University of Warwick

Mathematics Institute

Undergraduate Research Support Scheme

THE n-QUEENS PROBLEM

An Activity Book

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Contents

1 An Introduction

Welcome to 'The n-Queens Activity Book'! Before we start, let me explain a little bit about the booklet you are holding. This is part of the University of Warwick Undergraduate Research Support Scheme (URSS), a programme designed to help students do independent research on a topic they find interesting, and then giving them the opportunity to present said research in a public engagement activity.

My topic consists on the n-queens problem, a mathematical puzzle with almost 200 years of history. With the help of my supervisor, Candy Bowtell, I have designed this activity book with the goal of walking you, the reader, through the history and the mathematics of the puzzle, as well as including some puzzles and exercises so that you can complete proofs yourself! But before we begin, I guess you will have some questions:

What is the n-queens problem?
The n-queens problem actually started as the 8-queens problem, first posed in 1848 by the German chess player Max Bezzel in the Berliner Schahezeitung [1]. Bezzel was interested in ways to place 8 queens on the standard chessboard (an 8×8 grid), so that no two can attack one another, that is, so that no two can lie in the same row, column or either diagonal. This naturally evolved into a generalization of n queens: how can we place n queens in an $n \times n$ chessboard such that none of them are attacking one another? And from there on, mathematicians just kept asking themselves more and more questions: in how many ways can we place the n queens? What if the queens could only move on one diagonal? What if they were placed on a torus instead? How can we find all possible solutions?... and much more.

Why should I care about it?
Even though it started as a recreational puzzle, the n-queens problem has numerous useful applications, and mathematicians are still doing research about it to this day! Just to give a few examples, the n-queens problem has applications in things like traffic control and neural networks, as well as in chemistry when investigating the structure of nucleic acids [1, § 2]. It is also of interest in physics and computer science. As you can see, it has become much more than just a fun pastime.

But I've never heard of it, how am I going to solve any of this?
That's okay! As long as you have a bit of motivation to learn about something new, you'll be fine. Yes, there will be puzzles and some concepts that might be new to you, you don't need to worry, because we have included solutions at the end of the notebook to help you better understand the puz-

zles, as well as hints sprinkled here and there. In addition, we have designed an Appendix where you can find a crash course on any of the topics which are Assumed Knowledge, in case you aren't familiar with them. The rest should all be self-contained, and we encourage you to try everything, who knows what you will learn!

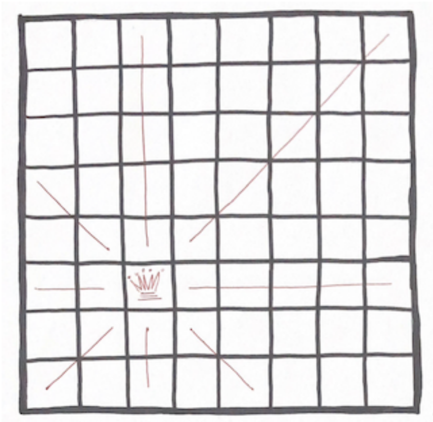
How am I supposed to approach the booklet?
Great question! The structure of this booklet is designed so that the most natural order of things is doing them as they come up. In other words, the best thing to do is try to complete everything in order, so that the build-up of results is more natural. For each chapter there will be a Preliminary Quiz, which is meant to get you in the "maths mood". Afterwards, you will see a list with the Assumed Knowledge for that chapter: if any of the concepts don't sound familiar, you can always check the Appendix to get a crash course!

1.1 Overview of Topics

In this activity book we will introduce and explain the main aspects of some of the variations of the n-queens problem. The n-queens problem originally asked how many ways of placing n non attacking queens in an $n \times n$ chessboard. We will first offer an Introduction to the problem and a set of the basic notions necessary to follow the content of the book. In Section 4 we focus on the original n-queens problem, and ask ourselves if, when and how many solutions exist. In Section 5 we explore the adaptation of the problem to a toroidal environment, and prove several results regarding the conditions for queens and semi-queens (a hybrid between a rook and a queen) solutions to exist. In Section 6 we look at the completion problem and discuss ways of viewing the problem as a graph theory problem. Finally, in Section 7, we ask ourselves what happens when we leave the two-dimensional plane and explore higher dimensions.

2 The 8-Queens Problem

This will be a very small section with the goal that we all get into the chess and math mindset, and everybody can get a good warm up that will prepare them for what is to come. First, we must remind ourselves of how a queen moves. It is the most powerful piece, and it can move any number of squares horizontally, vertically or diagonally. Here is an example:



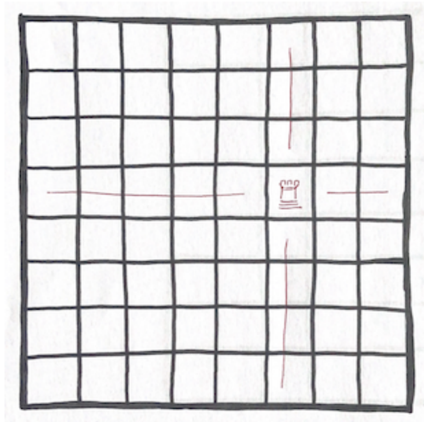


Figure 1: A queen and a rook on the 8×8 chessboard, and all the squares they can respectively reach.

We have very briefly stated what the n-queens problem consists of, but it is interesting to talk about rooks before we dive in deeper into the queens problem, so that we understand how to think about Combinatorics problems. As we know, rooks are chess pieces that can move any number of squares either vertically or horizontally. Suppose for a second that we want to solve the n-rooks problem, that is: is it possible to place n rooks on a $n \times n$ chessboard so that none of them attack each other? It is clear that if we place them all on one of the main diagonals, we have a solution, so there is a solution for all n.

Now, how many rook solutions exist? Well, we have n rooks, and each of them will be in a square (i, j) uniquely determined by its i^{th} row and j^{th} column. Let's start by placing a rook on the first column. Clearly there are n possible squares where we can put it. Now let's check the second column: we can place a rook in $n-1$ squares, since there is one square in the second column attacked by the rook we placed on the first column. Following a similar logic, there are $n-2$ possible squares to place a rook on the third column, and so on... Combining all of these choices gives us a total of

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 = n!$$

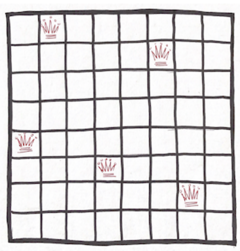
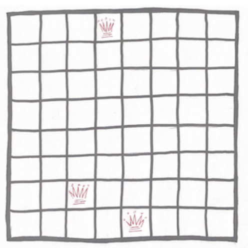
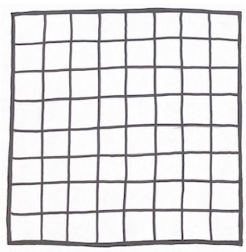
solutions for the n-rooks problem. Now lets get started with queens!

As mentioned before, the n-queens problem actually did not start as that, but as the 8-queens problem instead. A German chess player, Max Bezzel, proposed the puzzle in 1848 in the *Berliner Schachzeitung*, a chess magazine [1]. It seems quite natural that the first puzzle ever was asked for the 8×8 chessboard, since it is the board where actual chess matches are played! It would probably have been a bit weird if Bezzel had been more curious about 5×5 or 17×17 boards.

The reason we ask about 8 queens in the standard chessboard is because we already know that we cannot do any better than that, since we can place at most on queen on each of the rows. Knowing that limit exists, it is natural to ask what is the best we can do. It turns out we can indeed place 8 non-attacking queens in the 8×8 chessboard. However, it is not obvious whether we can place n non-attacking queens in the $n \times n$ chessboard for all values of n, which is where a lot of the interest is born, and what we will see later!

The puzzle caught the attention of many mathematicians, including the ever famous Gauss, who tried to count how many solutions existed (and wasn't quite right with his answer!). Trying to determine whether there was a solution was the first step, and it rapidly evolved into trying to find and count all possible solutions. So let us start getting the hang of it and try to identify solutions ourselves! Here comes our first puzzle:

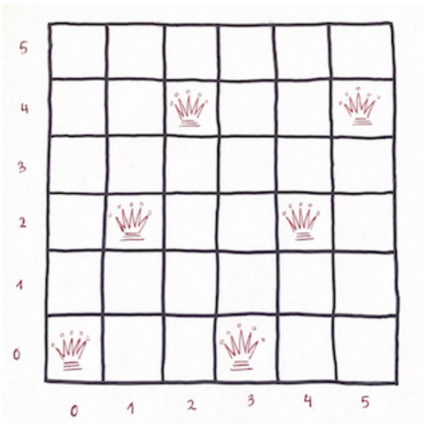
Puzzle I.
(a) *In the partially complete chessboards, place 3 and 5 other queens (respectively) such that you obtain an 8-queens solution.*
(b) *In the empty chessboard on the right, place 8 queens that will form an 8-queens solution. Make sure that it is not one of the solutions we have already seen!*

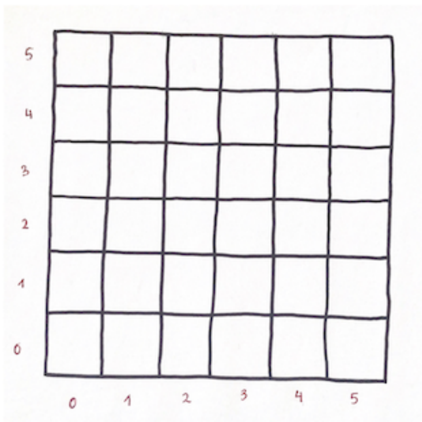




Hopefully now this refresher got us all in the chess mindset and it's a bit more clear what solutions are, and how we ended up with this puzzle to begin with. Up next, we will dive into the math side of things, and we will cover the basics on how to express solutions as functions, how to talk about certain parts of the chessboard, and much more. Turn the page to find out!

