The Fundamental Groupoid: a categorical approach to the fundamental group and the Seifert-Van Kampen theorem

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Introduction and motivation

A topological space is a complex mathematical object. It is uniquely defined by its collection of open sets, a subset of its power set, which can be exceedingly large. To establish a meaningful comparison between two topological spaces, the task is to identify a map between them that preserves the structural properties induced by these open sets. The pursuit of finding such maps and demonstrating that they preserve open sets can be a very hard task. Nevertheless, it is an immediate approach to directly establish the topological equivalence of two spaces.

For this reason, in order to understand how to compare and distinguish topological spaces, it is sensible to employ a more indirect strategy. One such approach involves the study of algebraic invariants associated with topological spaces. These algebraic invariants are mathematical structures that can be manipulated via algebraic methods and remain unchanged under certain topological transformations, such as homeomorphism or homotopy equivalence.

The fundamental group

One of the first examples of algebraic invariants is the fundamental group, which was introduced by Henri Poincaré in 1895. The fundamental group of a topological space X based at a point $x_0 \in X$, denoted by $\pi_1(X, x_0)$, is the group structure induced by the homotopy classes of loops inside the space, based at the point x_0. The classical approach to the theory of fundamental groups is through the study of covering theory, which examines the relationship between covering spaces of a given topological space.

Another approach, which represents the underling spirit of this project, is to develop the theory of fundamental groups within a categorical framework. This approach not only allows us to generalise the concept of fundamental to a more general object, called the fundamental groupoid, which removes the need to restrict ourselves to pointed topological spaces, but more importantly, it provides a formal setting to elucidate how the relationship between topological spaces and groups, as distinct mathematical entities, enables a more concrete understanding of the behaviour of this specific algebraic invariant on these objects.

Aim of the project

The primary goal of this project is to provide a comprehensive presentation of the fundamental groupoid, along with the necessary category theory required to define it and prove its fundamental properties. This will lead to the proof of the Seifert-Van Kampen theorem for fundamental groupoids. From there, we will discuss basic notions of 2-category theory: this will allow us to explore the categorical properties of the fundamental groupoid when viewed as a costack over the category of 2-groupoids. We will prove that, for a "nice" class of topological space, the fundamental groupoid is a terminal object in this category: this provides a purely categorical description of the fundamental groupoid. The focal point of this project is the discussion of this result, originally formulated by Ilia Pirashvili in 2015.

The fundamental groupoid

Definition 1: Let X be a topological space. The fundamental groupoid $\Pi_1 X$ is the category with $\mathrm{Ob}(\Pi_1 X) = X$, $\mathrm{Hom}_{\Pi_1 X}(x, y) = \mathrm{Hom}_{\mathbf{P}X}(x, y)/\sim$ and composition $*_{\Pi_1 X}$ is the concatenation of equivalence classes of paths and for any $x \in X$.

By considering paths up to homotopy, we obtain a more flexible and rich algebraic structure that encodes information about the connectivity and topology of the space.

Remark 2: The fundamental groupoid of a topological space X contains information about the fundamental groups of X at each of its points. Indeed, for any $x \in X$, the set of morphisms $\mathrm{Hom}_{\Pi_1 X}(x, x)$ corresponds to homotopy classes of loops based at x, which is precisely the definition of $\pi_1(X, x)$.

Theorem 3: Let X be a topological space and suppose that A is a representative in X, then the inclusion functor $\Pi_1 X[A] \hookrightarrow \Pi_1 X$ defines an equivalence of categories. In particular, if X is path connected $\mathbf{B}\pi_1(X, x_0)$ is equivalent to $\Pi_1 X$, and more generally, the canonical functor

$$\coprod_{x \in \pi_0(X)} \mathbf{B}\pi_1(X, x) \longrightarrow \Pi_1 X$$

is an equivalence of categories, where $\pi_0(X)$ denotes the set of path connected components of X.

