WEAK NON-LINEAR SOUND IN 2D WAVEGUIDES

HARRY TURNBULL, supervised by
DR ED BRAMBLEY and MR FREDDIE JENSEN at the
UNIVERSITY OF WARWICK, as part of the
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A waveguide (or duct) is any structure which 'guides' sound waves by restricting the direction of propagation. Such structures are everywhere - organ pipes, elephant trunks, engine intakes, and air conditioning ducts are all examples of waveguides; understanding how sound behaves in such objects would therefore be very useful.

There is already a method in the literature that deals with linear acoustics in waveguides but, in practice, this isn't a helpful model. In many applications, the amplitude of sound passing through waveguides is large enough that non-linear effects cannot be ignored. For example, the linear model detects no difference between sound in a trombone and sound in a euphonium.

More recently, this model was extended to a weakly non-linear regime (i.e. where we only neglect terms which are cubic or higher in the pressure and velocity, instead of neglecting quadratic or higher terms as in the linear case). As a result, we get mode-mixing - the fundamental feature in non-linear acoustics that gives rise to effects such as wave-steepening and shocks.

This is the model that we shall work with.

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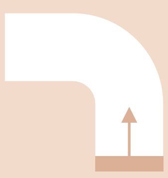
THE MODEL

In fluid dynamics, sound waves are described as periodic perturbations in pressure which don't transfer heat. Here, we are modelling sound propagation in air, so we shall also ignore viscosity. Under these assumptions, we can derive two differential equations (corresponding to conservation of mass and momentum) that relate the pressure and velocity to each other.

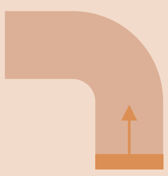
We also assume that our waveguide is rigid - it does not move in response to changes in pressure. This corresponds to imposing a boundary condition that needs to be taken into account when solving the aforementioned equations. By performing some manipulations, we can use this boundary condition to eliminate a component of the velocity, leaving a system of differential equations for the pressure and transverse velocity (i.e. the velocity in the direction parallel to the duct walls).

At this point, the natural thing to do would be to prescribe the velocity and pressure of the air at the inlet and solve these equations along the duct. However, the equations we have are unstable; the key step in this model is to introduce a new quantity called the admittance, which relates the pressure to the transverse velocity in a specific way.

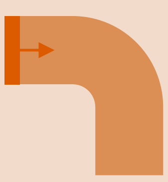
This admittance has a linear and non-linear part - substituting these into the equations we have yields three new equations, which we solve in the following way:



(1) The first equation only involves the linear part of the admittance, so we choose this at the outlet and solve backwards through the duct.



(2) The second involves both the linear and non-linear admittances - we prescribe a non-linear admittance at the outlet and use the linear admittance determined in (1) to solve this equation backwards along the duct.



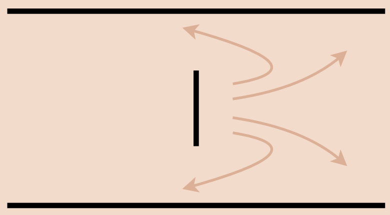
(3) The third involves both admittances and the pressure - after setting the pressure at the inlet, we can use the admittances found above to solve this equation forwards and determine the pressure (and hence the sound) through the duct.

THE PROBLEM

Note that our model requires us to specify the linear and non-linear admittances at the outlet. In fact, this is the essence of what makes solutions to these equations stable - certain choices of admittances ensure that exponentially growing components of the pressure are suppressed, making our solution both physical and stable. In a sense, the admittances at the outlet tell us something about the geometry of the outlet.

Simplifying the geometry of the waveguide leads to a characteristic choice of the admittances. However, the corresponding sound in the model behaves as if it propagates down an infinitely long straight duct (with a width equal to the width of the outlet) after propagating through the waveguide we are studying. Clearly, this is not realistic. Instead, it would be better to choose admittances corresponding to the waveguide opening out into free space. We shall call these choices the open-end admittances. When the non-linear method (see left) was introduced, some values for the open-end admittances were derived but led to poorly behaved solutions. The purpose of this project was to find and justify the choice of open-end admittances which give rise to more physical solutions.

Taking inspiration from the original derivation of the open-end admittances, we considered the duct exit as a source (with a width equal to the width of the outlet) placed at the centre of a much larger, infinitely long, straight duct of constant width. Note that this is therefore only an approximation of the duct opening out into free space, but this choice means that the admittances along the bigger duct are characteristic, giving us extra information to solve for the open-end admittances.



In effect, we suppose that sound does not interact with the walls of the waveguide after it has left the outlet. In the original derivation, the pressure at all points in the duct was taken to be the average of two other solutions corresponding to different geometries, ensuring that the pressure to the left of the source was zero. However, the resulting velocity solution was not continuous (and thus not physical). We instead suppose that both the pressure and velocity solutions are continuous directly above and below the source; to close the system, we also prescribe the impedance (the inverse of admittance) to the left of the source.

We can then use these constraints to determine the open-end admittances in terms of the chosen impedance and the width of the bigger waveguide.

OUTPUT

We can still ensure that the pressure to the left of the source is zero (as in the original derivation) by taking the impedance to be zero. However, computing the open-end admittances in this case numerically is challenging as one of the operators involved is not well-defined.

Here, we get around this issue by choosing the impedance to be small but non-zero. This has the effect of perturbing operators so that they are invertible; in particular, we can compute the open-end admittances. Using this choice, we solved for the sound surrounding the outlet - so far, only the linear pressure has been plotted (see the QR code above).

In many ways, the plots behave as we expect; there are reflections off the walls of the larger duct and beaming at high frequencies. However, there is sound to the left of the source, and these results naturally give rise to several questions:

(1) Is the sound to the left of the source a result of setting the impedance to be non-zero, or is it because of interference due to the geometry?

(2) Are we overprescribing the problem? Can we compute the impedance from the other constraints?

(3) Does plotting the non-linear sound at the outlet produce sensible results?

(4) The values of the admittances depend on the width of the big duct - how does varying this change the resulting sound through the waveguide?