3 Key Definitions and Notation

First, let us define a system with which to clearly describe queen placements: a function $f : \{0, \ldots, n-1\} \longrightarrow \{0, \ldots, n-1\}$ taking each value of a column to a certain row, and thus uniquely determining square in the chessboard. For example, the function $f : \{0, \ldots, n-1\} \longrightarrow \{0, \ldots, n-1\}$ defined by $f(i) = 2i \mod n$ would represent the following placement (which is not a solution!), for $n = 6$:





It is always good to keep in mind that the chessboard is symmetric horizontally, vertically and diagonally (when we ignore the colours of the squares, which we don't care about in this booklet), so the same functions described above could also be defined from rows to columns, and the result would be the same, but with a symmetric reflection.

Puzzle II.
In the 6×6 *chessboard above, draw the function we have in the example, that is* $f : \{0, \ldots, n-1\} \longrightarrow \{0, \ldots, n-1\}$ *defined by* $f(i) = 2i$*, but taking instead each value of a row to a column. What type of symmetry do you see?*

☐ Horizontal

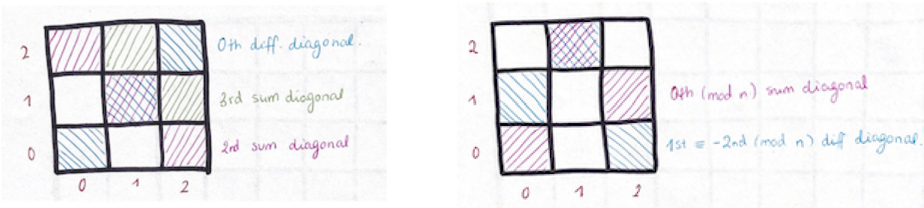
☐ Vertical

☐ Diagonal

Hence, it is easy to see that if f is a permutation, then no two queens (or semi-queens) will be in the same row or column. In other words, we just need to worry about diagonals now. Let us define our two types of diagonals:

Definition 1. We define the two main diagonals of an $n \times n$ chessboard as:

- The ℓ^{th} *sum diagonal* is the diagonal for which each square (i, j) in it satisfies $i + j = \ell$ (or $i + j = \ell \mod n$ in the case of the toroidal board).
- The ℓ^{th} *difference diagonal* is the diagonal for which each square (i, j) in it satisfies $i - j = \ell$ (or $i - j = \ell \mod n$ in the case of the toroidal board).



(a) In the standard chessboard.

(b) In the toroidal chessboard.

Figure 2: Some examples of several sum and difference diagonals in two types of chessboards.

Please note that these are the only key definitions specific to the n-queens problem and the chess setting. There are other key definitions that are more general or lean more towards the mathematical side of things. So don't worry if you see words that aren't familiar! Here is a list of all those other important concepts, whose definitions you can find in the Appendix.

- Types of Functions (injective, surjective, bijective...)
- Modular Arithmetic
- Proofs by Induction and Contrapositive
- Permutations
- Big-O and little-o Notation
- Graph Theory

4 The n-Queens Problem

Let us focus on the problem that gives a title to this notebook: the n-queens problem! Before we begin, there will be a short preliminary quiz, aimed at getting you thinking about the problem, and introducing some of the results we will focus on shortly. No pressure at all, you are not expected to know any of the answers to the questions, but if you do, that's awesome.

It is worth highlighting a bit of notation that will appear in the rest of this chapter. We will denote by $Q(n)$ the number of n-queens placements that exist. Let us try to put this notation to practice in the preliminary quiz!

Preliminary Quiz

1. For what values of n does an n-queens solution exist? In other words, for what values of n does it hold that $Q(n) > 0$?

☐ For all values of n

☐ Only when n is a multiple of 8

☐ For all n except 2 and 3

2. How many 8-queens solutions are there? In other words, what is the value of $Q(8)$? (Note: you're not expected to know this! It is just meant for you to take a guess, and later in the chapter we will give you the answer.)

☐ 12

☐ 76

☐ 92

Assumed Knowledge

- Modular arithmetic

4.1 Does a Solution exist?

Now lets talk about the actual n-queens problem, with no variants in chessboards or pieces whatsoever. In the same way that Bezzel asked back in 1848 if there was a solution to the 8-queens problem, it feels natural to ask the same question when we generalize the problem to n queens.

It's not obvious for which values of n a solution might exist, so let's start by considering some small values.

Puzzle III.
Is there an n-queens solution for $n = 1$? What about $n = 2$? Can you answer this question for the values $n = 3, 4, 5$ as well? ***Hint: What is the best way to prove that a solution exists? And to prove that it doesn't?***

This type of manual proof can be great for smaller values of n, but it would be a bit unfortunate if I asked you to manually show that there is a solution for $n = 21$, right? For that reason, we need to come up with a way of *building* solutions that is more efficient, and wastes less of our time. And that is what we will do in the remainder of this subsection. We will be following a proof carried out by Beasly and Clark separately [9], and presented by Rivin, Vardi and Zimmerman.

Theorem 4.1. *[9, Section 2] For $n = 1$ and for all $n > 3$, it holds that $Q(n) > 0$.*

We will now sketch a proof of this Theorem to show constructions for all values of $n \geq 4$. However, we will leave it to you to convince yourself that these are indeed solutions for every case.

Proof sketch. This proof will be split into three different parts, according to the residue classes of n mod 6. Put very simply, the residue classes of n mod 6 are the remainders that can be left when we divide n by 6. In this case, the residue classes of n mod 6 are $0, 1, 2, 3, 4$ and 5.

Let us start with the residue classes 1 and 5 mod 6, which means the cases where n has one of the following forms: $n = 6m + 1$ or $n = 6m + 5$, respectively, where m is an integer (we are not going to be touching decimals at all!). In this case, it is easy to prove the existence of a solution, because there is a function that works as a solution no matter what square you start in.

PUZZLE IV.
Which of the following functions is a solution valid for the residue classes 1 *and* 5 mod 6*?* ***Hint: try them all in a small chessboard satisfying the residue class conditions, such as*** 5×5 ***or*** 7×7***.***

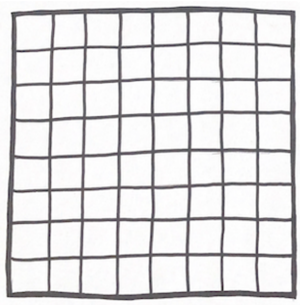
- $f(i) = 4i$
- $f(i) = 3i$
- $f(i) = 2i$

We will now focus on the residue classes 0 and 4 mod 6, that is, cases where $n = 6m$ or $n = 6m + 4$, with m an integer. This one is fun! The following is a very simple construction method that gives a solution. Take an $(n + 1) \times (n + 1)$ chessboard, and construct a solution using the method for the first case, starting to place queens in the bottom left corner (the $(0, 0)$ position). Now simply delete the queen in the $(0, 0)$ position, as well as its row and column, and you end up with a solution. Why does this work? When we construct a solution for the $n+1 \times n+1$ board, we use the method for the residue classes 1 and 5 mod 6, which means that n belongs to either of the 0 or 4 mod 6 residue classes, respectively. Hence, if in the $n + 1 \times n + 1$ chessboard no two queens are attacking one another, taking one queen out will mean that this condition still holds. Try it yourself!

Finally, we have the residue classes 2 and 3 mod n, that is, cases where $n = 6m + 2$ or $n = 6m + 3$, with m an integer. For the case $n = 6m + 2$, we will construct the following solution:

$$f(k) = \begin{cases} 2k + (n-2)/2 \mod n & \text{if } 0 \leq k \leq (n-2)/2 \\ n - 1 - f(n - 1 - k) & \text{if } n/2 \leq k \leq n - 1 \end{cases}$$

PUZZLE V.
Draw an example on the 8×8 *chessboard to check that the function above is in fact a solution.*



This solution, as you might have noticed, has no queens on the main diagonal. This means we can simply add a row and a column to create an $(n+1) \times (n+1)$ chessboard, and then add a queen on the new corner. Since the board went from $n \times n$ for $n = 6m + 2$ to $(n+1) \times (n+1)$, we have just constructed a solution for $n = 6m + 3$. Since we have provided a construction method for every residue class $\mod 6$, we have covered every possible case so we are done!

