Functional inequalities and their applications

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Introduction

Functional inequalities form a keystone of analysis with the more celebrated ones such as Cauchy Schwarz for inner products or Holder's inequality for L^p spaces being instrumental to developing new mathematical results in all areas, not just limited to purely analysis. Since probability and geometry are closely related to analysis, it seems reasonable that we can use analytic and geometric ideas to build functional inequalities which can show us something about probability. These are some of the ideas behind Bakry-Emery theory, which uses geometric ideas to show various analytic and probabilistic properties of classes of functions which satisfy certain properties.

Our tools used to develop these ideas are centred around Markov semigroups, which are a set of operators which evolve time for a given function f. This set of operators obeys similar rules to groups eg $P_s \circ P_t = P_{t+s}$ is similar to the composition property and $P_0 = \mathrm{Id}$. We build more ideas like invariant measures $d\mu(x)$ which satisfy

$$\int_E f d\mu = \int_E P_t f d\mu \qquad (1)$$

for suitable $f : E \to \mathbb{R}$, $(E, \mathcal{F})$ is a topological space equipped with a σ-algebra, and infinitesimal generators L which are closely related to the Markov semigroup $(P_t)_{t\geqslant 0}$ since L commutes with P_t. After this, we can define the associated carré du champ operator Γ for L,

$$\Gamma(f) = \frac{1}{2}[L(f^2) - 2fLf] \qquad (2)$$

Using the fact that for any $t > 0$, $P_t(f^2) \geqslant (P_t f)^2$ and taking $t \to 0$ shows that $\Gamma(f) = L(f^2) - 2fLf \geqslant 0$ from the commutation property of P_t and L. Combining the state space E, the invariant measure μ and the carré du champ operator Γ allows us to state we have obtained a Markov Triple being (E, μ, Γ), which is very important for inequalities we will cover.

Another important observation is for L, specifically if L is a diffusion operator. If L is a diffusion operator, then for any smooth $\phi : \mathbb{R} \to \mathbb{R}$ we have

$$L\phi(f) = \phi'(f)Lf + \phi''(f)\Gamma(f) \qquad (3)$$

The two examples

The examples that will be focused upon today are the heat equation in Euclidean space $\mathbb{R}^n$, given by the diffusion operator

$$L_H f = \triangle_x f \qquad (4)$$

and the Ornstein Uhlenbeck operator, given by

$$L_{OU} f = \triangle_x f - x \cdot \nabla_x f \qquad (5)$$

Both of these operators have a large number of uses across science, in particular in mathematical physics. The heat equation is one of the oldest and simplest PDEs that mathematicians have studied, and is one of the few PDEs which has a explicit kernel densities given by

$$p_t(x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(\frac{|x^2 - y^2|}{4t}\right) \quad t > 0, x \in \mathbb{R}^n \qquad (6)$$

Being able to have explicit kernel densities allows us to validate the theorems and lemmas we prove in a general context by direct calculation and checking these are true. The heat equation also is a good first model for looking at diffusion processes as well as it having many nice smoothing/regularising properties. The heat equation has the Markov triple of $(\mathbb{R}^n, dx, \Gamma)$ where dx is lebesgue measure on $\mathbb{R}^n$ and $\Gamma(f) = |\nabla_x f|^2$.

Why does L_{OU} matter? The Ornstein Uhlenbeck operator is closely related to the Laplacian operator, it is actually the generalisation of L_H for infinite dimensions. Indeed, it is developed for separable Banach spaces, which the Laplacian doesn't work since the Laplacian idea of measure, being standard Lebesgue measure, doesn't make sense in an infinite dimensional space. However, the measure associated with L_{OU} is the standard Gaussian, which does make sense over separable Banach spaces. It has the Markov triple $(\mathbb{R}^n, \mu, \Gamma)$ where μ is the standard Gaussian measure on $\mathbb{R}^n$ and $\Gamma(f) = |\nabla_x f|^2$.

