Beilinson Spectral Sequence in $D^b(\mathbb{P}^n)$

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Derived categories are the natural environments to do computations using homological algebra. In this project we restrict our attention to $D^b(\mathbb{P}^n)$, the bounded derived category of coherent sheaves on $\mathbb{P}^n$. We construct and apply the computational tool Beilinson spectral sequence in $D^b(\mathbb{P}^n)$. We further study some generalisations and some reverse problems regarding our computed examples.

Exceptional collections on $\mathbb{P}^n$

To understand $D^b(\mathbb{P}^n)$, we begin by asking two elementary questions: can we decompose the category and can we compare it with other categories? The answer to both questions is yes and exceptional collections play a crucial role in both answers.

Definition

An object E in a k-linear derived category is exceptional if $\text{Hom}^\bullet(E,E)=k$. A collection of exceptional objects $(E_1,...,E_n)$ is strongly exceptional if $\text{Hom}^\bullet(E_i,E_j)=k$ and $\text{Hom}^s(E_i,E_j)=0$ for all $i<j$ and $s\neq 0$.An Exceptional collection is full if it generates the derived category.

Exceptional collections are good for studying Fano varieties since their derived categories tend to have a lot of exceptional objects. On the other hand derived categories of Calabi-Yau varieties cannot have any exceptional objects.
Given a full exceptional collection $(E_1,...,E_n)$, one can obtain a semi-orthogonal decomposition by taking the i-th subcategory in the decomposition to be $< E_i >$, which is the smallest subcategory closed under homological shifting and taking cones. Moreover, we get the following comparison result: $D^b(X)$ is equivalent to $D^b(mod-A)$ where A is the endomorphism ring of $\bigoplus_{i=1,...,n}E_i$.

Theorem

***The collection of** $\mathcal{O}(-n),...,\mathcal{O}$ **is a full strong exceptional collection in** $D^b(\mathbb{P}^n)$**. In other words,** $<\mathcal{O}(-n),...,\mathcal{O}> = D^b(\mathbb{P}^n)$.*

Practically, this tells us that any complex of coherent sheaves on $\mathbb{P}^n$ is quasi-isomorphic to a direct summand of a complex that is the cones and shifting of $\mathcal{O}(-n),...,\mathcal{O}$. However, the theorem does not tell us how to explicitly find such a resolution. For that, the Beilinson spectral sequence serves as a useful tool.

Beilinson Spectral Sequence

Consider the Fourier–Mukai transform $\Phi_{\mathcal{O}_\Delta}$ where $\mathcal{O}_\Delta$ is the sheaf of the diagonal in $\mathbb{P}^n\times\mathbb{P}^n$ sending $\mathscr{F}^\bullet$ to $\mathbf{R}q_*(\mathbf{L}p^*\mathscr{F}^\bullet\otimes^{\mathbf{L}}\mathcal{O}_\Delta)$ where q and p are the standard projections $\mathbb{P}^n\times\mathbb{P}^n\to\mathbb{P}^n$. This is simply the pushforward induced by the identity map. However we have a miracle on $\mathbb{P}^n$ that there exist a free resolution of the sheaf of the diagonal

$$\bigwedge^n(\mathcal{O}(-1)\boxtimes\Omega(1)) \longrightarrow \bigwedge^{n-1}(\mathcal{O}(-1)\boxtimes\Omega(1)) \longrightarrow \dots$$

$$\longrightarrow \mathcal{O}(-1)\boxtimes\Omega(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n} \longrightarrow \mathcal{O}_\Delta.$$

Recall the standard result $\mathbf{R}^s F(A^r)\Rightarrow\mathbf{R}^{s+r}F(A^\bullet)$. We plug in $F=q_*$ and $A^\bullet=\mathbf{L}p^*(\mathscr{F}\otimes^{\mathbf{L}}\mathscr{L}^\bullet)$ where $\mathscr{L}^\bullet$ is the resolution of the diagonal, then we have

$$E_1^{r,s}=H^s(\mathbb{P}^n,\mathscr{F}(r))\otimes\Omega^{-r}(r)\Rightarrow\begin{cases}\mathscr{F} & \text{if } s+r=0\\ 0 & \text{otherwise}\end{cases}$$

$$E_1^{r,s}=H^s(\mathbb{P}^n,\mathscr{F}\otimes^{\mathbf{L}}\Omega^{-r}(-r))\otimes\mathcal{O}(r)\Rightarrow\begin{cases}\mathscr{F} & \text{if } s+r=0\\ 0 & \text{otherwise}\end{cases}$$

which is the Beilinson spectral sequence.