□

4.2 How many Solutions exist?

Great question, we're not sure! But let me elaborate on that. We only know how many solutions there are up until $n = 27$. Below is a table with the number of *total solutions* and *fundamental solutions* for all n up to 27. Let's quickly define what a fundamental solution is.

Definition 2. The *fundamental solutions* of the n-queens problem are those where the symmetry operations of rotation and reflection are not taken into account. For example, if you have a solution and you reflect it horizontally, both placements are the same fundamental solution.

As you can see, the number of solutions increases very, very rapidly.

n	fundamental	all
1	1	1
2	0	0
3	0	0
4	1	2
5	2	10
6	1	4
7	6	40
8	12	92
9	46	352
10	92	724
11	341	2680
12	1787	14200
13	9233	73712
14	45752	365596
15	285053	2279184
16	1846955	14772512
17	11977939	95815104
18	83263591	666090624
19	621012754	4968057848
20	4878666808	39029188884
21	39333324973	314666222712
22	336376244042	2691008701644
23	3029242658210	24233937684440
24	28439272956934	227514171973736
25	275986683743434	2207893435808352
26	2789712466510289	22317699616364004
27	29363495934315694	234907967154122528

Figure 3: The number of total and fundamental solutions of the n-queens problem for $n = 1$ up to 27.

Hopefully after taking a look at this table it makes sense why we don't yet know the number of solutions for other values of n. Even with a computer, it takes some time to check all of them! Fortunately, there is a very recent result by Michael Simkin [10] which gives us an approximation for the number of solutions, and how this number behaves as $n \to \infty$, that is, as n goes to infinity.

In short, Simkin proved that the number of n-solutions is approximately $(0.143n)^n$. However, if you want to know about it a bit more in-depth, we are going to elaborate on it now. This is the actual theorem which Simkin proved:

Theorem 4.2. *[10, Theorem 1.1] There exists a constant* $1.94 < \alpha < 1.9449$ *such that*

$$\lim_{n\to\infty} \frac{Q(n)^{\frac{1}{n}}}{n} = e^{-\alpha}.$$

We can also write the following string of equivalences:

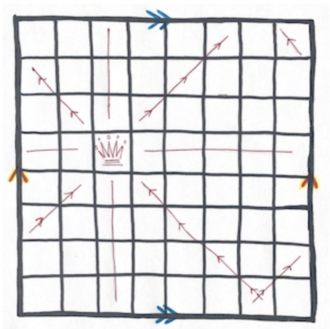
$$\lim_{n\to\infty} \frac{Q(n)^{\frac{1}{n}}}{n} = e^{-\alpha} \Leftrightarrow \frac{Q(n)^{\frac{1}{n}}}{n} = (1 \pm o(1))e^{-\alpha} \Leftrightarrow Q(n) = ((1 \pm o(1))ne^{-\alpha})^n.$$

The second equivalence simply comes from rearranging the expression in order to isolate $Q(n)$. And what does that $o(1)$ exactly mean? Well, it's part of what we call the *Bachmann-Landau notation*, a notation designed by mathematicians with the goal of representing in a concise way how functions behave asymptotically, that is, for extremely large or small values of the variables.

In our case, $o(1)$ simply represents a function that grows more slowly than the constant function 1. So in the first equivalence we are using the little-o notation in order to get rid of the limit. We have $o(1)$, which acts as an approximation to the expression on the other side of the equality, in the same way as limits do. If you want a few more formal definitions about big-O and little-o notation, head to the Appendix and you can find them there!

5 The Toroidal Version

We will now consider the toroidal version of the n-queens problem. There is only one key change from the standard to the toroidal version: instead of placing our queens in a regular chessboard, we will place them in a toroidal chessboard, that is, a chessboard embedded in a torus. We will transform our $n \times n$ chessboard into a torus by identifying opposite sides with one another. It will look something like this:



(a) The blue and yellow arrows represent the same edge.

(b) If you want to think about it in 3D, our queens now live in this doughnut!

Figure 4: Two different ways of drawing the toroidal chessboard. As you can see, queens can reach (and therefore attack) more squares on the toroidal board.

Notice that the diagonals in a toroidal chessboard (unlike in the standard chessboard) all have the same number of squares, n to be precise. This provides the regularity we always find in rows and columns, which also all have n squares each. This ultimately turns the structure of the chessboard more 'symmetrical', making it more natural to think about it mathematically.

Fun fact! Toroidal chessboard and the game of toroidal chess actually exist (along with several other variants of chess), and people do play. Now let's define a torus formally:

Definition 3. A *torus* is a surface defined as the product of 1-spheres. We will only consider the usual torus, which is the surface of a doughnut; it is generated by completely revolving one circle in the three-dimensional space.

It is important to mention some notable mathematicians who have come up with very important results in the toroidal n-queens problem. Among them, we will highlight George Pólya (1887-1985) and Torleiv Kløve (1943-) [8]. We would tell you the key results these two mathematicians developed, but you are actually going to walk through the proofs yourselves, so you will find out later!

As we know from Section 3, we can neatly express queen placements and solutions via a function f such that the coordinates of each queen are given by $(i, f(i))$. Let us expand this to our definition of toroidal solutions:

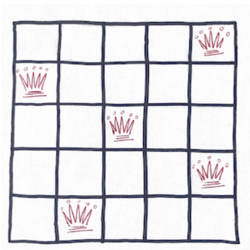
Definition 4. A function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is a *toroidal n-queen solution* if $i \mapsto f(i)$, $i \mapsto f(i) - i$ and $i \mapsto f(i) + i$ are all injective.

If some of the notation seems a bit confusing, check the Appendix for some extra definitions!

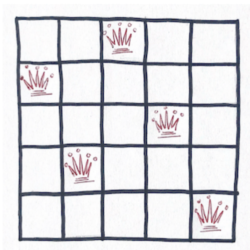
Preliminary Quiz

Let us start with a preliminary quiz to test your mathematical intuition. You are not supposed to know anything about this, so don't be worried if you have to answer everything at random. More than one answer might be correct!

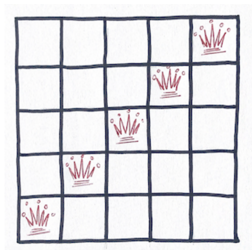
1. Which of the following is an n-queens toroidal solution for n=5?



(a)



(b)



(c)

2. Does a toroidal solution exist when n=4?

- ☐ Yes
- ☐ No

3. What is the relationship between toroidal solutions and standard solutions?

- ☐ Every toroidal solution is a standard queen solution
- ☐ Every standard queen solution is a toroidal solution
- ☐ The set of toroidal and standard queen solutions are the same
- ☐ There is no defined relationship between the two

Assumed Knowledge

- Proof by induction
- Proof by contrapositive
- Permutations
- Modular arithmetic (just the basics)
- Sums of consecutive numbers and consecutive squares

5.1 Toroidal Semi-Queens

Now that we have an idea of what the standard n-queens problem consists of, it is probably intuitive to deduce what the toroidal version will ask. Given an $n \times n$ chessboard, which we transform into a torus by identifying opposite sides, can we place n queens such that no two of them are attacking each other? If so, in how many different ways can we place the n queens? We also call this the modular n-queens problem, since we will be working using modular arithmetic. But first, let us start with an even simpler version of that: let us use semi-queens instead of queens.

Definition 5. A *semi-queen* is an imaginary chess piece that can move along rows, columns and sum diagonals.

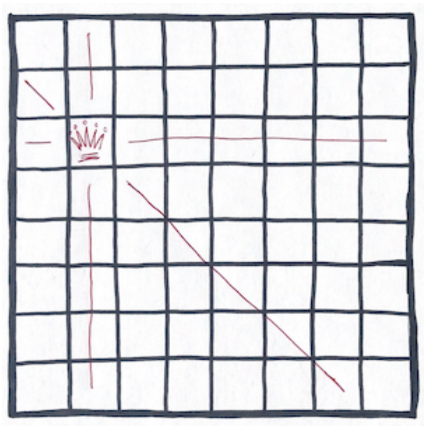


Figure 6: A semi-queen in the 8×8 chessboard and the squares it can reach.

Some of you might be thinking that it is a bit odd to consider and study an imaginary chess piece. However, it bridges the natural gap that exists between rook and queens (especially from a mathematical perspective). Hence, it is important to focus of this piece, since not only does it have interest in itself, but it will also make the transition to queens smoother.

Definition 6. A *toroidal n-semi-queens solution*, denoted $S(n)$, is a placement of n semi-queens in the toroidal (or modular) $n \times n$ chessboard, such that no two are attacking each other.

