FINITE DIFFERENCE METHODS IN AERO-ACOUSTICS

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UNDERGRADUATE RESEARCH SUPPORT SCHEME

WARWICK

1. Introduction

One major problem faced by engineers developing jet engines is adhering to stringent aircraft noise regulations. Engines need to be designed to minimise as much noise as possible. To solve such a problem, one cannot find the optimum conditions with pen and paper, and instead must produce computer simulations. Current techniques used in these problems can perform poorly for waves of rapidly varying amplitude (how loud the sound waves are). My research aims to find the optimum schemes while minimising the computational cost and preserving the correct amplitude for aerodynamically generated noise. This will involve a numerical technique called finite differences, which is used for solving partial differential equations modelling the flow of the sound waves.

2. Objective 1

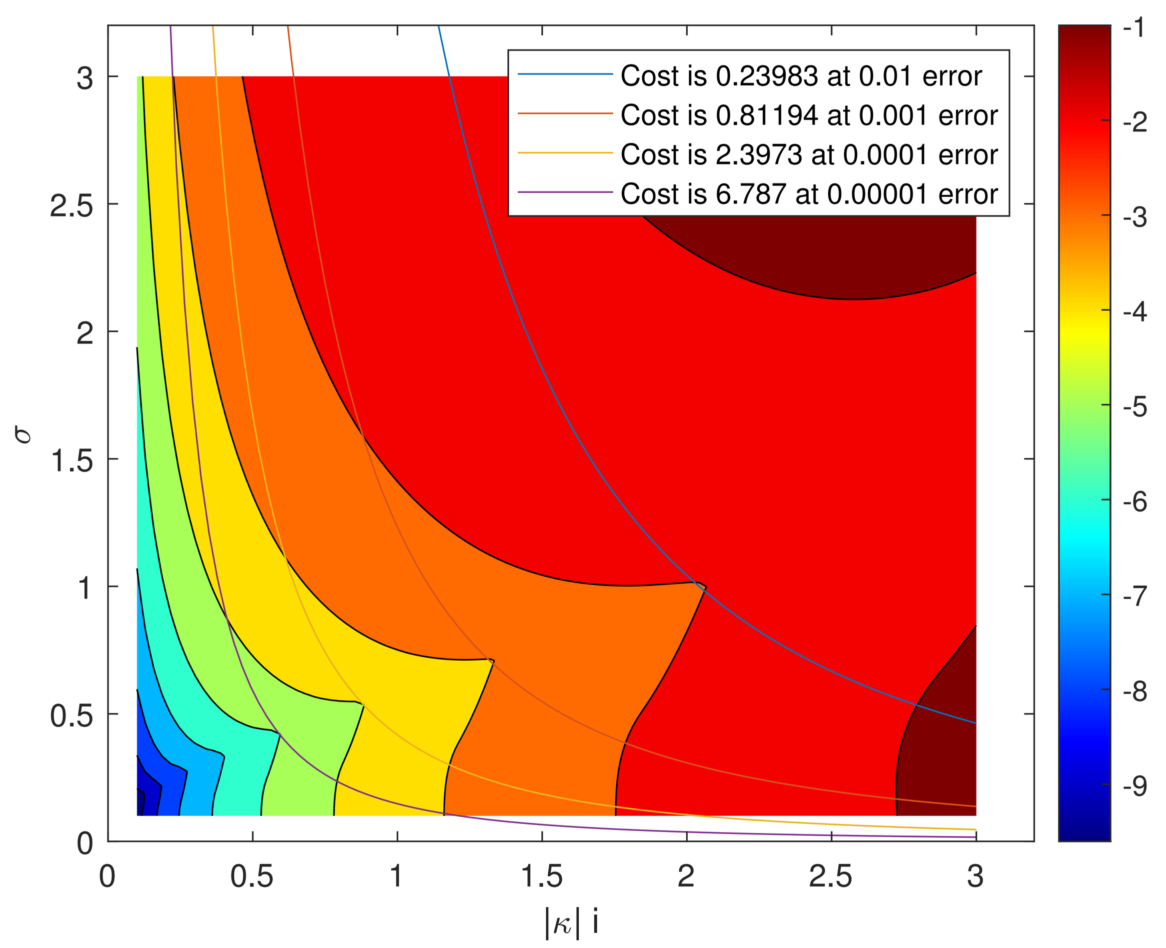
The two quantities one considers when determining the best scheme is the wave number (κ) and the CFL number (σ). The wave number is responsible for the amplitude of the wave, and when this is complex, it allows for the amplitude of the wave to vary correctly. The CFL number determines the number of points we sample on the wave, the more points, the more cost. A signal is a collection of waves of varying amplitude, frequency and speed. We begin with the simplest wave problem, the Advection Equation.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

We then analyse how well real and complex κ perform for this for a signal of varying maximum values $\hat{\kappa}$ and $\hat{\sigma}$.

