UNIVERSITY OF WARWICK

Connections, Curvature, and Symmetry:

a Study of G-Invariant Connection and Curvature Geometry on a Frame Bundle over a Symmetric Base Manifold

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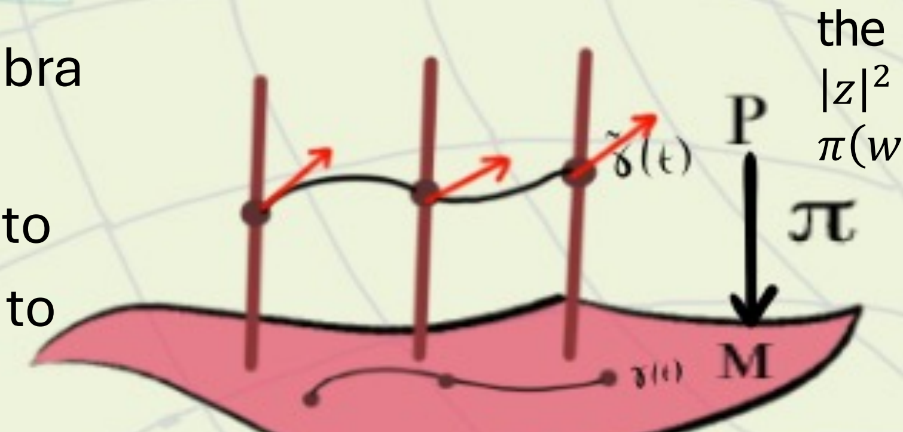
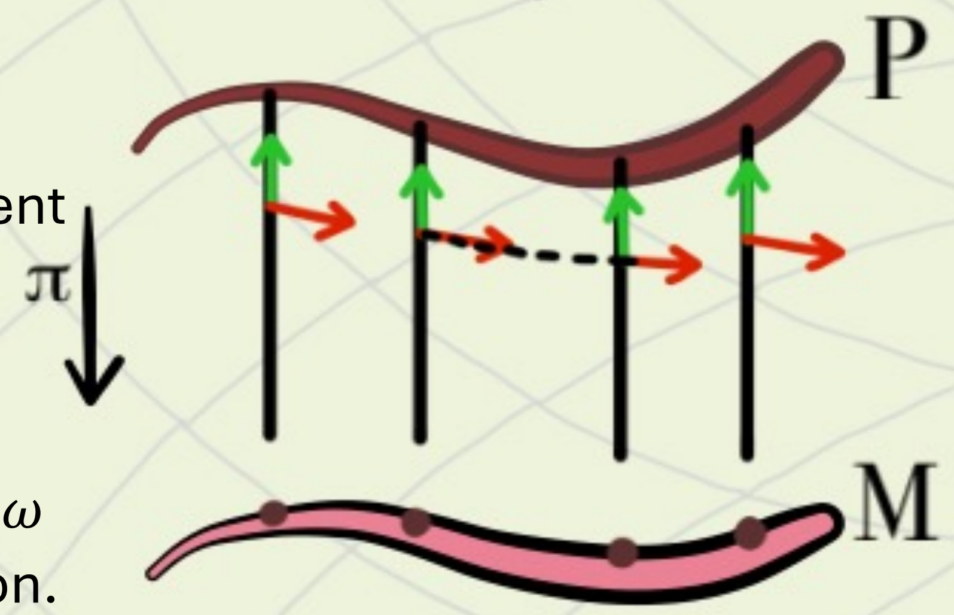
Connections on frame bundles describe how local frames twist as we move across a manifold, with curvature capturing global geometric structure. By imposing symmetry, G -invariant connections arise, linking geometry to group actions.

$$\begin{tikzcd} G \arrow[r] & Fr\,(M) \arrow[d] & GL(n,\mathbb{R}) \arrow[l] \\ & M & \end{tikzcd}$$

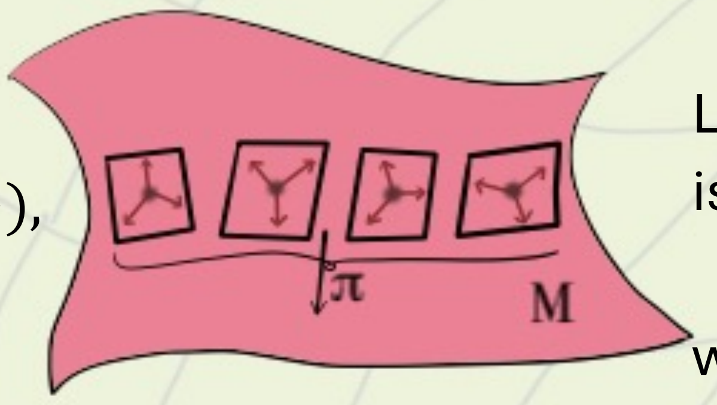
Bundle Geometry



The **curvature** of a connection is a Lie algebra valued 2-form, $F \in \Omega^2(P, \mathfrak{g})$, such that $F = d\omega + \frac{1}{2}[\omega, \omega]$. It measures the obstruction to flatness, or the failure of parallel transport to return to the starting point.