The first natural question we can ask ourselves is, **for what values of n does a toroidal semi-queens solution exist?** As we have seen in Section 2, if we consider the placements of queens as permutations, we ensure that no two are in the same row or column, so we only need to think about the diagonals. Let us begin the proof:

Proof. Let $\sigma(i)$ be a permutation of $\{0, \ldots, n-1\}$. This implies by definition that

$$\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} \sigma(i).$$

Now since semi-queens can only move along the sum diagonals, and we want no two semi-queens sharing a sum diagonal, we can define the following permutation:

$$\sigma'(i) = \sigma(i) + i$$

The fact that it is a permutation precisely means that no two queens share a sum diagonal either.

Puzzle VI.
Which of the following equalities are true, and which ones are false?

- ☐ (a) For any two permutations of n elements i and $\sigma(i)$, it holds that $\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} \sigma(i)$.

- ☐ (b) Let $\tau(i)$ and $\sigma(i)$ be two permutations of n elements. Then $\sigma(\tau(i)) = \tau(\sigma(i))$.

- ☐ (c) Let $\sigma(i)$ be a permutation of n elements. Then it holds that $\sum_{i=0}^{n-1} \sigma(i) = 2 \sum_{i=0}^{n-1} \sigma(i)$.

- ☐ (d) Let i, $\sigma(i)$ and $\sigma'(i) = \sigma(i) + i$ be permutations of n elements. Then $\sum_{i=0}^{n-1} \sigma'(i) = \sum_{i=0}^{n-1} \sigma(i) + \sum_{i=0}^{n-1} i$.

Spoiler alert! Some of the results above will be used now to continue the proof, so if you haven't attempted Puzzle VI we strongly recommend you do so before you continue reading.

Using some of the information above and the facts about permutations we stated at the beginning of the proof, we can write the following string of equalities:

$$\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} \sigma'(i) = \sum_{i=0}^{n-1} (\sigma(i) + i) = \sum_{i=0}^{n-1} \sigma(i) + \sum_{i=0}^{n-1} i = 2\sum_{i=0}^{n-1} i \mod n$$

But we can actually find an explicit expression for these sums:

Puzzle VII.
Find an explicit expression for the sum $\sum_{i=0}^{n-1} i$. ***Hint: consider the following way of writing the sequence, notice anything?***

$$\begin{array}{ccccccccc} 0 & + & 1 & + & \cdots & + & n-2 & + & n-1 \\ n-1 & + & n-2 & + & \cdots & + & 1 & + & 0 \end{array}$$

So rearranging the equality we get that

$$\frac{(n-1)n}{2} = \sum_{i=0}^{n-1} i \equiv 0 \bmod n$$

We know that $\frac{(n-1)n}{2} \equiv 0 \bmod n$ holds because the left hand-side contains n as a multiple! This implies that $\frac{n-1}{2}$ is an integer.

Puzzle VIII.
What would that imply for n?

☐ n is even

☐ n is odd

And just like that, our proof is complete [1, pp. 13–14] [9, pp. 632–633]! ☐

Let us now clearly state the theorem which we have just proved:

Theorem 5.1. *If n is even, then no modular n-semi-queens solutions exist.*

It is also worth saying that when n is odd, then a solution always exists. For example, $f(i) = -2i \mod n$ always works in this case. Recall that we need $f(i)$ and $i + f(i)$ to be permutations. In this case, $i + f(i) = i + (-2i) = -i \mod n$ is a permutation, and so is $f(i)$ because n is odd (try to convince yourself of this).

Here is a fun fact about the toroidal semi-queens problem. The problem of counting n-solutions for the toroidal semi-queens problems is actually equivalent to finding the number of additive triples of bijections $\{1, \ldots, n\} \to \mathbb{Z}/n\mathbb{Z}$. And what does this mean, exactly? Well, it's the number of pairs of bijections $\pi_1, \pi_2 : \{1, \ldots, n\} \to \mathbb{Z}/n\mathbb{Z}$ such that the pointwise sum $\pi_1 + \pi_2$ is also a bijection [5]. If this seems a bit confusing, check the Appendix for some extra definitions. They prove the following result:

Theorem 5.2. *[5, Theorem 1.2] Let n be an odd integer. Then*

$$s_n = (e^{-1/2} + o(1))n!^3/n^{n-1}$$

where $s_n/n!$ is the number of toroidal n-semi-queens solutions.

5.2 Toroidal Queens

We now know that there is never a toroidal n-semi-queens solution when n is even. What about full toroidal n-queens solutions? Similarly as before, the most natural questions to ask first are if any solutions exist, and if so how many.

In order to see when solutions exist, we will follow a proof by Kløve. It will be very similar to the one for the semi-queens, but since queens can move in more directions, it will be ever so slightly more challenging. In this case, we will state the theorem first. Note that this is also a result due to Pólya:

Theorem 5.3. *[8, Theorem 1] The modular n-queen problem has a solution if and only if $gdc(n, 6) = 1$.*

Proof. [8] Since we are going to prove a double implication, what we actually have to do is prove two different implications. We will first prove that if there exists a solution, then $gcd(n, 6) = 1$.

First, let us do a brief proof by induction:

Puzzle IX.
Use proof by induction to show that

$$\sum_{i=0}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6}$$

Now assume f is a solution, and define

$$S \equiv \sum_{i=0}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6} \bmod n \quad \text{and} \quad T \equiv \sum_{i=0}^{n-1} if(i) \bmod n$$

By definition $f(i)$ is injective (since it is a solution). But this means that we have an injective function of n elements in $\mod n$, which actually implies that $f(i)$ is bijective. We can thus write

$$\sum_{i=0}^{n-1} f(i)^2 \equiv S \bmod n$$

Recall that, again by definition, both $f(i)+i$ and $f(i)-i$ are also injective. So we will use a little trick and work with the squares of these expressions:

$$S \equiv \sum_{i=0}^{n-1} (f(i)+i)^2 = \sum_{i=0}^{n-1} f(i)^2 + 2\sum_{i=0}^{n-1} if(i) + \sum_{i=0}^{n-1} i^2 \equiv 2S + 2T \mod n$$

Puzzle X.
After seeing the example above, expand the following expression to deduce something about S in terms of S and T:

$$S \equiv \sum_{i=0}^{n-1} (f(i) - i)^2$$

Adding the two equalities together we obtain $2S \equiv 4S \mod n$, or, in other words, $0 \equiv 2S \mod n$.

Once we substitute in the formula for S we obtain the following:

$$\frac{(n-1)n(2n-1)}{3} \equiv 0 \mod n$$

Hence, we can deduce that $\frac{(n-1)(2n-1)}{3}$ is an integer. Now we will use proof by contrapositive to reach the conclusion that n is not, in fact, divisible by 3.

Puzzle XI.
Assume n is a multiple of 3. Then it is clear that $(n-1)$ is not divisible by 3, and similarly neither is $(2n-1)$, can you explain why?

We proved for the semi-queens that n must be odd (since it can't be even). This implies that this condition also holds for standard queens, since all of the movements a semi-queen can make, a queen can also make. All the reasoning we used when proving that n can't be even when finding toroidal n-semi-queens solutions, also applies to toroidal n-queens solutions. Now we have just shown that n is not divisible by 3, we can conclude that

$gcd(n, 6) = 1$.

Now we need to prove the other implication. Assume that $gcd(n, 6) = 1$.

Puzzle XII.
Find a function for a solution that works for all n such that $gcd(n, 6) = 1$, and write it down. You don't need to prove that the solution works for all such n, but it might be good to check for low values of n. ***Hint: think about the other chessboard pieces, you might find inspiration for a very useful movement!***

$\square$

6 The Completion Problem and Graph Theory

If you have got this far, I think it's fair to assume that you have completed part if not all of the content in the notebook prior to this chapter. And if my assumption is correct, then chances are you solved *Puzzle I*, where you had to come up with some solutions and complete others. This was no coincidence, as the so called **Completion Problem** is an area of the n-queens problem that is currently being researched. To understand it, we will introduce some new concepts.

Definition 7. A *partial n-queens configuration* is a placement of k queens (where $k < n$) on a $n \times n$ chessboard.

Following from this definition the completion problem asks the natural question: given a partial n-queens configuration, is it possible to add queens in order to obtain an n-queens solution? In order to consider this problem, we will reformulate it as a colouring problem in the field of graph theory. Let us start by defining what a graph is.

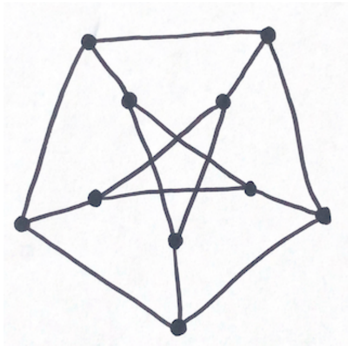
Definition 8. A *graph* is a pair $G = (V, E)$, where V is a set whose elements are called *vertices*, and E is a set of paired vertices, whose elements are called *edges*.

The vertices x and y of an edge $\{x, y\}$ are called the *endpoints* of the edge. The edge is said to *join* x and y and to be *incident* to x and y. A vertex may belong to no edge (*isolated vertex*). Sometimes, graphs are allowed to contain *loops*, which are edges that join a vertex to itself. We will not need to consider loops in this booklet.

