Change of Basis Example

Let us consider the linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T((x,y,z)) = (2x, y - z, z)$$

Let $E = \{e_1, e_2, e_3\}$ denote the standard basis of $\mathbb{R}^3$ and let

$$F = \{f_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, f_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$$

be another basis. With respect to $E$, the linear map can be written as the matrix

$$[T]^E_E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

If we wish to express $T$ with respect to our other basis, $F$, we are interested in the mapping $T: (\mathbb{R}^3, F) \to (\mathbb{R}^3, F)$. We can see that this is the composition of the following linear maps:

$id : (\mathbb{R}^3, F) \to (\mathbb{R}^3, E)$ with associated matrix $[id]^F_E$,

$T : (\mathbb{R}^3, E) \to (\mathbb{R}^3, E)$ with associated matrix $[T]^E_E$ and

$id : (\mathbb{R}^3, E) \to (\mathbb{R}^3, F)$ with associated matrix $[id]^E_F$.

Therefore, we can see that the map $T : (\mathbb{R}^3, F) \to (\mathbb{R}^3, F)$ is has the following associated matrix


This is simply because the composition of linear maps corresponds to matrix multiplication of the associated matrices. Note that the matrices $[id]^F_E$ and $[id]^E_F$ are change of basis matrices.

Exercises

(a) Check that $F$ is indeed a basis of $\mathbb{R}^3$

(b) Show that

$$[id]^F_E = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \\ -1/2 & 1 & -3/2 \end{pmatrix}$$

and $[id]^E_F = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(c) Use this to show that

$$[id]^F_E [T]^E_E [id]^E_F = \begin{pmatrix} 2 & -1/2 & 0 \\ 0 & 1 & 0 \\ -1 & -1/2 & 1 \end{pmatrix}$$

(d) Now, write the matrix that represents $T$ with respect to the basis $F$. Do this by writing the images $Tf_i$ with respect to the basis $F$.

(e) Compare (b) and (c)

(f) Use your change of basis matrices to write $v = e_1 + 2e_2 + 3e_3$ in terms of the basis $F$ and $w = f_1 + 2f_2 + 3f_3$ in terms of the basis $E$.

(g) Check (e) by solving the equations $v = \alpha f_1 + \beta f_2 + \gamma f_3$ and $w = af_1 + bf_2 + cf_3$ for the coefficients $\alpha, \beta, \gamma, a, b, c$.

(h) Use what we have done to calculate the matrix representation of $T$ with respect to $E$ in the domain and $F$ in the range. That is, find the matrix representing the linear map $T : (\mathbb{R}^3, E) \to (\mathbb{R}^3, F)$. Do the same for the map $T : (\mathbb{R}^3, F) \to (\mathbb{R}^3, E)$. 

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