Geometric Viewpoint on Sturm-Liouville Theory

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Introduction

The focus of this project lies at the intersection of two seemingly distinct fields: ordinary differential equations and Riemannian geometry. While they may appear unrelated at first glance, their connection is deeper than it initially seems.

In many modern physics and engineering systems, particularly when dealing with separable linear partial differential equations, a second order linear ordinary differential equation in a certain form occurs frequently, these are called Sturm-Liouville problems. Understanding the solutions in such a form is crucial in understanding a dynamic system when Sturm-Liouville problems occur. We will embark on an in-depth exploration of Riemannian geometry, uncovering the rich properties provided by the Riemannian metric. By the end, we'll bring everything together and demonstrate how tools from Sturm-Liouville theory can be applied to specific geometric surfaces. For those who are interested, a full documentation of the study can be found here:

Sturm-Liouville Theory

First, we will state the second order linear problem posed in Sturm-Liouville form as follow,

$$\frac{d}{dx}\Big(p(x)\frac{dy}{dx}\Big) + q(x)y = 0, \quad (a < x < b)$$

where the coefficient functions $p(x)$ and $q(x)$ are continuous and real valued in the closed and bounded interval $[a, b]$ and $p(x) \neq 0$ for $a \leq x \leq b$. If $p(x)$ has a continuous derivative the problem can be written in the standard form for a second order linear equation, this is usually assumed to be true in most cases unless it's pointed out otherwise. Even if $p(x)$ doesn't have a continuous derivative, we can still study the existence of solutions by converting it to a linear system in a phase plane.

Following from [3, sect 7], there are several notable properties of the Sturm-Liouville operator worth mentioning, including its symmetry with respect to the inner product on the vector space of integrable functions, the existence of eigenvalues and eigenfunctions proved using the Green's function, and the fact that solutions to the Sturm-Liouville problem can be expanded as an infinite series of eigenfunctions.

If we consider the differential operator $Ly := \frac{d}{dx}\Big(p(x)\frac{dy}{dx}\Big) + q(x)y$ for a class of function $y(x)$ defined in the interval $[a, b]$. We say L is regular if the coefficient functions $p(x)$ and $q(x)$ are real valued and continuous in the $[a, b]$, and $p(x) \neq 0$ for $a \leq x \leq b$. The differential operator being regular is also a property that is usually assumed. Ly makes sense if $y(x)$ has second order derivatives which can be extended to continuous functions in $[a, b]$. For complex valued functions u and v integrable in $[a, b]$ we define

$$\langle u, v\rangle := \int_a^b u(x)\overline{v(x)}dx.$$

Now if u and v are in $C^2[a, b]$ and they both have compact support in (a, b), using integrating by parts twice and the fact $\overline{v'(x)} = (\overline{v(x)})'$, we find $\langle u, Lv\rangle = \langle Lu, v\rangle$. However, without the condition that both u and v have compact support in (a, b), the result would differ as follows.

$$\langle u, Lv\rangle - \langle Lu, v\rangle = p(b)\begin{vmatrix} u(b) & u'(b) \\ v(b) & v'(b)\end{vmatrix} - p(a)\begin{vmatrix} u(a) & u'(a) \\ v(a) & v'(a)\end{vmatrix}.$$

But the symmetry property can be then recovered through a suitably chosen, homogeneous boundary conditions that are satisfied by both u and v.

Using the standard definitions for eigenfunctions and eigenvalues, several important properties arise, such as the fact that the eigenvalues of the operator L, along with the boundary operators U_1 and U_2, can be shown to be real. But proving the existence of the eigenvalues requires a more advanced theoretical framework, specifically complex analysis, which involves the use of Green's functions. As for the Green's function itself, it is derived by constructing a non-zero function that "almost satisfies" the homogeneous problem, which otherwise has no non-zero solution.