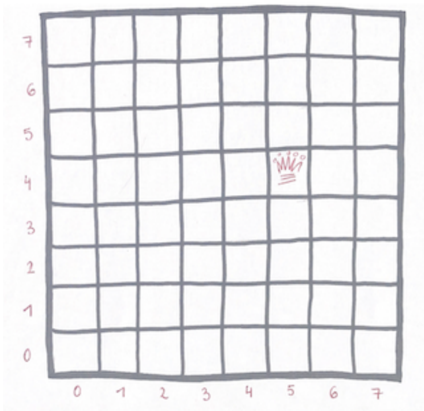
Figure 7: Three examples of graphs.

Preliminary Quiz

1. Can you colour the edges of the following graph such that no two edges of the same colour are incident to the same vertex? What is the least amount of colours you can use?



2. In the following chessboard, can you define the queen not by the row and column where it lies, but by the diagonals?



Assumed Knowledge

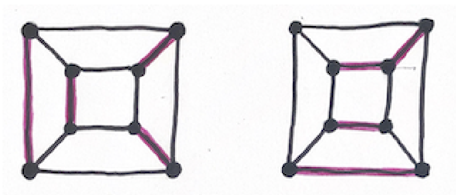
- Nothing new!

6.1 A Translation into Graph Theory

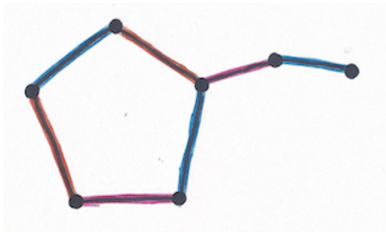
This subsection will be dedicated to explain how we can reformulate the n-queens problem into a graph theory problem, but first we need to become familiar with some new concepts that will be key to our exploration of the Completion Problem later on:

Definition 9. We define a *matching* in a graph G as a set of independent edges. In other words, a matching is a set of edges such that no two of them are incident to the same vertex.

Definition 10. An *edge colouring* of a graph G is an assignment of colours to the edges such that no two edges of the same colour are incident to the same vertex.



(a) The first is an example of a matching, the second isn't!



(b) An edge colouring with 3 colours.

Figure 8: In example (a), the highlighted pink edges are those we are selecting, and which belong to the 'matching' and 'no matching'.

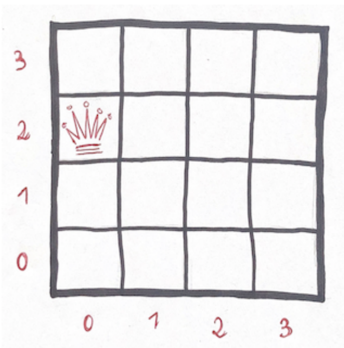
Something important to note is that the matching in picture (a) is actually a *perfect matching*, meaning its edges are incident to every vertex in the graph. With this clearer idea of some key terms in graph theory, it's time to describe how we translate the chessboard we all know and love into a graph.

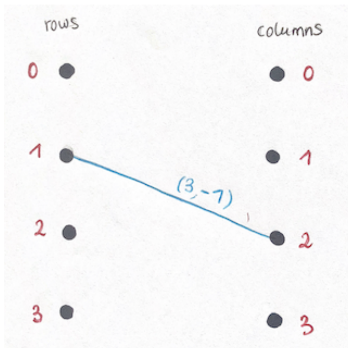
Let us construct our graph. As we know, the $n \times n$ chessboard has n rows and n columns, and these uniquely define a square in the chessboard. So we will form a graph with two vertex classes each of size n, one for the rows and one for the columns. Each edge between a "row vertex" and a 'column vertex" will represent a square in the chessboard, and we will have a total of n^2 edges. The resulting graph is a *complete bipartite graph*, meaning its vertices can be divided into two disjoint and independent sets A and B, that is, every edge connects a vertex in A to one in B. It is complete because we have drawn every possible edge satisfying these conditions.

In addition, as we know, the sum and difference diagonals also define a square, and in fact uniquely do so in the standard board (things can be a bit different in the toroidal environment). How can we include this information in the graph? Well, since each of the edges represents a square of the chessboard, we can assign a vector of two entries to each edge, where each entry indicates which sum diagonal and which difference diagonal the square belongs to [6, p. 3]. The point of also encoding this information is that this way we can ensure to find an n-queens solutions, since focusing only on rows and columns with the information in the paragraph above would only give us n-rooks placements.

Here is an example of the construction of the graph:

PUZZLE XIII.
Look at the chessboard and the graph below.
(a) *Highlight the square corresponding to the blue edge (whose vector is also written).*
(b) *In the graph on the left, draw the edge corresponding to the square with a queen inside, and write the vector corresponding to the diagonal to which the queen belongs.*





6.2 Linking it to Queens Solutions

Once we have completely understood our "mathematical translation", we can study queens solutions in the graph. As we know, in a solution any pair of queens are in different rows and columns.

PUZZLE XIV.
What does this mean in the queens graph? In other words, what does no two queens lying in the same row or column on the $n \times n$ chessboard correspond to in the complete bipartite graph with n vertices in each part?
Hint: We introduced new concepts in the last subsection for a reason!

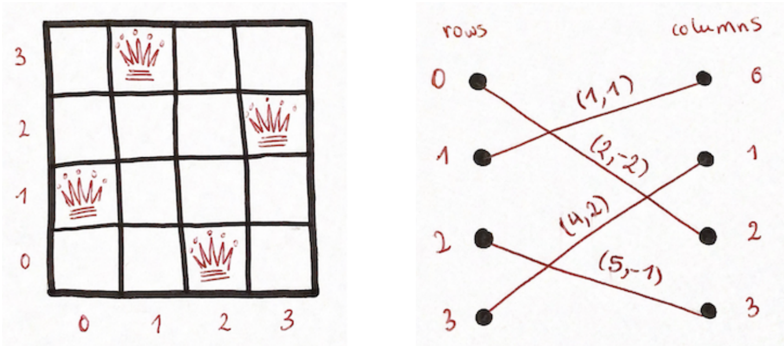


Figure 9: Example of a 4-queens solution and its corresponding graph (only the edges corresponding to squares with queens in them are shown for the sake of clarity).

Similarly, no two queens can be in the same diagonal. This means that if we look at the edges in the graph corresponding to the squares where queens live, no two of them will have vectors with the same entries, both the first and the second entry will be different (since they can't share either sum or difference diagonal). One can imagine the entries of the vector as being colours (each k^{th} sum diagonal has a colour, and each k^{th} difference diagonal has a colour), so every vector will be of two different colours. Note that the order here matters. For example, we can simultaneously have $(1,3)$ and $(3,1)$, since this doesn't imply two squares in the same diagonal. Hence, we are looking for all the *sum colours* to be rainbow, and all the *difference colours* to be rainbow, without needing the set of all sum and difference colours to be rainbow. An n-queens solution then translates into a *rainbow matching*.

Definition 11. A *rainbow matching* in a graph G is a matching such that each edge in the matching is of a different colour (or, in our case, a different vector of colours).

This way of expressing queens solutions (and the problem in general) might seem a bit random, but it is actually incredibly useful for some versions of the problem. Recall how, when dealing with the toroidal version, it was very natural to work with permutations and it allowed us to obtain results in a very concise way?

Well, in this case, mathematicians have a lot of tools and understanding of this abstract concept of matchings, which can be applied to the n-queens problem but can also model so many other different problems (apart from being of interest by themselves). Hence why we care so much about it! Thanks to this translation, mathematicians have been able to find bounds for the so called *n-queens completion threshold* $qc(n)$, which is the maximum integer with the property that any partial n-queens configurations of size at most $qc(n)$ can be completed into an n-queens solution.

For example, Glock, Munhá Correia and Sudakov have done research in this area and actually proved the following.

Theorem 6.1. *[6, Theorem 1.2] For all sufficiently large n, we have $n/60 \leq qc(n) \leq n/4$.*

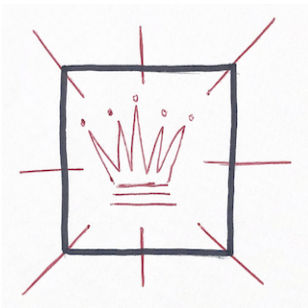
This means that if you place $n/60$ queens on the $n \times n$ chessboard, then someone else can always complete the chessboard with the $59n/60$ remaining queens and construct a solution (for a large enough n). In addition, there are examples where, if more than $n/4$ queens are placed, then there is no way to add the remaining queens. So we know that the threshold lies between those values, but we don't know its exact value (yet).

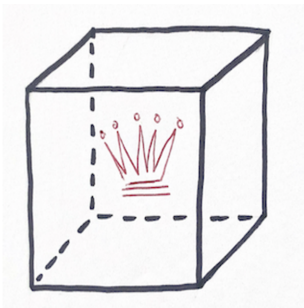
7 Latin Queen Squares and Upper Dimensions

In the previous pages in this booklet, we have been working with all types of variants of the original problem: we have changed the chess pieces and talked about rooks and semi-queens, we have answered questions about the existence of solutions, about the number of solutions, we have altered the chessboard to discover other shapes (topological variants) and worked on those problems as well... However, we have always stayed in the 2-dimensional space, and it's time to change that and see what happens when we explore different dimensions. But first, a little preliminary quiz!

