Assumed knowledge: MA241 Combinatorics and MA3J2 Combinatorics II.

Content

1. **Random graphs and applications.** The binomial random graph, the First Moment Method, the Probabilistic Method including applications to Ramsey numbers, a disproof of Hajos's conjecture, and high girth and high chromatic number.
2. **Thresholds.** Increasing properties and coupling arguments, thresholds for graph properties, the second moment method, minimum degree thresholds, and sharp thresholds.
3. **The evolution of the random graph up to connectivity.** Appearance of trees, the subcritical phase, Galton-Watson branching processes, the supercritical phase, and the connectivity threshold.
4. **Other random graph and digraph models.** The random graph $G_{n,M}$, the random graph process, directed random graphs, and random regular graphs.
5. **Small subgraphs and chromatic number.** Balanced and unbalanced graphs, thresholds for the appearance of balanced graphs, Janson's inequality, the lower tail of subgraph distributions, and the likely value of $\chi(G(n,p))$.
6. **Long paths and cycles.** Pósa rotation, expansion in random graphs, boosters, edge sprinkling and Hamiltonicity in $G(n,p)$.

Notes

- Course to be run in Term 1, 2024–2025, moderated by Richard Montgomery.
- An extended syllabus follows this proposal.

Resources

- Lecture notes for MA4M8 2023–24 Theory of Random Graphs, and accompanying sources.
- Lecture recordings from MA4M8 2023–24 Theory of Random Graphs.
- Problem sheets and model answers.
- Exams from 2023, 2024 and model answers.
- Support classes: 11am Fridays 25th October, 15th November, 6th December, plus one to be organised for Term 2, and a revision session.

While all aspects of the course not marked non-examinable are examinable, it should be noted that the aims of the course are that, having taken it, you should:

- Be able to follow the arguments in detail of the course throughout the lecture notes.
- Be able to make similar calculations to those in the course.
- Be able to construct similar coupling arguments to those appearing in the course, and explain clearly how they can be used to infer new results, including using conditioning arguments.

——————— 1 ———————

- Know the definition of the binomial random graph $G(n, p)$, a graph property, an isolated vertex, and be able to define what it means for a property to hold with high probability.
- Know Chernoff's bound, and be able to apply it in appropriate situations.
- Know Markov's Lemma, and be able to recognise when the first moment method is useful and be able to apply it: choosing an appropriate random variable X, and proving an upper-bound for $\mathbb{E}X$ which is $o(1)$.
- Be able to use the probabilistic method to prove results in extremal graph theory, including possibly modifying the graph (e.g. by deleting vertices in cycles) to get a good extremal graph after applying the probabilistic method.
- Know Facts 5 and 12 from the course notes, and be able to use them appropriately in calculations.
- Be able to define an independent set of vertices, the independence number $\alpha(G)$ of a graph G and the chromatic number $\chi(G)$ of a graph G.
- When p is fixed, be able to prove a lower bound for the chromatic number of $G(n, p)$ that holds with high probability.

——————— 2 ———————

- Know the definition of a threshold, an increasing property, a sharp threshold and a coarse threshold.
- Be able to prove that increasing the edge probability in $G(n, p)$ does not decrease the probablity $G(n, p)$ lies in a fixed increasing property.
- Be able to prove that every non-trival property has a threshold.
- Be able to use 'Big-O' notation comfortably: $\omega(1)$, $O(\log n)$, ...
- Know Chebyshev's lemma, and be able to recognise when the second moment method is useful and be able to apply it: choosing an appropriate random variable X, bounding $\mathbb{E}X$ and $\mathbb{E}(X^2)$, and showing from this that $\mathbb{P}(X = 0) = o(1)$.
- Be familiar with thresholds for minimum degree properties, and how to prove the values of these thresholds for minimum degree at least k for each fixed k, using the first and second moment method.

