Tessellations of the hyperbolic plane and triangular tilings on surfaces of genus 3

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1. Introduction

A tessellation or tiling is a complete covering of the surface by non-overlapping congruent polygons. In this project, we consider tilings of surfaces with hyperbolic triangles. The first part of this work involves a literature review of the field. We look ahead to third and fourth year modules, notably Introduction to Topology and Hyperbolic Geometry. This is followed by an investigation into classifying the groups corresponding to triangle tiling groups on surfaces of genus 3. But what is the motivation for looking into triangular tilings? It turns out tilings made up of triangles yield a large number of symmetries of the quotient surface, so are of interest when looking into groups of automorphisms of maximal order, and thus, highly symmetrical surfaces. These highly symmetrical surfaces are known as Hurwitz surfaces. We discuss **Klein's Quartic Curve** as a well-known example.

2. Research questions

The theoretical aspect of this project is guided by several motivating questions.

- A tessellation is formed from repeated congruent polygons. Which polygons can be used to tile the hyperbolic plane?
- How can we determine which portions of the hyperbolic plane can be wrapped up to form surfaces? And how do we determine the side-pairing transformations, or "gluing instructions"?

For the investigation, I aim to answer the question

- For a given genus g, say $g = 3$, what are the possible triangular surface tiling groups and what are their orders?

3. Tiling the hyperbolic plane

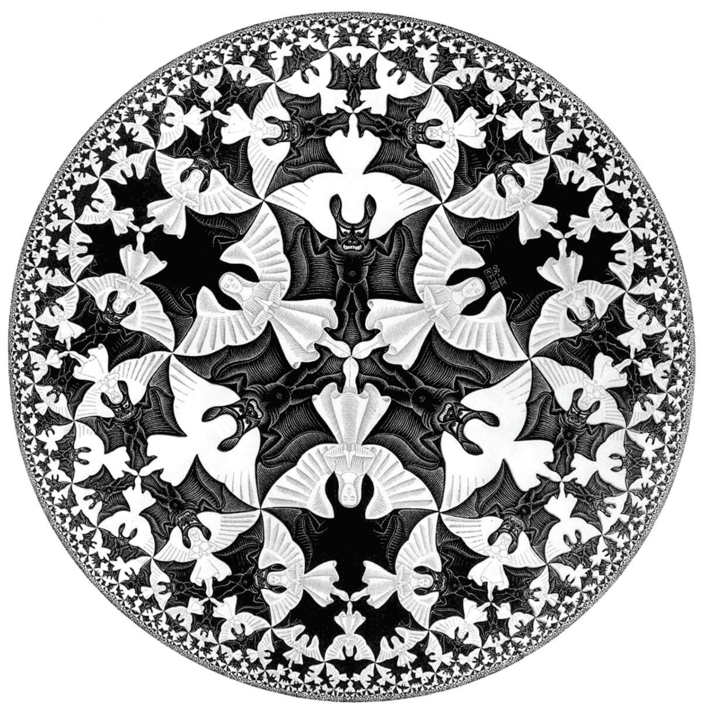
One of the models of the hyperbolic plane is the Poincaré disk, which is particularly convenient for viewing hyperbolic tilings. A famous example of a hyperbolic tiling is displayed in Escher's "Angels and Devils". Now we consider tilings of triangles. Given any natural numbers p, q and r, we consider a triangle with the angles $(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r})$. We call a triangle with these angles a (p, q, r)-triangle. Depending on the sum of the angles, this triangle is either Euclidean, spherical, or hyperbolic. For the hyperbolic case, we consider triangles which satisfy

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

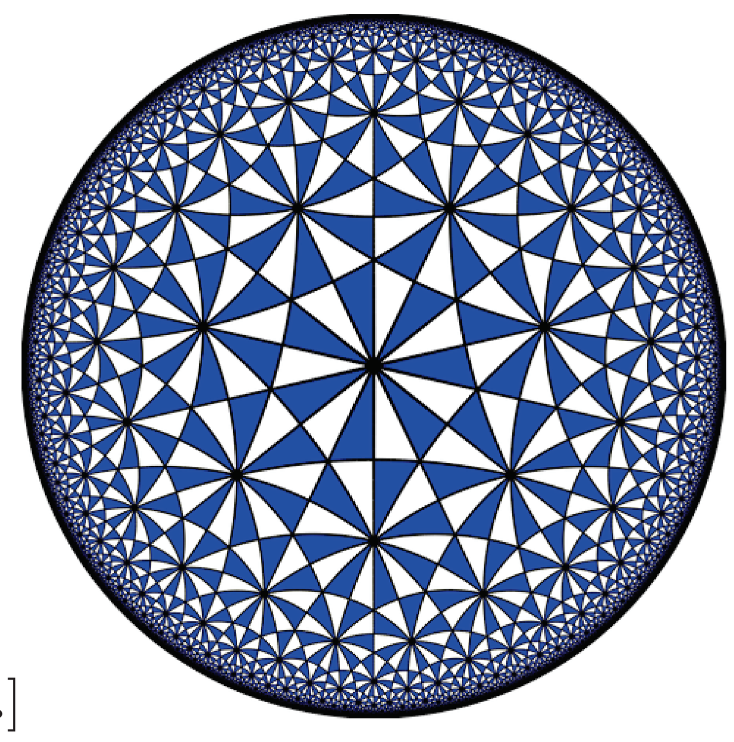
The (p, q, r)**-triangle group** is the infinite group generated by reflections in the triangle sides, and leads to a tiling of the hyperbolic plane by triangles. A triangle group of importance is the $(2, 3, 7)$ triangle group, with presentation

$$R_{2,3,7} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ca)^3 = (bc)^7 = 1 \rangle.$$

This gives the triangle of smallest area and the quotient curve with the most symmetry.
More generally, we may ask which polygons tile the hyperbolic plane without gaps or overlaps. A set of necessary (but not sufficient) conditions for a polygon P to tile the hyperbolic plane is given by **Poincaré's Polygon Theorem**. Given a convex, finitely sided hyperbolic polygon P, if each elliptic cycle of vertices satisfies the elliptic cycle condition, then the side-pairing transformations generate a **Fuchsian group**. This Fuchsian group then has fundamental domain P, which thus tessellates the plane.