In addition, the Ornstein Uhlenbeck operator is closely connected to the quantum harmonic oscillator potential which appears in quantum theory, which we can see here. The harmonic oscillator potential in $\mathbb{R}^n$ is given by

$$Hf = \triangle_x f - |x|^2 f \qquad (7)$$

This is a symmetric operator with respect to lebesgue measure. Let $(K_t)_{t\geqslant 0}$ be the associated Markov semigroup for H. We also observe that $U_0 = \exp(-\frac{|x|^2}{2})$ satisfies the equation $HU_0 = -nU_0$. Then define

$$R_t f = \frac{e^{nt}K_t(U_0 f)}{U_0}$$

with associated generator

$$Lf = nf + \frac{H(U_0 f)}{U_0}$$

Using the diffusion property

$$H(U_0 f) = -nU_0 f + U_0 \triangle_x f + 2\nabla_x U_0 \cdot \nabla_x f$$

This yields that by the chain rule

$$Lf = \triangle_x f - 2\nabla_x \log(U_0) \cdot \nabla_x f \qquad (8)$$

which is precisely L_{OU}. [1]

Functional inequalities

The Poincaré inequality is one of the simplest functional inequalities that is studied, and can be used to prove many useful properties of functions satisfying $d\partial_t f = Lf$. The Poincaré inequality is mainly concerned with placing a bound on the variance of function f with respect to probability measure μ and it can be used to show that functions that obey the Poincaré inequality converge to a steady state as they evolve in time. The Markov triple (E, μ, Γ) is sufficient to state the Poincaré inequality, we don't actually need to state L.

The Markov triple (E, μ, Γ) satisfies the Poincaré inequality $P(C)$ if for each f sufficiently nice

$$\mathrm{var}_\mu(f) = \int_E f^2 d\mu - \left(\int_E f d\mu\right)^2 \leqslant C\int_E \Gamma(f) d\mu \qquad (9)$$

This can be used to prove that $\mathrm{var}_\mu(P_t f) \to 0$ as t gets large, which shows that it will converge to steady state μ over time.

For our examples, L_{OU} satisfies the Poincaré inequality $P(1)$, which makes sense given that its associated invariant measure μ is the standard Gaussian measure on $\mathbb{R}^n$ which is a probability measure, so the tails will converge. However, L_H does not satisfy the Poincaré inequality, which again makes sense when we look at the measure associated with L_H since lebesgue measure does not make sense on an infinite space, which is a clear requirement when we look at the Poincaré inequality. [1], [3]

The logarithmic Sobolev inequality is more complicated than the Poincaré inequality, but it is stronger than the Poincaré inequality. The Markov triple (E, μ, Γ) satisfies the log Sobolev inequality $LS(C, D)$ for positive C, D with μ a probability measure on E if

$$\begin{aligned}\mathrm{Ent}_\mu(f^2) &= 2\int_E f^2 \log(f) d\mu - \int_E f^2 d\mu \log\left(\int_E f^2 d\mu\right) \\ &\leqslant 2C\int_E \Gamma(f) d\mu + D\int_E f^2 d\mu\end{aligned} \qquad (10)$$

If $D = 0$ then we abbreviate $LS(C, 0)$ to $LS(C)$ and this is a tight log Sobolev inequality. Else, for $D > 0$ this is called a defective inequality. Supposing that $f = 1 + \epsilon g$ obeys $LS(C)$, then by considering the Taylor expansion of $P_t f$, we find that g satisfies $P(C)$. Then using the invariance of μ and that μ is a finite measure, we see that f also satisfies a Poincaré inequality.

Looking back to our examples, L_{OU} satisfies $LS(1)$ (L_{OU} also satisfies $P(1)$) and L_H does not satisfy a logarithmic Sobolev inequality. Again, this makes sense as Lebesgue measure on $\mathbb{R}^n$ is not a finite measure, but the standard Gaussian measure on $\mathbb{R}^n$ is. [2]

Summary and conclusions

In the case of the Poincaré inequality and the logarithmic Sobolev inequality, observations on looking to bound L and the finiteness of μ are enough to obtain these implications for the Ornstein Uhlenbeck semigroup. This also works nicely for operators in the form

$$Lf = \triangle_x f + \nabla_x V \cdot \nabla_x f \qquad (11)$$

where V is a sufficiently nice function. This observation is a combination of knowing what μ is and the bounds on L, one of which is the curvature dimension condition. This can be used for operators over infinite dimensions, which we have seen in the case of the Ornstein Uhlenbeck operator. Many of the operator bounds in quantum field theory can be shown as equivalent to functional inequalities, some of which are shown in this poster.

References

[1] Analysis and geometry of Markov diffusion operators, D. Bakry, I Gentil and M Ledoux, 2014

[2] Logarithmic Sobolev inequalities, in American Journal of Mathematics L.Gross, 1975

[3] Analysis, E Lieb and M Loss, 2001