Introduction To Curve Shortening Flow

Curve Shortening Flow(CSF) is a type of geometric flow (the one-dimensional case of mean curvature flow) that determines the evolution of an immersed curve on a plane through time (We will be focusing on the behaviour of embedded closed curves). Namely, it says that every point on the curve moves along its inward pointing normal at a speed proportional to its curvature at that point. We are going to prove a very powerful theorem pertaining to the long term behaviour of this flow, which is as follows: A smooth, simple & closed curve undergoing curve-shortening flow will

- Remain smoothly embedded without any self-intersections
- Eventually become, and henceforth remain, convex
- All the points of the curve will move "inwards", and the shape of the curve will converge to a circle as the whole curve shrinks to a single point.

This is called **Grayson's Theorem**. The result came into being via contributions made by three individiuals; Michael Gage, Richard S. Hamilton, and Matthew Grayson. Gage was the first to make progress towards this result; he proved convergence to a circle for convex curves that contract to a point. In particular, he utilised isoperimetric ratios to do so. After these two papers, Gage & Hamilton together further proved that all smooth convex curves eventually contract to a point without forming any other singularities. Shortly after, Grayson proved that every non-convex curve will eventually become convex. We will be presenting a much more streamlined proof of the theorem, which was first presented by Ben Andrews & Paul Bryan. Let us start off by first defining CSF.

Definition 0.1. A smooth family of curves $\tilde{\gamma}_\tau = \tilde{X}(S^1, \tau)$ in the plane $\mathbb{R}^2$, with a given initial curve $\tilde{\gamma}_0$ given by an immersion $\tilde{X}_0 : S^1 \to \mathbb{R}^2$ is said to evolve by curve shortening flow if

$$\frac{\partial \tilde{X}}{\partial \tau} = -\tilde{\kappa}\mathbf{N} \tag{1}$$

is satisfied, where $\tilde{\kappa}$ is the curvature of the curve $\tilde{\gamma}_\tau$ and $\mathbf{N}$ is the outward pointing unit normal.

To see CSF in action, we give two curves that satisfy the flow, both of which will be revisited later on.

Grim Reaper: $x = -\log\cos(y), |y| < \frac{\pi}{2}$. This curve moves inwards without changing its shape. In particular, any curve obtained by translating the grim reaper is such that the flow shifts the curve in the direction of the symmetry axis of the curve without changing its shape or orientation. The grim reaper is the only curve with this property.

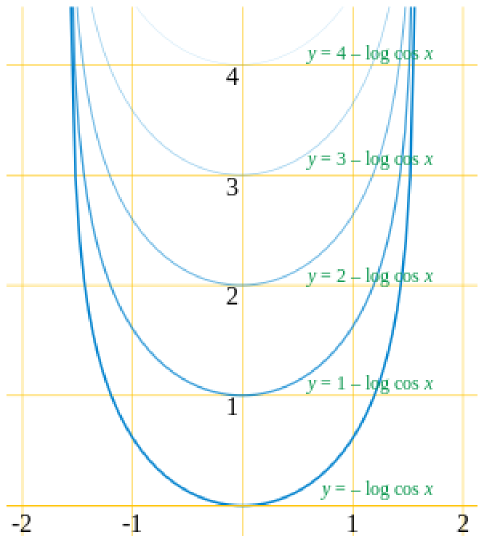

Figure 1: A grim reaper curve on the y axis. Our grim reaper lies on the the x axis instead; this shape but horizontal.

Paperclip: $\cosh(x(t)) = e^{-t}\cos(y(t)), |y| < \frac{\pi}{2}$. This solution contracts to the origin with circular asymptotic shape as $t \to 0$. Furthermore as $t \to -\infty$ the curve becomes an oval consisting of two grim reaper curves joined at the ends. In other words, it converges to the parallel lines $y = \pm\frac{\pi}{2}$, while near the maxima of curvature it is asymptotic to the grim reaper given by $\{x = -t + \log 2 + \log\cos y\}$.

