The Bernstein Problem for Minimal Surfaces

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Introduction

Geometric measure theory is the study of geometric objects using a measure theoretic framework. It lends itself nicely to the study of minimal surfaces, which are surfaces of least area among all surfaces with a given boundary. In three dimensions, this mimics the soap film that forms when a closed piece of wire is dipped in soapy water and pulled out.

This project explored the basics of geometric measure theory, leading to the study of area minimising graphs and the Bernstein problem. This asks whether minimal hypersurfaces embedded in $\mathbb{R}^{n+1}$ that are the graph of a function must be affine linear. This turns out to be true for $n \leq 7$, but not higher dimensions. Here, the key ideas of the proof are illustrated for the case $n = 3$.

Geometric Measure Theory

A **measure** is a generalisation of the notion of length, area or volume to arbitrary sets. For any subset A of a larger set, a measure μ assigns to it a value $\mu(A)$. For something to be a measure it must map the empty set to zero and satisfy the property of countable additivity. This simply means that taking the measure of a union of disjoint subsets is the same as taking the sum of the measures of the subsets.

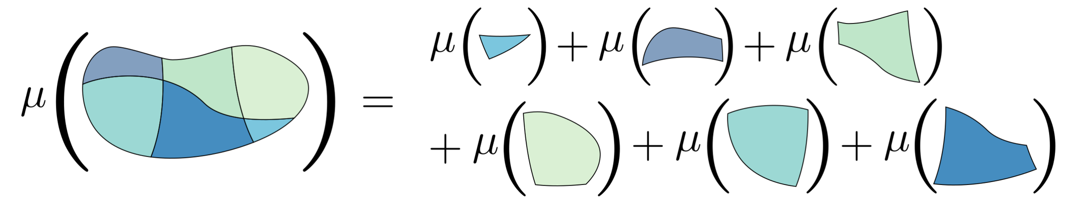

Figure 1. An illustration of countable additivity

Two important measures in geometric measure theory are the **Lebesgue** and **Hausdorff measures**. These measures are fundamental to the study of geometric objects as they are invariant under isometries (rotations, reflections and translations), matching the way volume behaves. Whilst the Lebesgue measure $\mathcal{L}^n$ measures n-dimensional subsets of $\mathbb{R}^n$, the Hausdorff measure $\mathcal{H}^m$ measures m-dimensional subsets of $\mathbb{R}^n$ for $m \leq n$. The two measures coincide when $m = n$.

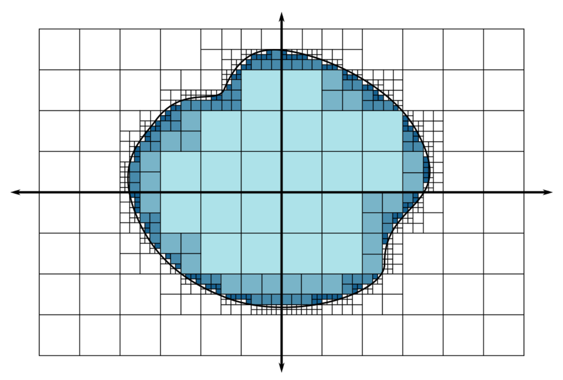

Figure 2. Measuring a subset using Lebesgue measure

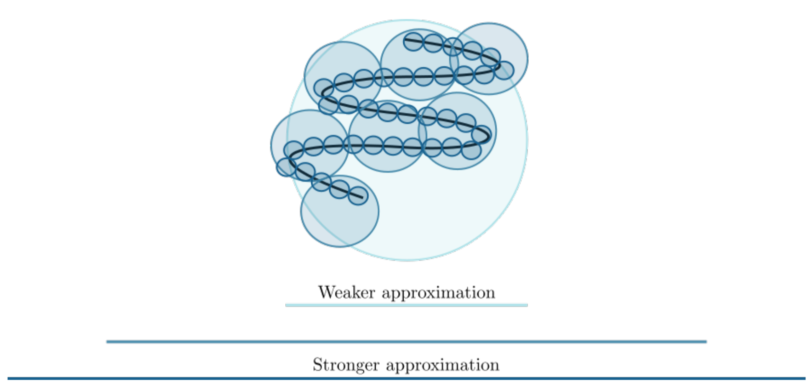

Figure 3. Measuring a curve using Hausdorff measure

Submanifolds of Codimension 1

A n-dimensional submanifold M of $\mathbb{R}^{n+l}$ is a subset of $\mathbb{R}^{n+l}$ that locally looks like $\mathbb{R}^n$. The codimension of a submanifold is the difference in dimension between the ambient space and the dimension of the submanifold. From now on we assume M is a n-dimensional submanifold embedded in $\mathbb{R}^{n+1}$. The tangent space T_yM at a point y is the linear subspace containing all tangent vectors to curves on M passing through y. This gives us a local orthonormal basis for $y \in M$.

