Bounding the number of solutions to linear diophantine equations in an application to PDEs

Kate Thomas
University of Warwick

Relevance from PDEs

Following the recommendations from [1], which uses techniques from number theory to provide an upper bound for the size of $E^{n,n'}_{\beta,\gamma,N,T}$ ([1, p. 11]), I see how these methods apply to similar sets in higher dimensions, starting with the 3 dimensional one and seeing what still works and what goes wrong. The upper bound for the size of $E^{n,n'}_{\beta,\gamma,N,T}$ is used to bound the quantity $\sup_{x\in\mathbb{T}} |K_N(t,x)|$ (defined in [1, p. 4], see Proposition 3.2). In turn, bounds on $K_N(t,x)$ are used to analyse Strichartz estimates for tori [1, Section 4]. The number theoretic techniques we use here are bounding the number of solutions to linear diophantine equations in two and three variables.
The quantity we investigate is μ_2, defined as follows:

$$\mu_2(t) = \text{measure}\{(\alpha_{ij}) \in E : \alpha_{i1}n_1^{(j)} + \ldots + \alpha_{id}n_d^{(j)} = t^{-1}m_i^{(j)} + O\left(\frac{1}{tN}\right)$$
$$\text{for all } i,j = 1,\ldots d \text{ and some } n_i^{(j)} \lesssim N, m_i^{(j)} \lesssim tN\}$$

where E^d is the set of symmetric $d\times d$ matrices (α_{ij}) with $|\alpha_{ij}| \leq 2$ and the smallest absolute eigenvalue $|\lambda_1| > 1$.

Counting points and solving linear diophantine equations

In order to bound the number of solutions to the compatibility conditions we need to have tools to count integer points within certain intervals, as well as how to parameterise solutions to some linear diophantine equations.

Lemma 1: *Let $x \in [a,b]$. Then the number of integer solutions for x is at most $1+b-a$*

I have also used the following two lemmas from [2] to parameterise the solutions (if they exist) to two and three variable linear diophantine equations, so we can vary the parameter(s) in order to bound integer solutions within a given range.

Lemma 2: *Let $a,b,n \in \mathbb{Z}, ab \neq 0$ and let $g = gcd(a,b)$. Then $ax+by=n$ has a solution in $\mathbb{Z}^2$ iff $d|n$, giving solutions*

$$(x,y) = \left(x_0 + \frac{kb}{d}, y_0 - \frac{ka}{d}\right)$$

parameterised by $k \in \mathbb{Z}$ with (x_0,y_0) a particular solution of $ax+by=n$

The idea to solve a three variable linear diophantine equation is to group two variables together as a new variable, solve solutions for the resulting two variable equation and then consider solutions to the equation for the two grouped variables. This parameterises all solutions to the original equation in two parameters which we may vary in $\mathbb{Z}$:

Lemma 3: *Let $a,b,c,d \in \mathbb{Z}$ such that $abc \neq 0$, and set $g = gcd(a,b,c)$. The equation $ax+by+cz=d$ has a solution in $\mathbb{Z}^3$ iff $g \mid d$, in which case the solutions are given by*

$$(x,y,z) = \left(x_0 + \frac{kb}{\delta} - u_0 l, y_0 - \frac{ka}{\delta} - v_0 l, z_0 + \delta l\right)$$

where $(k,l) \in \mathbb{Z}^3$, $\delta = gcd(a,b)$, (u_0,v_0) is a particular solution of $au+bv=\delta c$, (z_0,t_0) is a particular solution of $cz+\delta t = d$ and (x_0,y_0) is a particular solution of $ax+by=\delta t_0$.

Studying the 3x3 case

$$|t(\alpha_{11}n_1^{(1)} + \alpha_{12}n_2^{(1)} + \alpha_{13}n_3^{(1)}) - m_1^{(1)}| \leq 1/N \qquad |t(\alpha_{11}n_1^{(2)} + \alpha_{12}n_2^{(2)} + \alpha_{13}n_3^{(2)}) - m_1^{(2)}| \leq 1/N$$
$$|t(\alpha_{12}n_1^{(1)} + \alpha_{22}n_2^{(1)} + \alpha_{23}n_3^{(1)}) - m_2^{(1)}| \leq 1/N \qquad |t(\alpha_{12}n_1^{(2)} + \alpha_{22}n_2^{(2)} + \alpha_{23}n_3^{(2)}) - m_2^{(2)}| \leq 1/N$$
$$|t(\alpha_{13}n_1^{(1)} + \alpha_{23}n_2^{(1)} + \alpha_{33}n_3^{(1)}) - m_3^{(1)}| \leq 1/N \qquad |t(\alpha_{13}n_1^{(2)} + \alpha_{23}n_2^{(2)} + \alpha_{33}n_3^{(2)}) - m_3^{(2)}| \leq 1/N$$

$$|t(\alpha_{11}n_1^{(3)} + \alpha_{12}n_2^{(3)} + \alpha_{13}n_3^{(3)}) - m_1^{(3)}| \leq 1/N$$
$$|t(\alpha_{12}n_1^{(3)} + \alpha_{22}n_2^{(3)} + \alpha_{23}n_3^{(3)}) - m_2^{(3)}| \leq 1/N$$
$$|t(\alpha_{13}n_1^{(3)} + \alpha_{23}n_2^{(3)} + \alpha_{33}n_3^{(3)}) - m_3^{(3)}| \leq 1/N$$