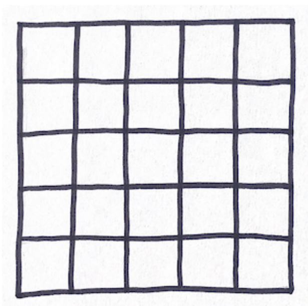
Preliminary Quiz

1. As we know, the standard 2-dimensional queen can move in 8 different directions. In how many directions will a 3-dimensional queen be able to move? ***Hint: the picture below should help you out!***





2. In the following board, fill up the squares with the numbers $1, 2, 3, 4$ *and* 5 *in a way such that the same number is not repeated in any row or column.*



Assumed Knowledge

- Nothing new!

7.1 A 3-Dimensional Excursion

So, as we have mentioned before, in this section we will touch upon a higher dimensional version of the n-queens problem. Since we like to go step by step, natural to start with 3-dimensions. Let us think then of an $n \times n \times n$ cube, or in other words, n $n \times n$ chessboards stacked on top of each other. As with all different variants of the problem, there is a first intuitive question to ask: is it possible to place n^2 queens in the $n \times n \times n$ cube (that is, n queens in each chessboard) such that no two of them attack each other?

You might have noticed that oftentimes in math we like to reformulate problems in order for them to be more digestible. For example, we used permutations when talking about the toroidal version and we worked with graph theory when the completion problem arose. Here we will do a similar thing, and use a new concept called a *Latin queen square* in order to more easily work with the 3-dimensional version of the n-queens problem. Let us provide a proper definition, in this case by van Rees [11, pp. 267–268].

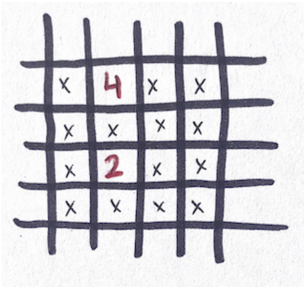
Definition 12. A *Latin queen square* of order n is an arrangement of the n elements $\{1, 2, 3, \ldots, n\}$ in an $n \times n$ array satisfying that:

1. Each element occurs at most once in each row, column or diagonal (where a diagonal is either a sum or a difference diagonal)
2. Any elements i and j (where $i \neq j$) are not placed a distance $|i - j|$ apart in any row, column or diagonal (where again both sum and difference diagonals are valid)

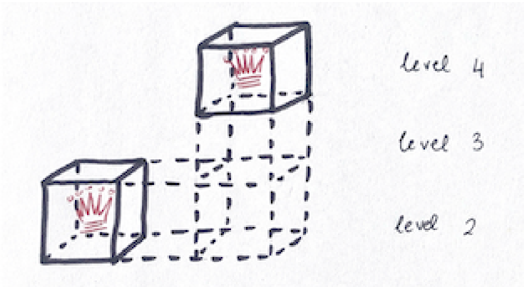
The way to interpret the above is that each number from 1 to n describes the 'level' where the queen is located: there are n queens on level 1, n queens on level 2, and so on... Lets explore why this definition works. We need to check what both properties mean, and why that is enough for us to be certain that our definition is accurate.

Property 1 is quite straightforward: in each level, no two queens must be in the same row, column or diagonal. This is simply the property we have been asking of queen solutions in 2 dimensions all this time.

Moving on to Property 2 we need a bit more of work to do a proper interpretation. The most simple way is probably an example, so lets do that. Let $i = 2$ and $j = 4$, and suppose that they are placed distance $|2 - 4| = 2$ apart on a column. If we look at the picture below we see that this actually corresponds to the queens being in an attacking position, but on different levels, which is precisely the n-queens cube solution condition that we haven't covered with Property 1. We can find similar examples for the cases where the symbols are distance $|i-j|$ apart in a row or diagonal.



(a) Seen on a latin queen square.



(b) Seen on the $n \times n \times n$ cube.

Figure 10: Two different ways of visualizing the example mentioned above.

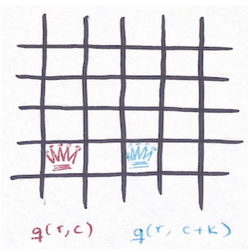
7.2 Any Solutions? How many?

As it is usually the case with math, there are several ways of defining something. Here, we are going to define a *Latin queen square* again, but this time with a definition due to Klarner [7]:

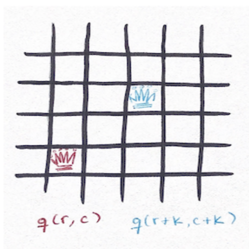
Definition 13. A *Latin queen square* is a square matrix $Q = [q(r,c) : 1 \leq r, c \leq n]$ whose entries belong to $\{1, 2, 3, \ldots, n\}$ and satisfy that for all k with $1 \leq k \leq n-1$, it holds that:

1. $q(r+k, c) - q(r,c) \neq 0, \pm k$
2. $q(r, c+k) - q(r,c) \neq 0, \pm k$
3. $q(r+k, c+k) - q(r,c) \neq 0, \pm k$

(a) Property 1



(b) Property 2



(c) Property 3

In this case, the three conditions mean exactly the same as in the first definition, but their distribution is different. For example, Property 1 includes two conditions: that the expression given is different from 0, and that it is different from $\pm k$. The expression given represents two entries of the Latin queen square which are in the same column. If the distance between these entries is 0, then they are the same entry and the expression doesn't really make sense. If the distance between the entries is k, we are in exactly the same situation as with Property 2 in the previous definition! So we have two queens on different levels attacking each other. Similarly, Property 2 means the same but for two entries in the same row, and Property 3 does so for two entries on the same diagonal.

And how can Latin queen squares help us understand the n-queens cube problem? Well, based on the definitions given, we can state a theorem that explains it very clearly:

Theorem 7.1. *[11, Theorem 1] The existence of a Latin queen square of order n is equivalent to the placing of n^2 non-attacking three-dimensional queens into an $n \times n \times n$ cube.*

So we have two ways to think about the n-queens problem in 3-dimensions. As in all of the previous versions of the problem, the starting point is to even just ask for which values of n a solution exists. Klarner [7] tells us *some* values for which a solution exists:

Theorem 7.2. *[11, Theorem 2.6] If $gcd(n, 210) = 1$ then n^2 queens can be placed in the $n \times n \times n$ cube such that no two of them are attacking each other (i.e. a Latin queen square of order n exists).*

In other words, if all the prime divisors of n are greater than 7 (if they aren't $2, 3, 5$ or 7), then there exists a solution for the n-queens problem in the $n \times n \times n$ cube. The proof for this result is not too complicated, but it won't be included in this booklet. This theorem, however, gives us no information for the values n such that $gcd(n, 210) > 1$, and mathematicians do not know whether solutions exist for any of these values!

In fact, van Rees [11] conjectured that there are no Latin queen squares of order n where $gcd(n, 210) > 1$. However, Vasquez and Habet [13] conceived an algorithm to find superimposable solutions for the $n \times n$ chessboard and used it to prove that there are n superimposable solutions for $n = 15, 16, 18, 20, 21, 22, 24, 28$ and 32. Even with all this information, mathematicians haven't yet been able to categorize all cases for which there exists (or doesn't) a Latin queen square of order n, and there is still research being done about it!

8 Answers

The n-Queens Problem Preliminary Quiz

1.
This is actually a trick question (sorry)! In a way, there are two answers that are correct. There are a total of 92 solutions, and a total of 12 *fundamental solutions*. What is the difference? A *fundamental solution* is one that doesn't take into account symmetries or rotations, meaning that if you find a solution and decide to rotate it or reflect it in any way, it will still be the same fundamental solution. Fun fact! 76 is actually the number of solutions that Gauss himself thought existed, which would later be proved wrong (nobody's perfect) [3].

2.
The correct answer is that there exists at least one solution for all n except 2 and 3. This is proven in the notebook, so we won't give a detailed explanation here.

The Toroidal Version Preliminary Quiz

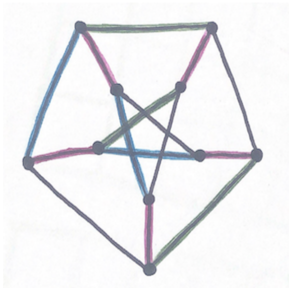
1.
Only placement (b) is a toroidal n-queens solution, since there are two queens in the same column in (a), and all of them are in the same diagonal in (c). However, what you might have not noticed is that (c) is actually a toroidal n-semi-queens solution, which, if you've gotten this far, you already know what it is!