Riemannian Geometry

A manifold, in simple terms, can be thought of as a surface that "locally resembles" $\mathbb{R}^n$ with the standard metric. A good introduction to this concept can be found in the first chapter of [6]. Using the definition stated in [1, sect 1.A.1], a subset $M \in \mathbb{R}^{n+k}$ is an n-dimensional sub-manifold of class C^p of $\mathbb{R}^{n+k}$ if, for any $x \in M$, there exists a neighbourhood U of x in $\mathbb{R}^{n+k}$ and a C^p submersion $f : U \to \mathbb{R}^k$ such that $U \cap M = f^{-1}(0)$. U is an open subset of $\mathbb{R}^{n+k}$, so for any $y \in U$, the differential df_y is a linear map:

$$df_y : T_y\mathbb{R}^{n+k} \to T_{f(y)}\mathbb{R}^k$$

where $T_y\mathbb{R}^{n+k} \cong \mathbb{R}^{n+k}$ and $T_{f(y)}\mathbb{R}^k \cong \mathbb{R}^k$. f is a submersion, meaning its differential df_y is surjective at each point $y \in U$, that is the map df_y covers the entire tangent space $T_{f(y)}\mathbb{R}^k$, which is isomorphic to $\mathbb{R}^k$. f being a submersion ensures that the pre-image $f^{-1}(0)$ near x is a smooth n-dimensional sub-manifold of $\mathbb{R}^{n+k}$. The below diagram will give a better understanding of the definition.

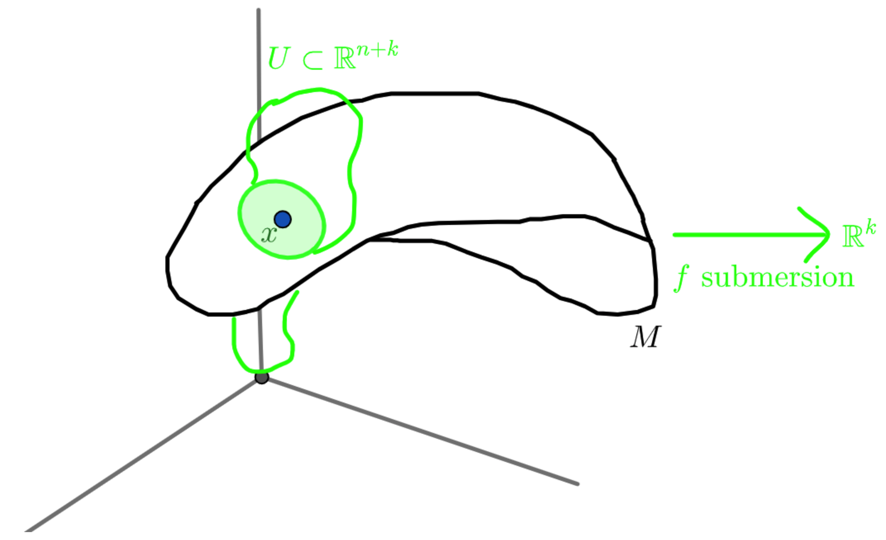

Fig. 1: Big fancy graphic.

Notice that $f^{-1}(0) \cap U = M \cap U$, informally, M is locally the zero set of a submersion into $\mathbb{R}^k$. For a differentiable manifold M, the tangent space T_pM at a point $p \in M$ is the vector space of tangent vectors to the manifold at the point p. Intuitively, we can think of it as describing all possible directions in which one can move from the point p within the manifold. If we limit ourselves to the Euclidean space, we can define the tangent vector to be the derivative of a smooth curve c at origin drawn on the manifold (i.e. $c'(0)$).

The primary goal of this phase of the project is to ultimately establish a well-defined Riemannian metric. We will give the definition from [2, p. 23] here and then discuss the meaning of it.

Definition. A Riemannian metric on a smooth manifold M is a 2-tensor field $g \in \mathcal{T}^2(M)$ that is symmetric (i.e. $g(X, Y) = g(Y, X)$) and positive definite (i.e. $g(X, X) > 0$ if $X \neq 0$). A Riemannian metric thus determines an inner product on each tangent space T_pM, which is typically written $\langle X, Y\rangle := g(X, Y)$ for $X, Y \in T_pM$. A manifold together with a given Riemannian metric is called a Riemannian manifold.

Riemannian metric could be considered as a smooth choice of inner product on each tangent space of a point of M and a $\binom{k}{l}$-tensor field $\mathcal{T}_l^k$ on M is a smooth choice of a $\binom{k}{l}$-tensor on each tangent space of a point of M. A tensor of type $\binom{k}{l}$, also called a k-covariant, l-contra-variant tensor, is a multi-linear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{l \text{ copies}} \times \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

The space of all mixed $\binom{k}{l}$-tensors on V is denoted by $T_l^k(V)$. In standard Euclidean space $\mathbb{R}^n$, the usual dot product gives a Riemannian metric. In fact, every smooth manifold M is associated with a Riemannian metric.