3. Results 1

Figure 1: imaginary κ



This graph analyses the case which minimises cost for a particular fixed error when κ is purely imaginary. For a signal with the following maximums $\hat{\kappa}$ and $\hat{\sigma}$, we can derive the following equations for computational cost and error.

$$c = N_{op}\frac{1}{\hat{\sigma}\hat{\kappa}^2}$$

$$e = \frac{1}{\hat{\sigma}\hat{\kappa}} \max_{(\kappa,\sigma)\in[0,\hat{\kappa}]x[0,\hat{\sigma}]} |r(\tilde{\kappa},\sigma) - e^{-i\kappa\sigma}|$$

To derive such equations, we combined spatial and temporal schemes as aimed. These were Dispersion Relation Spatial discretisation (DRP) followed by a time stepping Runge Katta scheme (RK4) [1].

4. Relevant Background

Any complex wave number κ can be written as $\kappa = |\kappa|e^{(i\theta)}$. The more we vary θ, the more wave frequencies are permitted. Hence in order to extend the previous error metric, I proposed a few like the following:

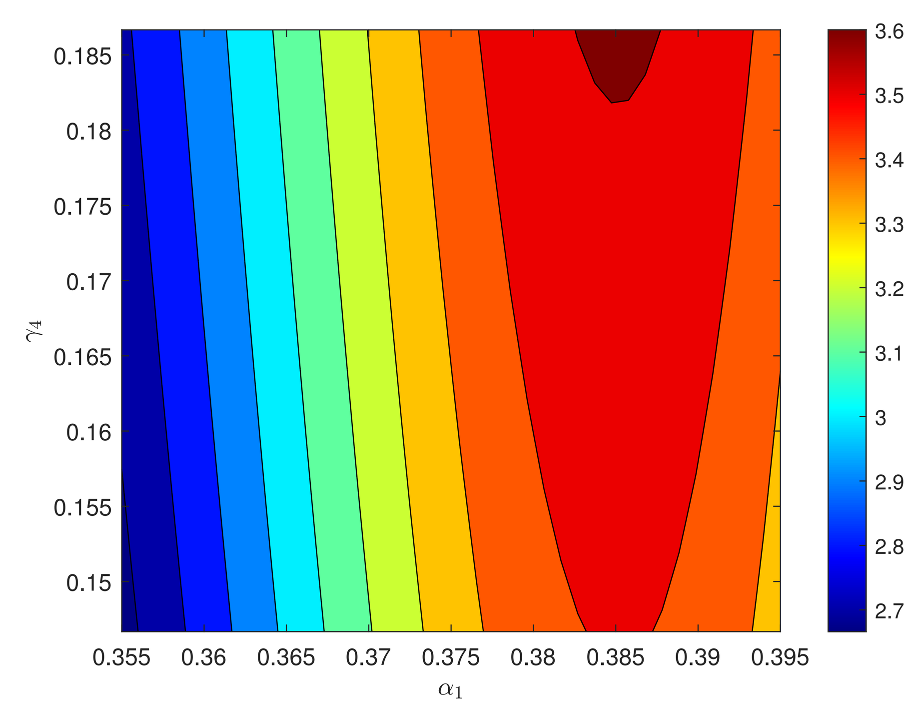
$$E = \frac{1}{\theta_1}\int_0^{\theta_1} \max_{\tilde{\theta}\in[0,\theta]} \frac{1}{\hat{\sigma}\hat{\kappa}} \max_{(0,\hat{\kappa})\text{x}(0,\hat{\sigma})} \left|\frac{r(\tilde{\kappa},\theta,\sigma)}{r_e(|\kappa|,\theta,\sigma)} - 1\right| d\theta$$

The finite differences schemes applied mathematicians often gravitate towards are called *Maximal Order schemes*, which are the best schemes for a very general problem. I aimed to amend these schemes slightly so that they preform better for aero-acoustics problems specifically. The spatial scheme I amended was called *C13* and the temporal scheme called *RK4*. The schemes are determined by coefficients α_1 for *C13* and γ_4 for *RK4*.