[**Figure 1.**]

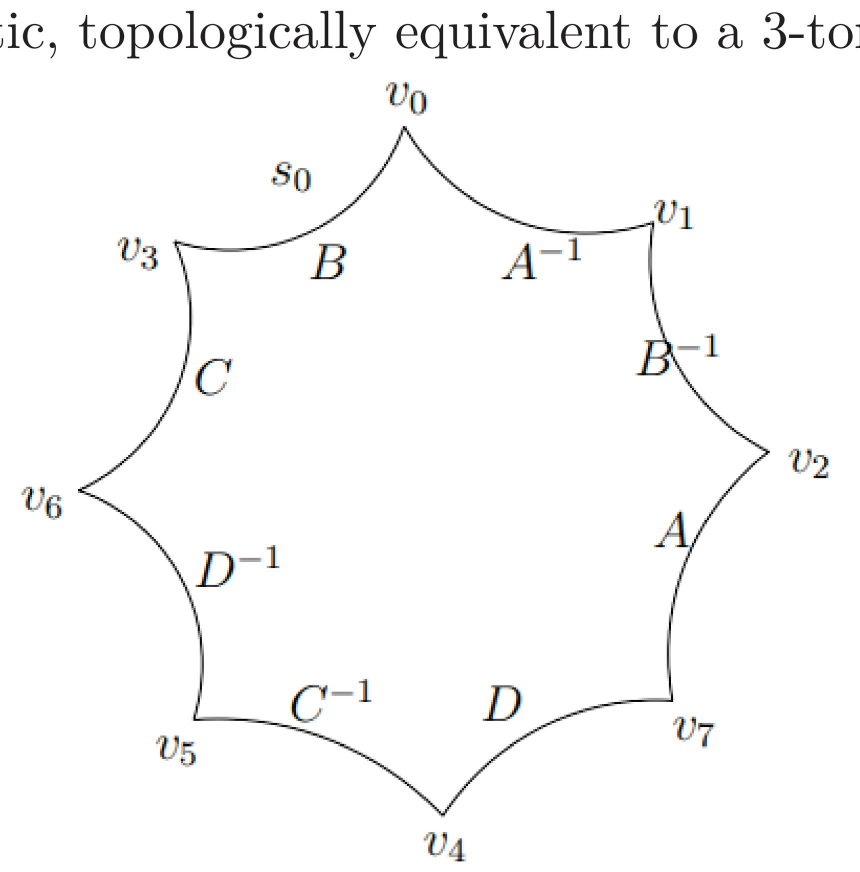


[**Figure 2.**]

1. Escher's famous woodcut, "Angels and Devils", also known as *Circle Limit IV*.
2. The Poincaré disk tiled with $(2, 3, 7)$-triangles.

4. Wrapping to form surfaces

The hyperbolic octagon forms an important example of how **side-pairing transformations** give rise to hyperbolic surfaces. Gluing the sides of the octagon by the side-pairing transformations described gives rise to a **quotient surface** of the form $\mathbb{H}^2/\Gamma$. It is difficult to visualise, but this creates a torus of genus 2, or bitorus. In a similar way, we can identify edges in a 14-gon to create the Klein Quartic, topologically equivalent to a 3-torus.



[**Figure 3.**]

3. Side-pairing transformations marked on a hyperbolic octagon with internal angles $\frac{\pi}{4}$, giving the relation $D^{-1}C^{-1}DCB^{-1}A^{-1}BA = id$.

5. Finding and classifying triangular tiling groups

We can construct finite groups from the infinite groups of reflections just described. This is the quotient of $R^+_{2,3,7}$, the orientation-preserving subgroup of $R_{2,3,7}$ of index 2, by a fixed-point free normal subgroup π_1 of $R^+_{2,3,7}$. The **Riemann-Hurwitz formula** describes a relationship between the genus g of the surface, the order of the tiling group $|G|$, and the orders of the generating rotations of $R^+_{2,3,7}$:

$$\frac{2g-2}{|G|} = 1 - (\frac{1}{p} + \frac{1}{q} + \frac{1}{r}).$$

I wrote code in Python to verify that $|G| = 168$ is in fact the order of the largest automorphism group, and given a genus g of the quotient surface, list possibilities for automorphism groups.

6. Further research

Throughout this project I was learning to use the computer algebra software MAGMA. I attempted to use the ALTAS database to further eliminate and classify triangular surface tiling groups. It proved to be a difficult task, but hopefully as I gain more experience using the software I will be able to complete this for genus 3. We could also extend this research by investigating for which orders $|G|$ is it possible to construct a triangular tiling group. For example, it is known that there are infinitely many odd order non-abelian tiling groups, however we can ask the converse; whether it is true that for all odd orders, there exists a non-abelian tiling group.

7. References

[**Figure 1.**] M. C. Escher, `https://www.wikiart.org/en/m-c-escher/circle-limit-iv`

[**Figure 2.**] Claudio Rocchini, `https://en.wikipedia.org/wiki/File:Order-3_heptakis_heptagonal_tiling.png`

[**Figure 3.**] Caroline Series, *p. 106*, **MA448** *Lecture Notes*, University of Warwick, 2008