Important observation

Theorem 3 establishes that the fundamental groupoid and the fundamental group of a topological space provide equivalent information on its path-connected components. While this equivalence may not be surprising, it is far from uninteresting. Indeed, despite the fundamental groupoid being essentially the same algebraic invariant as the fundamental group, it offers an alternative framework where it can be investigated. In the project, we explored the advantages of this approach by proving the fundamental groupoid version of the Seifert-Van Kampen theorem. This result not only recovers the standard result for fundamental groups, but allows us to develop a theory of fundamental groupoids that does not rely on covering theory. This observation motivated our further exploration in Part II of the paper, where our aim was to provide a "purely" categorical description of the fundamental groupoid.

The Seifert-Van Kampen Theorem for Π_1

Theorem 4: Let X be a topological space, and let U_1, U_2 be subspaces of X such that $U_1^\circ \cup U_2^\circ = X$ and $U_1 \cap U_2$ non-empty. Let us denote by i_1, i_2 the inclusions of $U_1 \cap U_2$ into U_1 and U_2 respectively, by j_i the inclusions of U_1 into X and by j_2 the inclusion of U_2 into X. Let $A \subseteq X$ be a representative in X, U_1, U_2 and $U_1 \cap U_2$. Then, the following square exhibits $\Pi_1 X[A]$ as pushout in $\mathbf{Grpd}$.

$$\begin{array}{ccc} \Pi_1 U_1 \cap U_2[A] & \xrightarrow{\Pi_1 i_1[A]} & \Pi_1 U_1[A] \\ {\scriptstyle \Pi_1 i_2[A]}\downarrow & & \downarrow{\scriptstyle \Pi_1 j_1[A]} \\ \Pi_1 U_2[A] & \xrightarrow[\Pi_1 j_2[A]]{} & \Pi_1 X[A] \end{array}$$

2-Category Theory and Π_1 as a costack in $2\mathrm{Grpd}$

Informally, a 2-category is a category in which, for any pair of objects, the corresponding class of morphisms is itself a category. In the context of our investigation, this concept proves to be advantageous as it allows us to develop a theoretical framework capable of establishing a non-trivial universal property for the fundamental groupoid which can provide a categorical axiomatisation of Π_1 for certain classes of topological spaces.

Theorem 5: Let X be a topological space, the 2-functor $\Pi_1 : \mathscr{O}(X) \to \mathbf{2Grpd}$ induced by $U \mapsto \Pi_1(U)$ is a cosheaf and costack of groupoids.

Theorem 6: Let X be a topological space such that any open set U can be covered by simply connected subsets whose pairwise and triple intersections are also simply connected. Then the costackification of the constant strict 2-functor $\mathscr{C} : \mathscr{O}(X) \to \mathbf{2Grpd}$ induced by sending $U \mapsto \mathbf{1}$ is Π_1. In particular, Π_1 is a 2-terminal object in $\mathbf{Cost}_{\mathfrak{U}_X}(X)$.

Conclusion: motivation for further research

The categorical perspective we used to investigate the fundamental groupoid provided us with a comprehensive understanding of its behaviour within a sufficiently nice class of topological spaces. A natural question that arises is how much we can weaken the condition on the topological space in order to keep the same property we proved in Theorem 1.4. It is important to observe that these conditions must remain somewhat restrictive: a pathological topological space failing to have sufficiently many paths would inherently limit the ability of the fundamental groupoid to gather any non-trivial information about the space. In such cases, the very construction of a fundamental groupoid could be meaningless. For this reason, to approach this question, it seems appropriate to restrict ourselves to spaces that are (at least) path connected but not necessarily locally path connected.

Another interesting point, closely related to the above discussion, involves the possibility of extending the definition of fundamental groupoids to generalisations of topological spaces, such as locales or sites. Indeed, we could explore the potential definition of the fundamental groupoid as the costackification of the constant strict 2-functor from a fixed site to the trivial groupoid. Of course, one would need to study under which conditions the costackification exists. Such investigations into broadening the scope of fundamental groupoids to encompass diverse more general structures hold the promise of yielding valuable insights into the intricate relationship between topology and category theory.

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