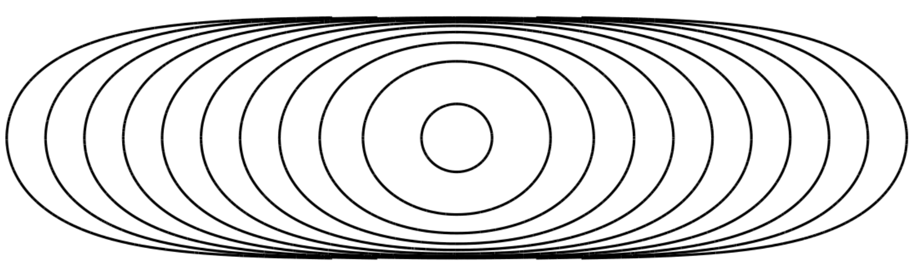

Figure 2: The paperclip solution for range $t < 0$.

Grayson's Theorem Proof Strategy

Henceforth, our study of CSF will be done on a rescaled version which normalizes area; Given a solution $\tilde{X}$ of (1), we define $X : S^1 \times [0, T) \to \mathbb{R}^2$ by

$$X(p,t) = \sqrt{\frac{\pi}{A[\tilde{\gamma_t}]}}\tilde{X}(p,\tau)$$

where $A[\tilde{\gamma_t}]$ is the area enclosed by the curve $\tilde{\gamma_\tau}$, $t = \int_0^\tau \frac{\pi}{A[\tilde{\gamma_{\tau'}}]} d\tau'$ and $T = \int_0^{\tilde{T}} \frac{\pi}{A[\tilde{\gamma_{\tau'}}]} d\tau'$. This results in the rescaled curves $\gamma_t = X(S^1, t)$ having $A[\gamma_t] = \pi$ for all t, and X evolving according to the normailzed equation

$$\frac{\partial X}{\partial t} = X - \kappa \mathbf{N}. \tag{2}$$

We call this area normalized curve shortening flow. Restating **Grayson's theorem**, but under this normalized curver shortening flow, we get that when $\tilde{\gamma_0}$ is an embedded closed curve,

- The evolving curves will eventually become (and stay) convex
- The evolving curves will converge to a circle of area π as $t \to \infty$.

We now define an isoperimetric profile, which will be our main tool to prove the theorem.

Definition 0.2. Let Ω be an open subset of $\mathbb{R}^2$ of area A with smooth boundary curve γ. The isoperimetric profile of Ω is the function $\Psi : (0, A) \to \mathbb{R}^+$ defined by

$$\Psi(\Omega, a) = \inf\{|\partial_\Omega K| : K \subseteq \Omega, |K| = a\}$$

Here $\partial_\Omega K$ denotes the boundary of K as a subset of Ω, which is given by the part of the boundary of K as a subset of $\mathbb{R}^2$ which is not contained in γ. Furthermore if $\partial\Omega$ is compact, then for each $a \in (0, A)$, equality in the infimum is attained for some $K \subseteq \Omega$, so that we have $|K| = a$ and $|\partial_\Omega K| = \Psi(a)$, and in this case $\partial_\Omega K$ consists of circular arcs of some fixed radius meeting γ orthogonally.

Our strategy to prove the theorem is to bound the curvatures of the evolving curves, namely to show that $0 \leq \kappa_{\gamma_t} \leq 1$ is satisfied for all t eventually, by two (time dependent) functions each defined in terms of two seperate model solutions to (2) respectively. One model solution for the lower bound to curvature, and one for the upper. This ensures the curves eventually become convex, and also that as $t \to \infty$ by the behaviour of curve shortening flow we then get convergence to a circle. We now outline in steps how we reach the *upper* bound on the curvature:

1. Establish a relation between the isoperimetric profile of a region and its curvature

2. Determine conditions under which there exists a function that bounds the isoperimetric profiles of evolving curves below at each time step
3. Construct a model region which has known and straight-forward isoperimetric regions
4. Use the model region to construct a function that satisfies the conditions, and therefore bounds the isoperimetric profiles of the evolving curves from below at each time step
5. Utilize the relation to therefore deduce a bound on curvature from the bound on the isoperimetric profiles
6. Provide an explicit model solution to (2) which produces the model region and therefore produce a function that bounds the curvature of our evolving curve.

We will then repeat these steps with a slight variation to deduce the *lower* bound on curvature as well.