2.
A toroidal solutions does **not** exist for $n = 4$, as proved by Theorem 5.2 in the notebook.

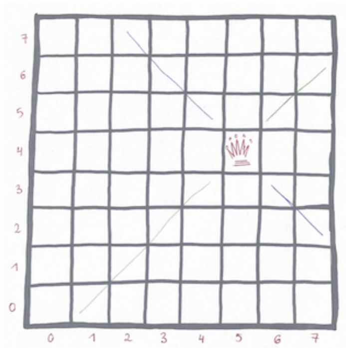
3.
Every toroidal solution is a standard queen solution. The reason behind this is that we only have toroidal solutions for n such that $gcd(n, 6) = 1$ (n is not a multiple of 2 or 3), but we have standard solutions for all n but 2 and 3. The way a queen can attack in the toroidal board includes all the ways a queen can attack in the standard board, plus some extra attacks! So a placement of non-attacking queens in the toroidal board is still a placement of non-attacking queens in the standard board.

The Completion Problem and Graph Theory Preliminary Quiz

1.
There are several ways of colouring the edges of the graph, below you can find an example. As we can see, the minimum number of colours we need is 4 (which is interesting in terms of optimization, even though we could choose one individual colour for each of the edges).



2.
As we can see, the queen is in the position $(4, 5)$, that is the 4th row and 5th column. Hence, the sum diagonal will be the $4+5 = 9th$ sum diagonal, and the difference diagonal will be the $4-5 = -1st$ difference diagonal.



Latin Queen Squares and Upper Dimensions Preliminary Quiz

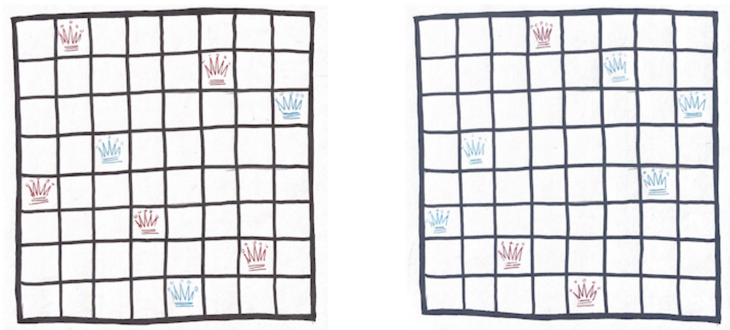
1.
Let's count! The queen can go through either of the 6 faces, 12 edges or 8 corners of the cube, which make up a total of 26 directions.

2.
There are many solutions for this one! Here is one example:

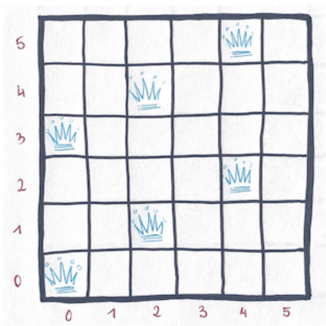
1	5	3	4	2
5	3	4	2	1
3	4	2	1	5
4	2	1	5	3
2	1	5	3	4

Solutions to all Puzzles

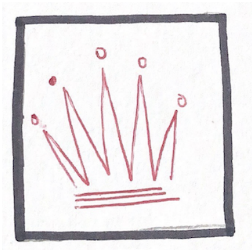
Solution to Puzzle I.
(a) Look at the picture below, the new queens are in a different colour.
(b) There are many solutions for this one! Since it would be a bit bulky to include all of them here, try to check yourself manually or look it up online!

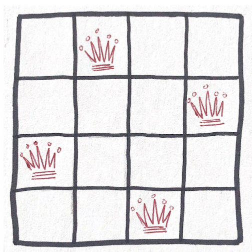


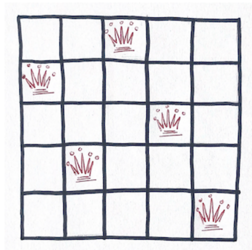
Solution to Puzzle II.
The drawing bit is quite straightforward: you can just calculate each value individually by plugging $0, 1, 2, 3, 4$ and 5 into the function and it will give you the second coordinate of each queen. Once we superimpose both placements, it's easy to see that we have a **diagonal** symmetry.



Solution to Puzzle III.
There exist queen solutions when $n = 1, 4$ and 5, below you have some examples. Clearly, when we place a queen in the 2×2 chessboard, it attack all other squares, so it's impossible to add a second queen. Similarly, no matter where we place a queen in the 3×3 chessboard (corner, edge or center), it only allows to place at most one other queen (except if we place the first queen in the center, a case where all other squares are attacked), so we cannot place 3 queens to achieve a solution.

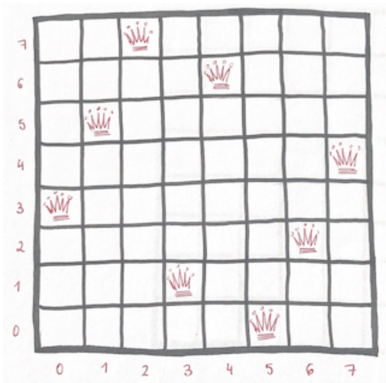






Solution to Puzzle IV.
The correct function is $f(i) = 2i$, which coincidentally is equivalent to all the queens being a 'knight's move' apart!

Solution to Puzzle V.
Find the correct placement below.



Solution to Puzzle VI.

☐ (a) **True.** Since the permutations are just rearrangements of the values $(1, 2, \ldots, n-1, n)$, no matter the order in which we add $1 + 2 + 3 + \cdots + (n-1) + n$, by commutativity the final value will be the same.

☐ (b) **False.** Here is a counterexample. Define $\tau(i)$ by taking $(1, 2, 3, 4) \mapsto (2, 4, 3, 1)$, and $\sigma(i)$ by taking $(1, 2, 3, 4) \mapsto (3, 4, 1, 2)$. Then $\sigma(\tau(i))$ is defined by $(1, 2, 3, 4) \mapsto (4, 2, 1, 3)$, but $\tau(\sigma(i))$ is defined by $(1, 2, 3, 4) \mapsto (3, 1, 2, 4)$.

☐ (c) **False.** (This contains spoilers for Puzzle VII) Take for example the *identity permutation* $\sigma(i)$, which takes $(1, 2, \ldots, n) \mapsto (1, 2, \ldots, n)$. Then

$$\sum_{i=0}^{n-1} \sigma(i) = \frac{n(n-1)}{2} \neq n(n-1) = 2\frac{n(n-1)}{2} = 2\sum_{i=0}^{n-1} \sigma(i)$$

☐ (d) **True.** This is more easily seem by writing

$$\sum_{i=0}^{n-1} \sigma'(i) = \sum_{i=0}^{n-1} (\sigma(i) + i) = \sum_{i=0}^{n-1} \sigma(i) + \sum_{i=0}^{n-1} i$$

Solution to Puzzle VII.
Let's take a look at the hint given:

$$\begin{array}{ccccccccc} 0 & + & 1 & + & \cdots & + & n-2 & + & n-1 \\ n-1 & + & n-2 & + & \cdots & + & 1 & + & 0 \end{array}$$

We want to find the value of the sum of either of the rows (since they are the same but in opposite order). When placing one row on top of the other, we see we can add $0+(n-1)$, $1+(n-2)$, and so on until $(n-1)+0$. Each sum of pairs adds up to $n-1$, and since each row has n elements, we have n pairs that add up to $n-1$. So the total value of summing up both expressions is $n(n-1)$. Hence, since each row adds up to the same value, and we want the value of one of the row, we can conclude that

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$

Solution to Puzzle VIII.
We know that $\frac{n-1}{2}$ is an integer, so we deduce that $n-1$ is even (so it's divisible by 2). This implies that n is **odd**.

Solution to Puzzle IX.
As with any proof by induction, we first check that the base case holds, so let $n = 1$. Then

$$\sum_{i=0}^{n-1=0} i^2 = 0^2 = 0$$

so the base case holds. Now let $n = k+1$ and let's assume that the equality

$$\sum_{i=0}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6}$$

is true for all $n \leq k$. Then we have the following:

$$\sum_{i=0}^{(k+1)-1} i^2 = \sum_{i=0}^{k} i^2 = \sum_{i=0}^{k-1} i^2 + k^2 =$$

$$\frac{(k-1)k(2k-1)}{6} + k^2 = \frac{(k^2-k)(2k-1)+6k^2}{6} =$$

$$\frac{2k^3-k^2-2k^2+k+6k^2}{6} = \frac{2k^3+3k^2+k}{6} =$$

$$\frac{k(2k^2+3k+1)}{6} = \frac{k(k+1)(2k+1)}{6} =$$

$$\frac{((k+1)-1)(k+1)(2(k+1)-1)}{6} = \frac{(n-1)n(2n-1)}{6}$$

where in the last equality we have used that $n = k+1$. This is our desired result, and the proof is complete!