Connection to Sturm-Liouville Operators

Let $M^{p+q-1} \subset \mathbb{R}^p \times \mathbb{R}^q$ be a hyper-surface of $\mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^{p+q}$, we say M is $O(p) \times O(q)$-invariant if

$$M = A \cdot M := \{A\overrightarrow{z} : \overrightarrow{z} \in M\}, \qquad \forall A \in O(p) \times O(q).$$

It follows that M is determined by its intersection with the plane

$$P = \{(x^1, 0, \ldots, 0, y^1, 0, \ldots, 0) : x^1, y^1 \in \mathbb{R}\} \subset \{(\overrightarrow{x}, \overrightarrow{y}) \in \mathbb{R}^p \times \mathbb{R}^q\} = \mathbb{R}^p \times \mathbb{R}^q.$$

Assume additionally that $M \cap P$ is the graph of some function $f : \mathbb{R} \to \mathbb{R}$

$$\text{i.e.} \quad M \cap P = \{(x^1, 0, \ldots, y^1 = (f(x^1), 0, \ldots)) : x^1 \in \mathbb{R}\} = \text{Graph}(f).$$

We can express the induced metric g (first fundamental form) and the second fundamental form A in terms of the function f and its derivatives. We can deduce that M is an minimal surface (i.e. the mean curvature $H = 0$) if and only if

$$\frac{f''}{1 + f'^2} + \frac{p-1}{x}f' - \frac{q-1}{f} = 0.$$

Let M be a hyper-surface as above, assume also that M is minimal (i.e. $H = 0$). Let $u : M \to \mathbb{R}$ be a smooth function on M which is also $O(p) \times O(q)$-invariant

$$\text{i.e.} \quad u(A\overrightarrow{z}) = u(\overrightarrow{z}) \qquad \forall A \in O(p) \times O(q).$$

Then there exists a smooth function $\tilde{u} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{u}(x) = u((x, 0, \ldots, f(x), 0, \ldots)) \qquad \forall x \in \mathbb{R}.$$

Where the following appears in the second variation formula for minimal surfaces,

$$\Delta_M u + |A_M|^2 u,$$

it can then become

$$\frac{\tilde{u}''}{1 + f'^2} + \frac{p-1}{x}\tilde{u}' + \frac{1}{1 + f'^2}\Big[\Big(\frac{f''}{1 + f'^2}\Big)^2 + \frac{f'^2}{x^2}(p-1) + \frac{q-1}{f^2}\Big]\tilde{u}.$$

The above can be expressed in Sturm-Liouville form.

$$\tilde{p}(x)\Big[\frac{d}{dx}\Big(p(x)\frac{du}{dx}\Big) + q(x)u(x)\Big].$$

Motivation for Further Study

Sturm-Liouville problems also occur within the field of geometric flow, more specifically, mean curvature flow. Mean curvature flow is the most natural evolution equation in extrinsic geometry, it arises naturally in the problems where the surface energy is relevant. Similarly, Sturm-Liouville operators appear in the second variation form of the area functional. The main object where the system could be established on are n-dimensional, complete hyper-surfaces immersed in $\mathbb{R}^{n+1}$, that is, pairs (M, φ) where M is an n-dimensional smooth manifold with empty boundary and $\varphi : M \to \mathbb{R}^{n+1}$ is a smooth immersion (the rank of the differential $d\varphi$ is equal to n everywhere on M). The manifold M gets, in a natural way, a metric tensor g turning it into a Riemannian manifold (M, g), by pulling back the standard scalar product of $\mathbb{R}^{n+1}$ with the immersion map φ. Some good references for this continuation of study would be [4] and [5].

References

[1] Sylvestre Gallot, Dominique Hulin, Jacques Lafontaine, et al. *Riemannian geometry*. Vol. 2. Springer, 1990.

[2] John M Lee. *Riemannian manifolds: an introduction to curvature*. Vol. 176. Springer Science & Business Media, 2006.

[3] Robert Magnus. *Essential ordinary differential equations*. Springer, 2023.

[4] Carlo Mantegazza. *Lecture notes on mean curvature flow*. Vol. 290. Springer Science & Business Media, 2011.

[5] Robert Osserman. *A survey of minimal surfaces*. Courier Corporation, 2013.

[6] Michael Spivak. "A comprehensive introduction to differential geometry, Publish or Perish". In: *Inc., Berkeley* 2 (1979).