Clique subdivisions in dense graphs with no bipartite holes

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Motivation

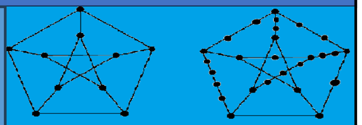
Extremal graph theory is a field which emerged just over 100 years ago with Mantel's theorem (1907) being one of the first known results. The theorem states what the maximum number of edges a graph can have if it contains no triangle. Another central set of questions in extremal graph theory consists of determining conditions which guarantee the existence of structures within a graph. A classic example of this is Dirac's theorem which establishes that every graph on $n \geq 3$ vertices and minimum degree at least $\frac{n}{2}$ contains a Hamiltonian cycle. In our setting we are assuming that G is a graph with two properties of being dense and having no bipartite holes and in such a setting we show that G necessarily contains a balanced spanning subdivision of the complete graph K_l for all $2 \leq l \leq c\sqrt{n}$

Theorem 1

For every $\mu > 0$, there are constants $c, \varepsilon \in (0,1]$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. If G is an n-vertex graph with $\delta(G) \geq \mu n$ and which is εn-joined, then G contains a spanning K_l-subdivision for all $2 \leq l \leq c\sqrt{n}$

Subdivisions

Given a graph G we say that another graph G' is a G-subdivision if G' can be obtained from G by replacing all edges in G by disjoint paths (possibly of length 1).



The Peterson graph and a subdivision of it

Notation

We say that an n-vertex graph G is dense if it has minimum degree $\delta(G) \geq \mu n$ for some $\mu > 0$

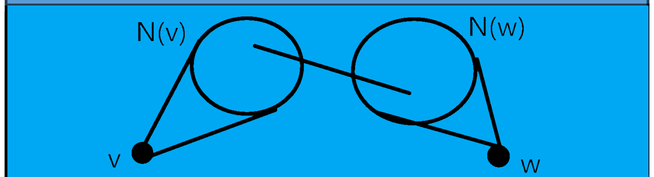
A graph G is m-joined if for all disjoint subsets A, B of $V(G)$ each of size at least m there exists an edge joining a vertex in A together to a vertex in B.

In this case we also say that G has no bipartite holes.

We say that a subdivision is balanced if each edge is replaced by a path of the same length.

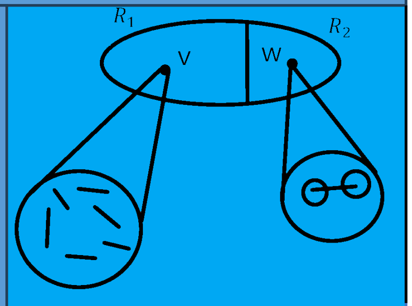
Sketch of proof

Pick vertices v, w in G. Then since $\delta(G) \geq \mu n$ and G is εn-joined we can find an edge between $N_G(v)$ and $N_G(w)$, giving us a path of length 3. Here we are using that $\mu \gg \varepsilon$. In this manner we can join any two vertices by short path.



We pick 3 subsets of $V(G)$ namely R_1, R_2 and R_3 with the property that $|R_1| = 2\theta n$ and that $\forall\, v$ in $V(G)$ we have $d_{R_1}(v) \geq \theta \mu n$ and similarly for the other two. We do this by choosing them at random one by one and showing that with high probability any random subset has the desired property and hence such a subset must exist, this is known as **The Probabilistic Method**. We call the 3 subsets above **reservoirs.** They are small subsets with very useful properties that we are going to set aside and come back to later. Given a graph G and a vertex v in $V(G)$, an **absorber** for v is a tuple (A, x, y) that consists of a subset A of $V(G - v)$ and vertices x, y in A such that $G[A]$ has a Hamiltonian x, y- path and $G[A \cup v]$ has a Hamiltonian x, y- path. For every r in $R_1 \cup R_2$ we find 100 disjoint edges in $N_{V(G)\setminus(R_1 \cup R_2 \cup R_3)}(r)$. Note that each such edge is an absorber for the corresponding vertex.

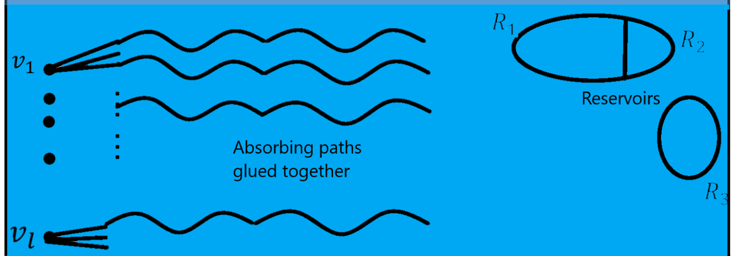
We can find these edges as can be shown on the right side of the diagram. Since the neighbourhood of w is large and G is $\mathcal{E}n$-joined.



We then glue together 100 of these edges to form absorbing paths that absorb some 100 vertices in $R_1 \cup R_2$. We then have the same number of absorbing paths as vertices in $R_1 \cup R_2$ and so we can later match them one-to-one in hopes of our subdivision being balanced.

Sketch of proof

In a subdivision, the vertices which correspond to vertices in the original graph are called **branching vertices** and the paths between them **branching paths.** Choose l vertices in $V(G)$ but not in $R_1 \cup R_2 \cup R_3$ and not in our absorbing paths. These will be our branching vertices. Call them $v_1, \ldots, v_l$ and greedily find $l - 1$ neighbours for each one, calling them $u_{1,1}, \ldots, u_{l,l-1}$. Next, we glue on the absorbing paths by using our technique of finding paths of length 3 and we spread them out as evenly as possible among the vertices $u_{1,1}, \ldots, u_{l,l-1}$. Note that we have more absorbing paths than vertices $u_{i,j}$ so each one gets several absorbing paths glued onto it.



Note that at this point, we have only used up a tiny proportion of the graph. Let G' denote the rest of the graph which we have not yet used. We use the following well-known lemma to find a very long path in G'.

Lemma 1

If H is a m-joined graph, then H contains a path of length at least $|H| - 2m$.

Sketch of proof

Next, we take such a long path M and split it up into $\frac{l(l-1)}{2}$ segments choosing each segment length appropriately. We then use the reservoir R_3 to join up the segments together with the absorbing paths to form our branching paths. We do this by making use of the nice properties of the subset R_3. After doing so we will have a small set of left-over vertices which in turn we incorporate into our subdivision by using the reservoirs R_1 and R_2. Then whatever is left over in $R_1 \cup R_2$ we take care of by using the original absorbing paths, but I will leave out the technicalities. This method is referred to as the **absorption method**. In the end we obtain our desired spanning subdivision of the complete graph K_l.