For a Lipschitz map $f : M \to \mathbb{R}$, the **tangential gradient** $\nabla^M f(y)$ is defined by

$$\nabla^M f(y) = \sum_{j=1}^{n} (D_{\tau_j} f(y))\tau_j$$

where $\tau_1, \ldots, \tau_n$ is any orthonormal basis for T_yM.

The **second fundamental form** is the bilinear form $B_y : T_yM \times T_yM \to (T_yM)^\perp$ such that

$$B_y(\tau, \eta) = -(\eta \cdot D_\tau \nu)\nu\Big|_y$$

for $\tau, \eta \in T_yM$. This is closely related to **mean curvature** $\underline{H}$ which is defined as

$$\underline{H}(y) = \sum_{i=1}^{n} B_y(\tau_i, \tau_i) \in (T_yM)^\perp.$$

Stable minimal surfaces have mean curvature equal to zero everywhere.

The Area Formula

The **area formula** is an important theorem in geometric measure theory that allows us to calculate the Hausdorff measure of the image of a Lipschitz map from $\mathbb{R}^n$ to $\mathbb{R}^m$, with $n \leq m$.

Let $u : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and define $M = \text{graph}(u)$. Then M is the image of the map $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ with $f(x) = (x, u(x))$. By the area formula,

$$\mathcal{H}^n(M) = \int_{\mathbb{R}^n} \sqrt{1 + |Du|^2}\, d\mathcal{H}^n.$$

We can actually extend the area formula to a locally Lipschitz map $f : M \to \mathbb{R}^P$ where M is a n-dimensional manifold and $P \geq n$.

Area Minimising Graphs

Here, we state some results about graphs that are minimal surfaces in $\mathbb{R}^{n+1}$. In this section, the function $u : \mathbb{R}^n \to \mathbb{R}$ is assumed to be Lipschitz and $M = \text{graph}(u) = \{(x, u(x)) : x \in \mathbb{R}^n\}$. Moreover, M is assumed to be area minimising, in particular, $\underline{H} = 0$.

- It can be shown that
$$\text{div}_{\mathbb{R}^n}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = 0.$$
This is done by analysing the first variation (derivative of the change in area) of local perturbations of M.
- Define $\hat{\nu}$ on $\mathbb{R}^{n+1}$ as $\hat{\nu}(x, x_{n+1} = \nu(x, u(x))$, where ν is the normal vector
$$\nu = \frac{1}{\sqrt{1 + |Du|^2}}(Du, -1).$$

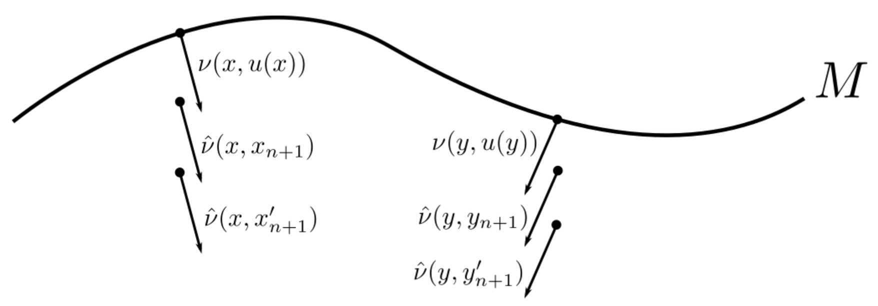

Figure 4. The normal field $\hat{\nu}$

Then $\text{div}_{\mathbb{R}^{n+1}}(\hat{\nu}) = 0$.
- Let M' be any graph such that $M' = M$ outside of a compact set $K \subset \mathbb{R}^{n+1}$. Then
$$\mathcal{H}^n(M' \cap K) \geq \mathcal{H}^n(M \cap K).$$
In other words, M locally minimises area.

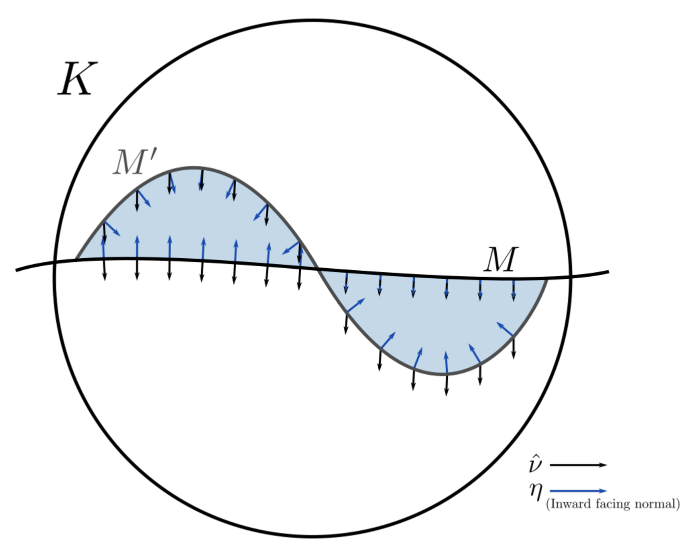

Figure 5. The region between M and M'

This can be proved by applying the Divergence Theorem to the shaded regions in the above figure.