5. Objective and Results 2

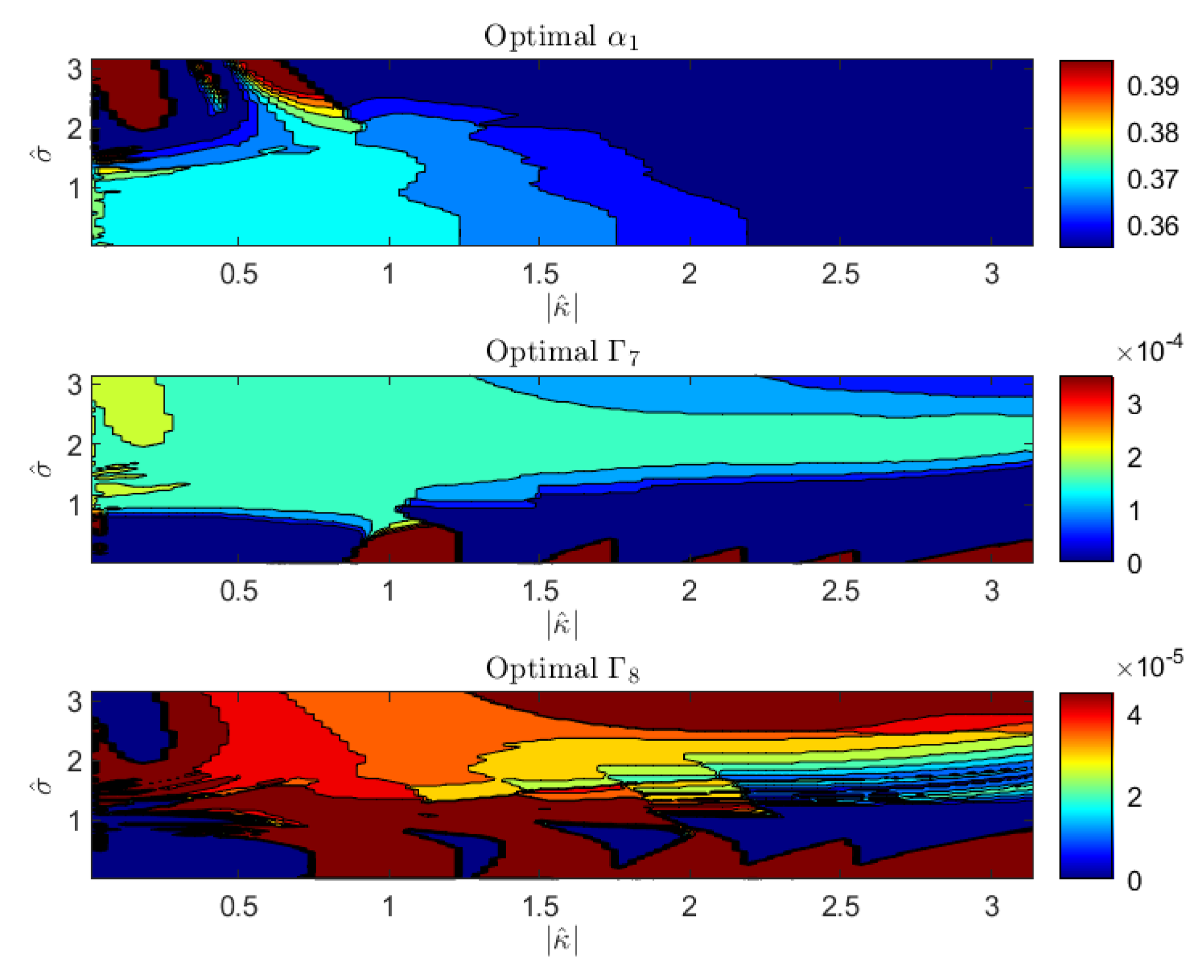
I investigated varying one in the spatial and one in the temporal, and plotted how the error changed for a fixed $|\kappa| = \pi$, $\sigma = \pi$ and $\theta_1 = \pi$. Multiple variations of these parameters were investigated.

Figure 3: log(E)



6. Objective and Results 3

Figure 2: Optimal Values for $\theta_0 = \frac{\pi}{4}$ and $\theta_1 = \frac{3\pi}{4}$



One of the final parts of my project was extending the optimisation of the temporal part of my scheme, by transitioning from *RK4* to *RK8*, which involved optimising over 2 temporal coefficients instead of 1. Hence the 3 values of interest became α_1, Γ_7 and Γ_8. With this extension, I always considered a more complex error metric, given by:

$$E = \frac{1}{\theta_1-\theta_0}\int_0^{|\hat{\kappa}|}\int_{\theta_0}^{\theta_l} \frac{fT}{\hat{\sigma}\frac{1}{P\hat{P}W}} \max_{\sigma\in(0,\hat{\sigma})} |\hat{\kappa}| \left|\frac{r(\tilde{\kappa},\theta,\hat{\sigma})}{r_e(|\kappa|,\theta,\hat{\sigma})} - 1\right| d\theta d|\hat{\kappa}|$$

This investigation allows engineers to more easily tailor their schemes to the problem at hand whilst remaining within their computational cost threshold. One example of a PDE [2] would be:

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial t} = -k_p(x)p \qquad \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = -k_v(x)v$$

References

[1] Nancy Walker. Joint optimisation of spatial and temporal derivatives for varying amplitude waves.

[2] Markeviciute. Optimisation of drp schemes for non-constant amplitude oscillations.

Overview

This is just a flavour at the more basic parts of my research, and the work completed will be continued by the MASDOC research group, led by Ed Brambley. Finite difference optimisation remains of continued importance as new mathematical applications rely on tailored computational software.

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