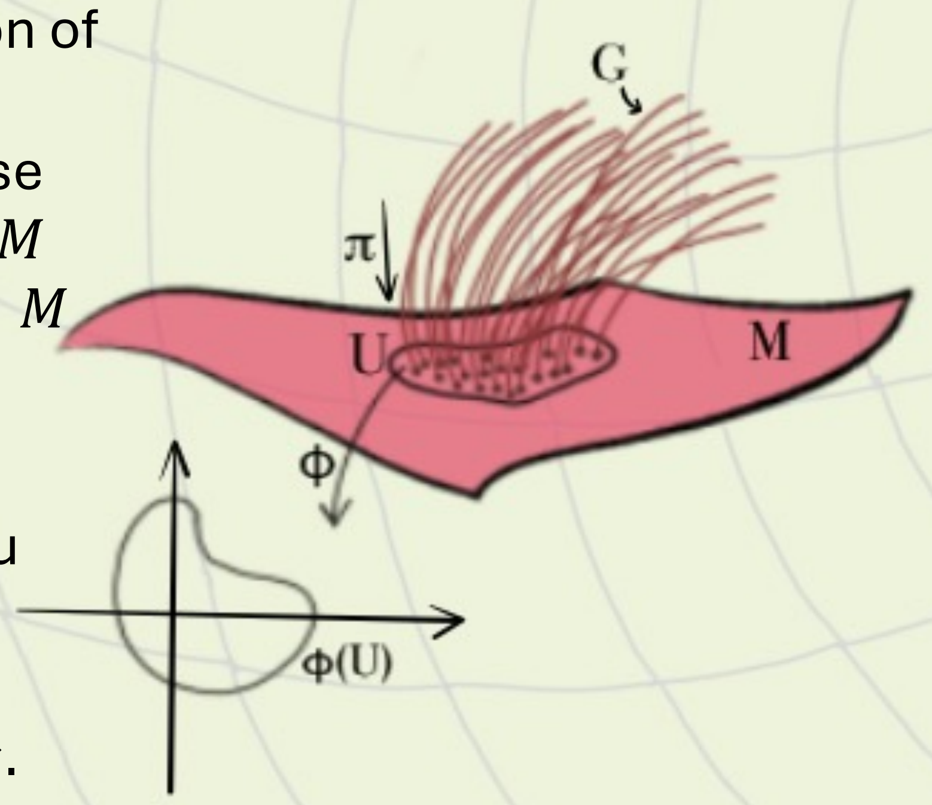
A **connection** is the non-unique prescription of how to move "horizontally" through the bundle, allowing comparison of fibres at different points. Technically, it is a Lie algebra-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ such that 1) $\omega(X^{\#}) = X$ for every fundamental vector field, and 2) $(R_g)^*\omega = Ad(g^{-1})\omega$ for all $g \in G$, where R_g is the right action.



A **frame bundle** is a principal $GL(n, \mathbb{R})$-bundle $FM = \coprod_{x\in M}\{(e_1, \dots, e_n) : (e_1, \dots, e_n) \text{ basis of } T_xM\}$. The group $GL(n, \mathbb{R})$ acts on frames by change of basis:

$$(e_1, \dots, e_n) \cdot A = (\sum_{i=1}^{n} v_i A_{i1}, \dots, \sum_{i=1}^{n} v_i A_{ij}),$$

for any $(v_1, \dots, v_n) \in Fr_{GL}(M)_x, A \in GL(n, \mathbb{R})$. Simply put, it is the space of all coordinate systems attached to each point of M.