Setting up the equations following the definition of μ_2 in the 3 dimensional case gives us the system of 9 inequalities above, which we may reduce to 3 matrix inequalities by the following method. We have an overdetermined system in α_{ij} (as it is a symmetric matrix) so by comparing solutions for $\alpha_{12}, \alpha_{13}, \alpha_{23}$ solved in two different ways each, we get 3 compatibility conditions. In order to analyse these, we group the equations as follows.

$$\boldsymbol{n} = \begin{pmatrix} n_1^{(1)} & n_2^{(1)} & n_3^{(1)} \\ n_1^{(2)} & n_2^{(2)} & n_3^{(2)} \\ n_1^{(3)} & n_2^{(3)} & n_3^{(3)} \end{pmatrix}, \boldsymbol{\alpha_1} = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \end{pmatrix}, \boldsymbol{\alpha_2} = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \\ \alpha_{23} \end{pmatrix}, \boldsymbol{\alpha_3} = \begin{pmatrix} \alpha_{13} \\ \alpha_{23} \\ \alpha_{33} \end{pmatrix}, \boldsymbol{m_i} = \begin{pmatrix} m_i^{(1)} \\ m_i^{(2)} \\ m_i^{(3)} \end{pmatrix} \quad \text{for i=1,2,3}$$

$$|t\boldsymbol{n}\boldsymbol{\alpha_1} - \boldsymbol{m_1}| \lesssim 1/N$$
$$|t\boldsymbol{n}\boldsymbol{\alpha_2} - \boldsymbol{m_2}| \lesssim 1/N$$
$$|t\boldsymbol{n}\boldsymbol{\alpha_3} - \boldsymbol{m_3}| \lesssim 1/N$$

Since we're able to assume the error term is sufficiently small, we may compare the integer parts of the solutions to these inequalities and thus treat the system as equalities. We may also assume det(n)$\neq 0$, allowing us to use the following method to solve for these.

$$t\boldsymbol{n}\boldsymbol{\alpha_i} = \boldsymbol{m_i}$$
$$\boldsymbol{\alpha_i} = \frac{1}{t}\boldsymbol{n^{-1}}\boldsymbol{m_i}$$
$$\boldsymbol{\alpha_i} = \frac{1}{t\det(\boldsymbol{n})}\text{adj}(\boldsymbol{n})\boldsymbol{m_i}$$

To derive the conditions, we compare respective rows of adj($\boldsymbol{n}$) for $i = 1,2,3$, giving the following:

$$\begin{aligned} (1) \quad & (n_3^{(2)}n_1^{(3)} - n_1^{(2)}n_3^{(3)})m_1^{(1)} + (n_1^{(1)}n_3^{(3)} - n_3^{(1)}n_1^{(3)})m_1^{(2)} + (n_3^{(1)}n_1^{(2)} - n_1^{(1)}n_3^{(2)})m_1^{(3)} \\ & = (n_2^{(2)}n_3^{(3)} - n_2^{(3)}n_3^{(2)})m_2^{(1)} + (n_3^{(1)}n_2^{(3)} - n_2^{(1)}n_3^{(3)})m_2^{(2)} + (n_2^{(1)}n_3^{(2)} - n_3^{(1)}n_2^{(2)})m_2^{(3)} \end{aligned}$$

$$\begin{aligned} (2) \quad & (n_1^{(2)}n_2^{(3)} - n_2^{(2)}n_3^{(1)})m_2^{(1)} + (n_2^{(1)}n_1^{(3)} - n_1^{(1)}n_2^{(3)})m_2^{(2)} + (n_1^{(1)}n_2^{(2)} - n_2^{(1)}n_1^{(2)})m_2^{(3)} \\ & = (n_3^{(2)}n_1^{(3)} - n_1^{(2)}n_3^{(3)})m_3^{(1)} + (n_1^{(1)}n_3^{(3)} - n_3^{(1)}n_1^{(3)})m_3^{(2)} + (n_3^{(1)}n_1^{(2)} - n_1^{(1)}n_3^{(2)})m_3^{(3)} \end{aligned}$$

$$\begin{aligned} (3) \quad & (n_1^{(2)}n_2^{(3)} - n_2^{(2)}n_3^{(1)})m_1^{(1)} + (n_2^{(1)}n_1^{(3)} - n_1^{(1)}n_2^{(3)})m_1^{(2)} + (n_1^{(1)}n_2^{(2)} - n_2^{(1)}n_1^{(2)})m_1^{(3)} \\ & = (n_2^{(2)}n_3^{(3)} - n_2^{(3)}n_3^{(2)})m_3^{(1)} + (n_3^{(1)}n_2^{(3)} - n_2^{(1)}n_3^{(3)})m_3^{(2)} + (n_2^{(1)}n_3^{(2)} - n_3^{(1)}n_2^{(2)})m_3^{(3)} \end{aligned}$$