Though resolution of the diagonal is a special feature on $\mathbb{P}^n$, we in fact always have a spectral sequence of Beilinson type as long as there exist a full exceptional collection in $D^b(X)$ for any smooth projective variety X. To state this theorem, we first Notice that the Beilinson spectral sequence express a certain duality between the exceptional collections $\mathcal{O}(-n),\mathcal{O}(-n+1)...,\mathcal{O}$ and $\Omega^n(n),\Omega^{n-1}(n-1),...,\mathcal{O}$. Mutation formalises this idea.

Mutation

Given an exceptional pair (E_1,E_2), the left mutation of E_2 by E_1 is defined using the following distinguished triangle

$$E_2 \longrightarrow L_{E_1}E_2 \longrightarrow \oplus_{n\in\mathbb{Z}}\text{Ext}^n(E_1,E_2)\otimes E_1 \longrightarrow E_2[1].$$

Note that $(E_2,L_{E_1}E_2)$ is again an exceptional pair. More generally, $(E_1,...,E_{i-1},L_{E_i}E_{i+1},E_i,...,E_n)$ is an exceptional collection, and we define the left dual of $(E_1,...,E_n)$ to be $(E_1^\vee,...,E_n^\vee)$ where $E_i^\vee = L_{E_1}...L_{E_{n-i}}E_{n-i+1}$. One can check that $(\Omega^n(n),\Omega^{n-1}(n-1),...,\mathcal{O})$ is indeed the left dual to $(\mathcal{O}(-n),\mathcal{O}(-n+1)...,\mathcal{O})$. Using this notation, Gorodentsev's theorem on generalized Beilinson spectral sequences says that there exist a spectral sequence

$$E_1^{p,q}=\bigoplus_{p+q=n}\text{Ext}^{n+i-1}(E_{n-p}^\vee,\mathscr{G})\otimes F^j(E_{p+1})\Rightarrow F^{p+q}(\mathscr{G})$$

for any $\mathscr{G}\in D^b(X)$ and $F: D^b(X)\to\mathcal{A}$ any covariant cohomological functor.

Example

We now demonstrate the computational value of Beilinson spectral sequence by finding a resolution of ideal sheaf of three points on $\mathbb{P}^2$. We use the dual collections $(\mathcal{O}(-3),\mathcal{O}(-2),\mathcal{O}(-1))$ and $(\mathcal{O},\Omega(2),\mathcal{O}(1))$ and take F to be the functor that takes an object in $D^b(\mathbb{P}^n)$ to its zeroth cohomology sheaf. When 3 points are colinear, computing $\bigoplus_{p+q=n}\text{Ext}^{n+i-1}(E_{n-p}^\vee,\mathscr{I}_{3points})$ gives the E_1-page

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \mathcal{O}^{\oplus 2}(-3) & \longrightarrow & \mathcal{O}^{\oplus 3}(-2) & \xrightarrow{f} & \mathcal{O}(-1) \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}(-1) \end{array}$$

Note that f has to be surjective, then $\ker(f)$ must be $\Omega(-1)$. Then together with the E_2-page we get the resolution

$$0\to\mathcal{O}^{\oplus 2}(-3)\to\Omega(-1)\oplus\mathcal{O}(-1)\to\mathscr{I}_{3points}$$

When 3 points are in general position (non-colinear), similar computation gives us the more trivial resolution

$$0\to\mathcal{O}^{\oplus 2}(-3)\to\mathcal{O}^{\oplus 3}(-2)\to\mathscr{I}_{3points}.$$

The reverse problem

We ask the reverse question: given $0\to\mathcal{O}^{\oplus 2}(-3)\xrightarrow{f}\mathcal{O}^{\oplus 3}(-2)$, what are the conditions on f needed to guarantee that this is a resolution for ideal sheaf of three points?
Let $\wedge^2 f$ be the morphism defined by all the two by two minors of f. A sufficient condition is to require that the degeneracy locus $M_1(f)=\{\wedge^2 f=0\}$ have codimension 2. When this is satisfied, we can show that $\ker(\wedge^2 f)=\text{im}(f)$ and apply Porteous formula. Consider the complex

$$0\to\mathcal{O}^{\oplus 2}(-3)\xrightarrow{f}\mathcal{O}^{\oplus 3}(-2)\xrightarrow{\wedge^2 f}\mathcal{O}\to 0.$$

We can compute the fundamental class $[M_1(f)]$ which, by Porteous formula is equal to the second Chern class

$$c_2(\mathcal{O}^{\oplus 3}(-2)-\mathcal{O}^{\oplus 2}(-3))=\{\frac{c(\mathcal{O}^{\oplus 3}(-2)}{c(\mathcal{O}^{\oplus 2}(-3)}\}_2$$

which works out to be $3\xi^2$. This means that the subscheme $\mathcal{O}_{M_1(f)}$ which is isomorphic to the rightmost cohomology is of codimension two and of length three. Then it must be the skyscraper sheaf of three points and the image of $\wedge^2 f$ must be the ideal sheaf of three points.

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