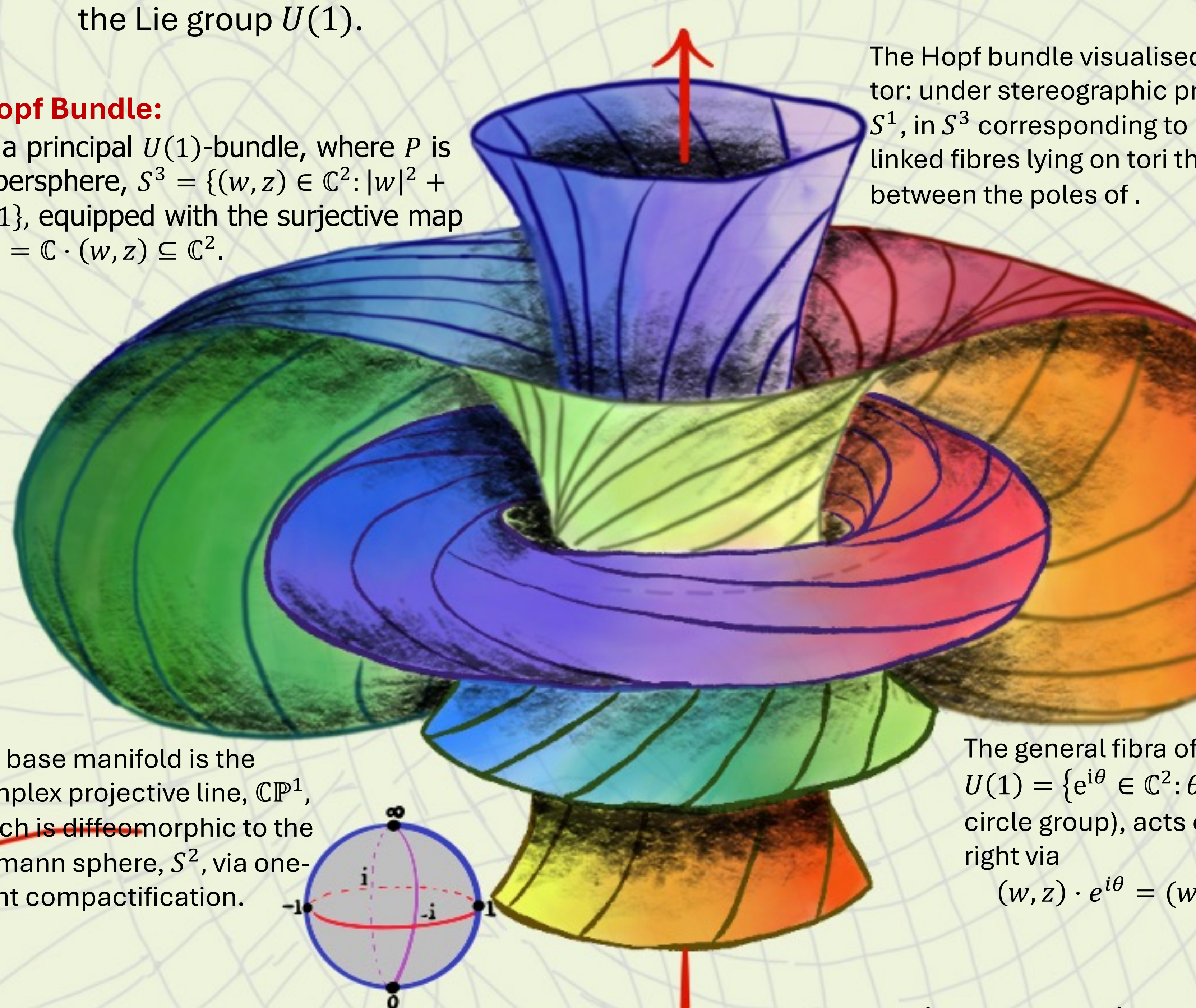


A **principal G-bundle** is a smooth manifold, P, with a free right action of a Lie group G and a smooth surjection $\pi: P \to M$, onto the base manifold, such that for each $x \in M$ there exists a neighbourhood $U \subset M$ with $\pi^{-1}(U) = U \times G$.

On the base is an n-dimensional **manifold**: a space such that if you zoom in enough, it is diffeomorphic to an n-dimensional Euclidean space, $\mathbb{R}^n$.

The Hopf Bundle: a model example

The investigation began by defining the conditions under which a connection and curvature on the frame bundle of the Hopf bundle is invariant under the action of the Lie group $U(1)$.



The Hopf Bundle:
This is a principal $U(1)$-bundle, where P is the hypersphere, $S^3 = \{(w,z) \in \mathbb{C}^2 : |w|^2 + |z|^2 = 1\}$, equipped with the surjective map $\pi(w,z) = \mathbb{C} \cdot (w,z) \subseteq \mathbb{C}^2$.

The Hopf bundle visualised as expanding nested tor: under stereographic projection, the circles, S^1, in S^3 corresponding to points on S^2 form linked fibres lying on tori that grow and shrink between the poles of .

The base manifold is the complex projective line, $\mathbb{CP}^1$, which is diffeomorphic to the Riemann sphere, S^2, via one-point compactification.