In the case where no entries of adj($\boldsymbol{n}$) are zero, each equation (1), (2) and (3) is a three variable linear diophantine equation. We also assume that any conditions on gcd's within this system hold such that it is possible for solutions to exist. The method then to bound the number of integer solutions for $m_i^{(j)}$ is then as follows, using the condition $m_i^{(j)} \lesssim tN$:

- Use that $m_i^{(j)} \lesssim tN$ and lemma 1 to find that there are at most $\lesssim 1+tN$ choices for each $m_i^{(j)}$. In particular, we apply this to the values $m_2^{(j)}$ on the RHS of (1) to find there are $\lesssim (1+tN)^3$ possible triples $(m_2^{(1)}, m_2^{(2)}, m_2^{(3)})$
- Now assuming values of $(m_2^{(1)}, m_2^{(2)}, m_2^{(3)})$ are fixed, we use results from linear diophantine equations to bound the number of solutions $(m_1^{(1)}, m_1^{(2)}, m_1^{(3)})$ to the LHS of (1), noting any gcds here are well defined as we have all entries of adj($\boldsymbol{n}$) $\neq 0$.
- Now fixing those values of $(m_1^{(1)}, m_1^{(2)}, m_1^{(3)})$ along with $(m_2^{(1)}, m_2^{(2)}, m_2^{(3)})$, we can find solutions for $(m_3^{(1)}, m_3^{(2)}, m_3^{(3)})$ either from (2) or (3). Using the same linear diophantine equations methods for each, we find two bounds on the number of solutions here and take a minimum.
- Finally we multiple to find an upper bound on the number of integer $m_i^{(j)}$ for $i,j = 1,2,3$.

This gives an upper bound for the number of possible integer solutions for $m_i^{(j)}$ in the desired range:

$$\lesssim (1+tN)^3\left(1 + \frac{tN}{\gcd((n_3^{(2)}n_1^{(3)} - n_1^{(2)}n_3^{(3)}), (n_1^{(1)}n_3^{(3)} - n_3^{(1)}n_1^{(3)}))}\right)\left(1 + \frac{tN\gcd((n_3^{(2)}n_1^{(3)} - n_1^{(2)}n_3^{(3)}), (n_1^{(1)}n_3^{(3)} - n_3^{(1)}n_1^{(3)}))}{(n_1^{(1)}n_3^{(3)} - n_3^{(1)}n_1^{(3)})}\right)$$
$$\min\left\{(1 + \frac{tN}{\gcd((n_3^{(2)}n_1^{(3)} - n_1^{(2)}n_3^{(3)}), (n_3^{(3)}n_1^{(1)} - n_1^{(3)}n_3^{(1)}))})(1 + \frac{tN\gcd((n_3^{(2)}n_1^{(3)} - n_1^{(2)}n_3^{(3)}), (n_3^{(3)}n_1^{(1)} - n_1^{(3)}n_3^{(1)}))}{n_3^{(3)}n_1^{(1)} - n_1^{(3)}n_3^{(1)}})\right.$$
$$\left.(1 + \frac{tN}{\gcd((n_2^{(2)}n_3^{(3)} - n_3^{(2)}n_2^{(3)}, (n_3^{(1)}n_2^{(3)} - n_2^{(1)}n_3^{(3)}))})(1 + \frac{tN\gcd((n_2^{(2)}n_3^{(3)} - n_3^{(2)}n_2^{(3)}, (n_3^{(1)}n_2^{(3)} - n_2^{(1)}n_3^{(3)}))}{n_3^{(1)}n_2^{(3)} - n_2^{(1)}n_3^{(3)}})\right\}$$

When some entries of adj($\boldsymbol{n}$) are zero, we have to take a more careful, case-by-case approach, using techniques to solve two variable linear diophantine equations where needed and adjusting bounds where necessary.

Where next?

In order to make these results usuable for the ensuing analysis to Strichartz estimates, we may need a cleaner formula or a more accurate upper bound. A more accurate bound would be obtained by focusing only on solutions that respect the simultaneous nature of the compatibility conditions. By reducing the following matrix to Hermite normal form, it may be easier to analyse and count the solutions, as well as determine when they exist, using results from [3].

$$\begin{pmatrix} r_2 & -r_1 & (0,0,0) \\ (0,0,0) & r_3 & -r_2 \\ r_3 & (0,0,0) & -r_1 \end{pmatrix}\begin{pmatrix} \boldsymbol{m_1} \\ \boldsymbol{m_2} \\ \boldsymbol{m_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

References

[1] Y. Deng, P. Germain, L. Guth, and S. L. Rydin Myerson, "Strichartz estimates for the schrödinger equation on non-rectangular two-dimensional tori," 2021.

[2] O. Bordellès, "Linear diophantine equations," in *Arithmetic Tales: Advanced Edition*. Springer International Publishing, 2020.

[3] A. Schrijver, *Theory of Linear and Integer programming*. Wiley-Interscience, 1986.

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