to worry about the integrands. We notice that they are of the form of a translation invariant operator acting on f and therefore can be written as $\phi_t \star f$ where ϕ_t is a test function depending on t of the form $\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{\phi}{t}\right)$. In terms of Fourier transforms $\hat{\phi}_t(\xi) = \hat{\phi}(t\xi)$. This links to the Littlewood-Paley decomposition test functions. In the cases above, the Fourier transform of the integrand becomes

$$\widehat{\tau_{te_j} f}(\xi) = (e^{it\xi_j} - 1)\hat{f}(\xi).$$

Further we have

$$t\widehat{\frac{\partial u}{\partial t}}(\xi) = t|\xi|e^{-t|\xi|}\hat{f}(\xi).$$

Thus we would want a generalised definition to be of the form

$$B^s_{p,q} = \left\{ f : \sum_{\text{finite no. } \phi} \left(\int_0^\infty \left(t^{-s} \|\phi_t \star f\|_{L_p} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\} \tag{6.3.2}$$

with possible restrictions on ϕ and s.

To have a look for restrictions on ϕ we will use the embedding $B^s_{p,1} \hookrightarrow W^s_p \hookrightarrow B^s_{p,\infty}$ from interpolation spaces. We find that $\hat{\phi}$ must vanish in a neighborhood of 0 and ∞. Thus we get the Tauberian character

$$\{t\xi : t > 0\} \cap \{\hat{\phi} \neq 0\} \neq \text{ for each } \xi \neq 0$$

or in a stronger form

$$\text{supp } \hat{\phi} = \{b^{-1} < |\xi| < b\} \text{ with } b > 1.$$

Usually $b = 2$ is chosen and we need a term $\|\Phi \star f\|_{L_p}$ where $\{\hat{\Phi} \neq 0\} = \{|\xi| < 1\}$. We notice that the $\hat{\phi}_t$ for the previous two definitions also share this property of vanishing at zero and infinity. We also notice that they have values such that $|\phi_t| \leq 1$ and this is a further condition in the Littlewood-Paley decomposition so we see the links between these definitions and the previous.

With stronger regularity conditions imposed this leads to the Littlewood-Paley definition as we take $\{\phi_j\} \in \mathcal{S}$ as well. As with this extra condition the Fourier transform will make sense for all $f \in \mathcal{S}'$.

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