The general fibra of group $U(1) = \{e^{i\theta} \in \mathbb{C}^2 : \theta \in \mathbb{R}\}$ (the circle group), acts on S^3 on the right via

$$(w,z) \cdot e^{i\theta} = (we^{i\theta}, ze^{i\theta})$$

$$\begin{tikzcd} U(1) \arrow[r] & Fr(S^3) \arrow[d] & GL(n,\mathbb{R}) \arrow[l] \\ U(1) \arrow[r] & S^3 \arrow[d] & \\ & S^2 \cong \mathbb{CP}^1 & \end{tikzcd}$$

Let $Fr(S^3)$ denote the frame bundle of S^3 whose points are the pairs $(p,(e_1,e_2,e_3))$, for $p \in S^3$, and (e_1,e_2,e_3) is an ordered basis of T_pS^3. For every $\lambda = e^{i\theta} \in U(1)$, we have the map $\tilde{R}_\lambda : Fr(S^3) \to Fr(S^3)$,

$$\tilde{R}_{\lambda(p,(e_1,e_2,e_3))} := (p \cdot z, ((dR_\lambda)_p e_1, (dR_\lambda)_p e_2, (dR_\lambda)_p e_3)),$$

where $(dR_\lambda)_p : T_pS^3 \to T_{p\cdot\lambda}S^3$ is the differential of R_λ at p. In fact, this map $\tilde{R}_\lambda$ defines a right $U(1)$-action on $Fr(S^3)$. This action also commutes with the standard right action of $GL(3,\mathbb{R})$ on frames, so each $\tilde{R}_\lambda$ is a principal bundle automorphism of $Fr(S^3)$ covering $R_\lambda : S^3 \to S^3$.

Keeping to the same notation as above, we can define a $U(1)$-invariant connection 1-form $\omega \in \Omega^1(Fr(S^3), \mathfrak{gl}(3,\mathbb{R}))$ to be such that for any $\lambda \in U(1)$ we have

$$\tilde{R}_\lambda^*\omega = \omega.$$

More geometrically, we can define an Ehresmann connection, a non-unique choice of horizontal distributions, to be $U(1)$-invariant if for every $\lambda \in U(1)$ again, and any $p \in Fr(S^3)$

$$(d\tilde{R}_\lambda)_p(H_p) = \mathrm{H}_{\mathrm{p}\cdot\lambda}.$$

That is this induced right action on the frame bundle preserves the horizontal distribution. A proof these definitions correspond can be followed via the QR code!

Finally, we look at what is means for the curvature 2-form, $F \in \Omega^2(Fr(S^3), \mathfrak{gl}(3,\mathbb{R}))$, to be $U(1)$-invariant. Given a fixed $U(1)$-invariant connection 1-form and any $\lambda \in U(1)$, the associated $U(1)$-invariant curvature 2-form is $(\tilde{R}_\lambda)^* F = d((\tilde{R}_\lambda)^*\omega) + \frac{1}{2}[(\tilde{R}_\lambda)^*\omega, (\tilde{R}_\lambda)^*\omega] = \mathrm{d}\omega + \frac{1}{2}[\omega,\omega] = F$.

General Framework

Using the Hopf analysis, we can generalize these definitions to the following case. Take an n-dimensional manifold M, and let G be a Lie group acting smoothly on M:

$$\Phi: G \times M \to M, \qquad (g,x) \mapsto g \cdot x.$$

The action of G on M lifts naturally to a left action on $Fr(M)$ by the pushforward of frames:

$$\tilde{\Phi}_g(x,(e_1,\dots,e_n)) = (g \cdot x, (d\Phi_g)_x e_1, \dots, (d\Phi_g)_x e_n)),$$

for any $g \in G$. The induced action covers the action on the base space $\pi \circ \tilde{\Phi}_g = \Phi_g \circ \pi$; and commutes with the right $GL(n,\mathbb{R})$-action, $\tilde{\Phi}_g(u \cdot A) = \tilde{\Phi}_g(u) \cdot A$; therefore, the frame bundle of M is G-invariant.

The Ehresmann connection is G-invariant if

$$(d\tilde{\Phi}_g)_p\,(H_g) = H_{\tilde{\Phi}_g(p)} \quad \forall\, g \in G, p \in Fr(M).$$

A connection 1-form $\omega \in \Omega^1(Fr(M), \mathfrak{g})$ is G-invariant if

$$\tilde{\Phi}_g^*\omega = \omega \qquad \forall g \in G.$$

Given that ω is a G-invariant connection 1-form, a G-invariant curvature 2-form $F \in \Omega^2(Fr(M), \mathfrak{g})$ is defined as follows

$$\tilde{\Phi}_g^*F = d(\tilde{\Phi}_g^*\omega) + \frac{1}{2}[\tilde{\Phi}_g^*\omega, \tilde{\Phi}_g^*\omega] = d\omega + \frac{1}{2}[\omega,\omega] = F.$$

Key Insights

This work explored how symmetry can simplify the study of connections and curvature on principal and frame bundles. By considering G-invariant structures, we saw that geometric quantities become easier to describe and compute, as the invariance conditions reduce the freedom in choosing a connection.

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