——————— 3 ———————

- Know the likely structure of $G(n,p)$ as p increases to $p = \frac{\log n}{n}$, as this passes from the empty graph, to disjoint edges, to increasingly large yet still small trees, to small trees and unicyclic components, to a giant component and small components, to a single connected component, and the values of p around which these transitions happen or relevant thresholds.
- Know the definition of a complex component and unicyclic component and be able to prove no complex components exist with high probability in the subcritical phase, while few vertices are in unicyclic components.
- Know the Breadth First Search Algorithm and be able to use it to analyse the components of $G(n,p)$. Be able to use this, and a coupling argument to show that with high probablity the components of $G(n,p)$ in the subcritical phase will have size $O(\log n)$.
- Be able to define a Galton-Watson branching process defined by X, and know when its extinction probability is < 1 based on the value of $\mathbb{E}X$.
- Be able to use coupling arguments to compare the growth of the component containing a vertex v in the Breadth First Search (BFS) algorithm running on $G(n,p)$ from v to the growth of a Galton-Watson branching process (to either bound below or above the size of the component) in the supercritical phase.
- Be familiar with how to use the BFS algorithm to show that that in the supercritical regime there are with high probability no components with size between $f_c \log n$ and $n^{2/3}$ (in $G(n,p)$ with $p = c/n$, for some f_c), and that there are is at most 1 component larger than $n^{2/3}$ with high probability.
- Be familiar with how to show the size of the giant component is concentrated around its mean in the supercritical phase and therefore give a good bound on its likely size when given the extinction probability of the required Galton-Watson branching process.
- Be able to show that when p is broadly around $\frac{\log n}{n}$, with high probability any two vertices with degree $\varepsilon \log n$ (for some $\varepsilon > 0$) are not neighbours. Be familiar with how to use this and the likely non-existence of medium-sized components to conclude that around $p = \frac{\log n}{n}$, $G(n,p)$ is likely one giant component, and hence give a sharp threshold for connectivity in $G(n,p)$.

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- Know the definition of $G_{n,M}$, the random graph process, directed graphs (digraphs), the binomial random digraph $D(n,p)$, and the random r-regular graph $G_r(n)$.
- Know what it means for a property to hold in almost every random graph process.
- Know a broad outline for showing that in almost every random graph process the first graph with minimum degree at least 1 is connected.
- Be able to couple $G_{n,M}$ and $G(n,p)$ to translate results for increasing graph properties in each direction.
- Be able to prove (using McDiarmid's coupling argument for one bound and a simpler coupling argument for the other) that if we direct the edges of a graph H to get F, then the threshold for a copy of H in $G(n,p)$ is the same as a copy of F in $D(n,p)$.

- Know the 'configuration method' for studying random regular graphs, including why use it and that, for each fixed r the relevant random configuration produces an r-regular (simple) graph with at least some constant probability.
- Know what a loop and a multiple edge is, and be able to prove for fixed r that there are likely to be few loops and multiple edges in $\pi(F)$ for a relevant uniformly at random chosen configuration.
- Be familiar with switching arguments and their use to prove that for each fixed r the relevant random configuration produces an r-regular (simple) graph with at least some constant probability.
- Be able to prove that for each fixed $r \geq 3$ $G_r(n)$ is with high probability connected (assuming the result on the probability a random configuration gives rise to a simple graph).
- Be able to define a perfect matching.

5

- Know what a balanced, unbalanced, and strictly balanced graph is.
- For any fixed balanced graph H, know and be able to prove what its threshold of appearance is in $G(n, p)$.
- Be familiar with Janson's inequality, and, given its statement, be able to apply it bound above the probability that no copy of H appears in $G(n, p)$.
- Be familiar with Janson's inequality, and be able to apply it when given its statement, in particular to show the likely existence of independent sets in large vertex subsets of $G(n, p)$, with p fixed.
- Given the result of Lemma 73 from the course, be able to prove, when $p \in (0, 1)$ is fixed, an upper bound for $\chi(G(n, p))$ that holds with high probability that matches asymptotically the lower bound you are able to prove holds with high probability.

6

- Know the threshold for Hamiltonicity in $G(n, p)$ and be able to related it to the existence of vertices with degree at most 1.
- Understand Pósa rotation, and be able prove Pósa's lemma (Lemma 83) using it.
- Know a typical definition of expansion useful for Hamiltonicity in $G(n, p)$.
- Understand that if $p = \frac{\log n + \log \log n + \omega(1)}{n}$, $G(n, p)$ is with high probability a $(2, \frac{n}{4})$-expander, and a broad outline for proving this.
- Know the definition of a booster, and be able to prove that every $(2, \frac{n}{4})$-expander has many boosters.
- Know a broad outline for using Pósa rotation, boosters, and edge sprinkling to find the sharp threshold for Hamiltonicity in $G(n, p)$.