Main References

[EG15] Lawrence Evans and Ronald Gariepy. *Measure Theory and Fine Properties of Functions*. CRC Press, Florida, 2015.

[Giu84] Enrico Giusti. *Minimal Surfaces and Functions of Bounded Variation*. Springer Science + Business Media, New York, 1984.

[Sim14] Leon Simon. *Introduction to Geometric Measure Theory*. Stanford University, 2014.

Bounds on Area Minimising Graphs

Here we explore some inequalities that use the fact that the graph M is locally area minimising.

- The Hausdorff measure of the intersection between M and a $(n+1)$-dimensional ball can be bounded by half the 'surface area' of the ball. More precisely, for $x \in M$
$$\mathcal{H}^n(M \cap \mathbb{B}^{n+1}(x, r)) \leq \frac{1}{2}\mathcal{H}^n(\partial\mathbb{B}^{n+1}(x, r)). \quad (1)$$

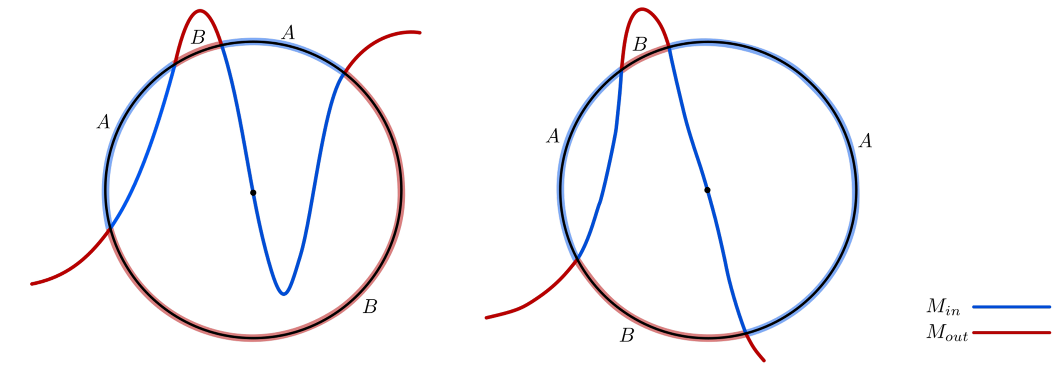

Figure 6. Examples of possible intersections between $\mathbb{B}^2(x, r)$ and 1-dimensional M

The cases shown in Figure 6 are not candidates for M as they do not satisfy inequality (1). However, they illustrate a method for proving (1) in which M is allowed to intersect $\mathbb{B}^{n+1}(x, r)$ in arbitrarily many places. The boundary $\partial\mathbb{B}^{n+1}(x, r)$ is then divided into regions A and B based on these intersections. The Divergence Theorem is applied to the regions enclosed by A and M_{in} or B and M_{in} (depending on which of A and B has smaller measure) to show that M must have smaller measure than its corresponding section of the boundary. We then use the fact that $\min\{\mathcal{H}^n(A), \mathcal{H}^n(B)\} \leq \frac{1}{2}\mathcal{H}^n(\partial\mathbb{B}^{n+1}(x, r))$ to complete the proof.
- Since M is locally area minimising, the second variation (second derivative of the change in area) is positive, which leads to the following bound on mean curvature:
$$\int_M |\nabla^M \zeta|^2\, d\mathcal{H}^n \geq \int_M \zeta^2 |B|^2\, d\mathcal{H}^n, \quad (2)$$
for ζ Lipschitz with compact support, where
$$|B|^2 = \sum_{i,j=1}^{n} |B(\tau_i, \tau_j)|^2.$$

Bernstein's Problem in $\mathbb{R}^3$

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function such that $M = \text{graph}(u)$ locally area minimising, then u is affine linear.

Proof of Bernstein's Problem

The key idea of the proof is what is known as the log cut-off trick. We consider the function

$$\zeta_j = \begin{cases} 1 & |x| \leq j \\ 2 - \frac{\log|x|}{\log j} & j < |x| < j^2 \\ 0 & |x| \geq j^2 \end{cases}$$

for all $j \in \mathbb{N}$. This is a Lipschitz function with compact support.

Using inequality (2), the integral of $|B|^2$ on $M \cap \mathbb{B}(j)$ (where ζ_j is non-zero) can now be bounded above by $\int_M |D\zeta_j|^2\, d\mathcal{H}^n$. Algebraic manipulation using shows that

$$\int_M |D\zeta_j|^2\, d\mathcal{H}^2 < \frac{A}{\log j}$$

for some constant A. So as j approaches ∞, it follows that $\int_{M \cap \mathbb{B}(j)} |B|^2 = 0$ and hence $|B|^2 = 0$. From here it is straightforward to show that M must be a plane.

Conclusion

In general, Bernstein's problem for a minimal n-dimensional graph embedded in $(n+1)$-dimensional space is true for $n \leq 7$, and false for all higher dimensions. The problem was posed in 1915, but only solved in its entirety in 1969, this marks over 50 years of work on the topic in which many new ideas were developed. There are many generalisations of this problem that are still being worked on today.