(A very short guide to)

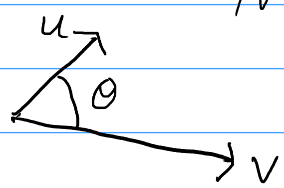
Symmetric Bilinear Forms and the Grothendieck-Witt Group

- Let V_F be a vector space over a field F (or more generally an R-module, if you know what that is)

- A bilinear form on V_F is a function $\beta : V_F \times V_F \to F$ that is linear in each argument i.e.:
$$\beta(u+v,w) = \beta(u,w)+\beta(v,w)$$
$$\beta(\lambda u, v) = \lambda\beta(u,v)$$
$$\beta(u, v+w) = \beta(u,v)+\beta(v,w)$$
$$\beta(u,\lambda v) = \lambda\beta(u,v) \,;$$
for $u,v,w \in V_F$, $\lambda \in F$.

- A bilinear form is called "Symmetric" if $\beta(u,v)=\beta(v,u)$

The usual dot product on $\mathbb{R}^n$ is an example of a symmetric bilinear form; $(- \cdot -) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
$$(u,v) \mapsto |u||v|\cos\theta$$



(From now on, we should really talk about objects called symmetric inner product spaces; these are symmetric bilinear forms satisfying certain non-degeneracy conditions. For the purpose of this short document I will omit this distinction.)

- We can "add" bilinear forms together.

Let U, V be vector spaces with bilinear forms β_1, β_2 respectively. Then we may define a new bilinear form, α, on the orthogonal sum of U and V, $U \oplus V$.

$$\alpha(u_1 \oplus v_1, u_2 \oplus v_2) = \beta_1(u_1, u_2) + \beta_2(v_1, v_2)$$

(We should now consider "isometry classes" of bilinear forms, but I will omit this.)

Fix a field F. This operation gives the set of pairs (V_F, β) the structure of a commutative monoid, where V_F is a vector space over F and β is a symmetric bilinear form on β.

This is almost a group, we just need to be able to "subtract" elements.
Luckily, for every commutative monoid there is an abelian group which may be constructed from it in the "most universal way", called the Grothendieck Group. Denote this as $GW(F)$. It also makes sense to talk about $GW(R)$, where R is a "commutative local ring". Fields are examples of commutative local rings.

When F is a field, $GW(F)$ is well understood.
In fact, the following is known:

- $GW(F)$ is generated as a group by the symbols $\langle u \rangle$, $u \in F\setminus\{0\}$ subject to the following relations:

1) $\langle uv^2 \rangle = \langle u \rangle$
2) $\langle u \rangle + \langle -u \rangle = \langle 1 \rangle + \langle -1 \rangle$
3) $\langle u \rangle + \langle v \rangle = \langle u+v \rangle + \langle (u+v)uv \rangle$ if $u+v \neq 0$.

My project has been investigating $GW(R)$. We have found that if R is a commutative local ring with "infinite residue field", then it is generated in the same way as above! In other words, the presentation when considering fields also holds when considering such a ring.
We have also shown that this is not true of local rings whose residue field is too small.
There is lots more to say but this is probably enough for now!