Solution to Puzzle X.
The complete string of equalities is the following:

$$S \equiv \sum_{i=0}^{n-1}(f(i)-i)^2 = \sum_{i=0}^{n-1} f(i)^2 - 2\sum_{i=0}^{n-1} if(i) + \sum_{i=0}^{n-1} i^2 \equiv 2S - 2T \mod n$$

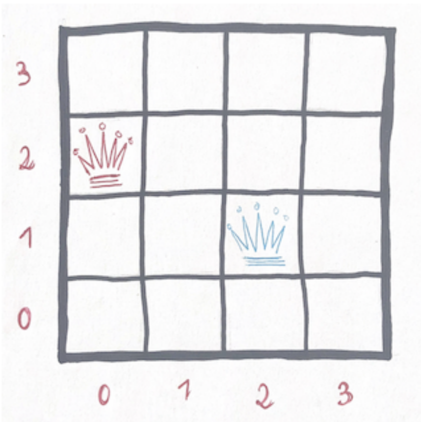
Solution to Puzzle XI.
Take a look at the following sequence:

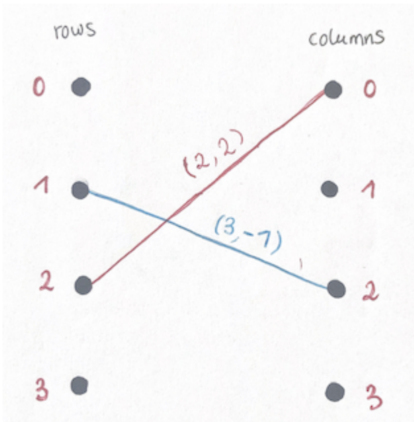
$$\mathbf{0}, 1, 2, \mathbf{3}, 4, 5, \mathbf{6}, 7, 8, \mathbf{9}, 10, 11, \mathbf{12}, 13, \ldots$$

All the multiples of 3 are in bold, and we can see that they are all distance 3 apart. Hence, if n is a multiple of 3, it is impossible for $n-1$ to be a multiple of 3. Similarly, n being a multiple of 3 implies that $2n$ is a multiple of 3, so it can't be that $2n-1$ is a multiple of 3.

Solution to Puzzle XII.
The correct answer is $f(i) = 2i$. Surprise! We have already encountered this function, which corresponds to all the queens being a 'knight's move' apart. If it doesn't look too familiar, check *Puzzle IV*, where we see in our proof of the existence of standard solutions that $f(i) = 2i$ gives a solution precisely for $n = 1$ or 5 mod 6.

Solution to Puzzle XIII.
Check the solution below:





Solution to Puzzle XIV.
Recall that each queen in a solution is in a different row and a different column, so when placing a solution, all the columns and rows have exactly one queen. Hence, in the graph, once we highlight all the edges corresponding to the squares where there is a queen, the edges will each join a unique pair of vertices from each side of the graph. In other words, we will get a *matching*, and more specifically a *perfect matching*, since no two edges are incident to the same vertex (no two queens are in the same row or column).

9 Appendix

9.1 Types of Functions

Definition 14. Let $f : X \to Y$ be a function. Then f is *injective* or *one-to-one* if distinct elements of X are mapped to distinct elements of Y. That is, if x_1 and x_2 are in X such that $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. This is equivalent to saying if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Definition 15. Let $f : X \to Y$ be a function. Then f is *surjective* or *onto* if very element of Y is the image of at least one element of X. That is, $image(f) = Y$. That is, for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Definition 16. Let $f : X \to Y$ be a function. Then f is *bijective* if it is injective and surjective; that is, every element $y \in Y$ is the image of exactly one element $x \in X$.

9.2 Modular Arithmetic

Modular arithmetic is a side of mathematics that is concerned with the remainders that arise when we divide one number by another. When we divide two integers, we will have an equation like this:

$$A/B = Q \text{ remainder } R$$

where A is the *dividend*, B is the *divisor*, Q is the *quotient* and R is the *remainder*. In this case, since we are only concerned with the remainders, we will use an operator called the *modulo operator*, usually written as $\mod$. In the example above, we could write:

$$A \equiv R \mod B$$

which means that the remainder that we get when we divide both A and R by B is the same, so in *modulo n*, they are equivalent. Lets see a couple examples with with actual numbers:

$$7/3 = 2 \text{ remainder } 1 \Rightarrow 7 \equiv 1 \mod 3$$
$$21/9 = 2 \text{ remainder } 3 \text{ and } 12/9 = 1 \text{ remainder } 3$$

so we can write $21 \equiv 12 \mod 3$.

A very important concept in modular arithmetic is that of the group $\mathbb{Z}/n\mathbb{Z}$. This is formally referred to as the *ring of integers modulo n*, and it is the set of all congruence classes of the integers for a modulus n. In other words, it is the set of all the possible remainders we can obtain when we divide an integer by n, that is $\{0, 1, 2, \ldots, n-2, n-1\}$.

9.3 Proofs by Induction and by Contrapositive

Definition 17 (*Proof by Induction*)**.** *Mathematical induction* is a method for proving that a statement $P(n)$ is true for every natural number n, that is, that the infinitely many cases $P(0), P(1), P(2), P(3), \ldots$ all hold.

A *proof by induction* consists of two cases. The first is the *base case*, which proves the statement for $n = 0$ without assuming any knowledge of other cases. The second case is the *induction step*, which proves that *if* the statement holds for any given case $n = k$, *then* it must also hold for the next case $n = k + 1$. These two steps establish that the statement holds for every natural number n. Oftentimes, the base case does not begin with $n = 0$, but with $n = 1$ instead.

Definition 18 (*Proof by Contrapositive*)**.** In logic, the *contrapositive* of a conditional statement is formed by negating both terms and reversing the direction of inference. In other words, the contrapositive of the statement "if X, then Y" is "if not Y, then not X".

So in a *proof by contrapositive* of a statement "if X, then Y", what is actually proved is "if not Y, then not X". This is a valid method because both statements are logically equivalent, that is, they mean the same thing.

9.4 Permutations

Definition 19. A *permutation of a set* is an arrangement of its members into a sequence or linear order, or if the set is already ordered, a rearrangement of its elements.

This concept is very easy to understand with an example. For example, let's see all the permutations of the set $\{1, 2, 3\}$:

$$123, 132, 213, 231, 312, 321$$

9.5 Big-O and little-o Notation

Definition 20 (*Little-o Notation*)**.** Let $f(n)$ and $g(n)$ be functions of n. We say that $f(n) = o(g(n))$ (as $n \to \infty$) if for every $\alpha > 0$, there exists $n_0 > 0$ such that $|f(n)| \le \alpha |g(n)|$ for every $n \ge n_0$.

Definition 21 (*Big-O Notation*)**.** Let $f(n)$ and $g(n)$ be functions of n. We say that $f(n) = O(g(n))$ (as $n \to \infty$) if there exists $M, n_0 > 0$ such that $|f(n)| \le M|g(n)|$ for every $n \ge n_0$.

If any of these definitions doesn't satisfy you or doesn't work for you, feel free to look on the Internet in order to find something that suits you better!

I would recommend *Khan Academy*, *Wolfram Alpha* and *Brilliant.org*.

In addition, if you want some more formal sources, the following are some books and papers that can also be useful if you want to dive into any of the concepts we have seen deeper: [4], [2] and [12].

References

[1] Jordan Bell and Brett Stevens. A survey of known results and research areas for n-queens. *Discrete mathematics*, 309(1):1–31, 2009.

[2] Candida Bowtell and Peter Keevash. The n-queens problem. *arXiv preprint arXiv:2109.08083*, 2021.

[3] Paul J Campbell. Gauss and the eight queens problem: A study in miniature of the propagation of historical error. *Historia mathematica*, 4(4):397–404, 1977.

[4] R. Diestel. *Graph Theory*. Graduate Texts in Mathematics. Springer Berlin Heidelberg, 2010. ISBN 9783642142789.

[5] Sean Eberhard, Freddie Manners, and Rudi Mrazović. Additive triples of bijections, or the toroidal semiqueens problem. *Journal of the European Mathematical Society (EMS Publishing)*, 21(2), 2019.

[6] Stefan Glock, David Munhá Correia, and Benny Sudakov. The n-queens completion problem. *Research in the Mathematical Sciences*, 9(3):41, 2022.

[7] David A Klarner. Queen squares. *J. Recreational Math*, 12(3):177–178, 1979.

[8] Torleiv Kløve. The modular n-queen problem. *Discrete Mathematics*, 19(3):289–291, 1977.

[9] Igor Rivin, Ilan Vardi, and Paul Zimmermann. The n-queens problem. *The American Mathematical Monthly*, 101(7):629–639, 1994.

[10] Michael Simkin. The number of n-queens configurations. *Advances in Mathematics*, 427:109127, 2023.

[11] GHJ Van Rees. On latin queen squares. In *Proceedings of the Tenth Manitoba Conference on Numerical Mathematics and Computing*, volume 2, pages 267–273, 1981.

[12] Ilan Vardi. *Computational recreations in Mathematica*. Addison Wesley Longman Publishing Co., Inc., 1991.

[13] Michel Vasquez and Djamal Habet. Complete and incomplete algorithms for the queen graph coloring problem. In *ECAI*, volume 16, page 226. Citeseer, 2004.