Staircase Scattering Problem

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Introduction

The Kukulkan Temple (pictured, picture from [5]) has the property that clapping, while standing at the bottom of the staircase as depicted, will produce an echo mimicing the call of the Mexican Quetzal, a local bird. This is just one motivating example of the broader problem of modelling waves interacting with a non-uniform boundary, which finds a wide variety of applications.

Despite the motivation arising from a 3D problem, the surface of the staircase as shown does not change as you move across the face, with the 2D cross-section being the same. In this project we will stick to the (relatively) simple case of modelling the scattering of a point source wave by an infinite staircase in 2D. The next step would of course be to model a semi-infinite staircase, which can be thought of as a temple of infinite height, before then modelling the finite staircase. But we will stick to the simplest case here, which actually proves to be quite challenging.

Setting up the problem

For this kind of modelling problem, a PDE called the Helmholtz equation is often used and will be here too. A difficulty however lies in our boundary conditions due to the infinite staircase, which are unpleasant. Prior work on this project in [1] & [4] first got around this problem of boundary conditions via using such a conformal map between the infinite staircase and the real line (called a *Schwarz-Christoffel mapping*), followed by co-ordinate transformations, to turn the problem from a nice PDE with difficult boundary conditions to a less-nice PDE with straightforward boundary conditions.

The importance of the conformality is that the conformal mapping preserves angles, so the wave components perpendicular and parallel to the straight line boundary will remain so after the conformal mapping. However, the mapping is not however isometric, which gives rise to an inhomogenous Helmholtz equation which is where the main challenge arises.

Post coordinate transforming our problem is now to solve, for φ, the following inhomogenous Helmholtz equation in $\mathbb{H}$ (the upper half plane),

$$(\nabla^2 + k^2)\varphi(a) = k^2(1 - |\tan(a)|)\varphi(a) + \delta(a - a_0) + \delta(a - \overline{a_0})$$

subject to the Von-Neumann boundary conditions $\frac{\partial \varphi}{\partial y} = 0$, $y = 0$. We must also impose the Sommerfield radiation condition, which can be written (parametrising φ)

$$\lim_{r \to \infty} (r^{\frac{1}{2}}(\frac{\partial}{\partial r} - ik)\varphi(r)) = 0$$

which means that the waves die out as they get further away from their source, which in this case is a point source.

An approximation scheme

If a solution to the above PDE even exists, like most 'real-world' modelling problems, an analytic solution is almost certainly impossible. So what we really want to study instead are efficient schemes of approximating a solution to the given PDE. The following, based on the exposition from [2], is such an iterative approximation scheme. Suppose $\varphi_0(a)$ solves the much simpler equation,

$$(\nabla^2 + k^2)\varphi_0(a) = \delta(a - a_0) + \delta(a - \overline{a_0})$$

subject to the same boundary conditions. Physically, this corresponds to a wave originating from a single point source, located in $\mathbb{H}$. The mirror image forcing term realises the Von-Neumann boundary conditions through the method of images. We take this to be our preconditioner, the starting term in what will become our scheme's iteration series. This preconditioner is used because the Sommerfield radiation condition is satisfied and because there is a well known analytic solution for the Green's function, in terms of $H_0^{(1)}$, the Hankel function of the first kind,

$$\varphi_0(a) = G(a;p) = \frac{i}{4}(H_0^{(1)}(k|a - p|) + H_0^{(1)}(k|a - \overline{p}|)).$$

φ_0, the starting term to our approximation, is a good approximation of our inhomogenous PDE 'very far away' from the real line. Take $\varphi_1 := \varphi - \varphi_0$ to be our 'error', the correction that is required the closer we get the boundary. Writing $\varphi = \varphi_1 + \varphi_0$, substituting into the inhomogenous equation and taking the difference, we obtain

$$(\nabla^2 + k^2)\varphi_1(a) = k^2(1 - |\tan(a)|)\varphi(a).$$

We now approximate φ_1, by substiuting $\varphi \approx \varphi_0$ into the right hand side, which can now be solved numerically, given that φ_0 is a known function, to approximate φ_1. Of course, as this is an approximation, we now have a second correction term, φ_2, so we write $\varphi = \varphi_0 + \varphi_1 + \varphi_2$. Substituting this and re-arranging, we obtain

$$(\nabla^2 + k^2)\varphi_2(a) = k^2(1 - |\tan(a)|)(\varphi(a) - \varphi_0(a)).$$

in a similar way to the previous step. As you would expect from the previous step, we again approximate φ_2 by substituting $\varphi - \varphi_0 \approx \varphi_1$ and solving numerically. If we iterate this n times, we will obtain sequence $\varphi_0, \varphi_1, ..., \varphi_n$, with the goal being that their sum provides a good approximation to φ, in other words that the resulting series converges to φ, at least pointwise. The name of this type of series is a Born series.

This of course gives rise to many questions. Under what conditions (if any) does this series converge? Is this series necessarily the 'best of it's kind' for the given type of approximation? Is this even the best kind of approximation to use? Can we approve this approximation in any way? Is there a better approximation that we can use?

Exploring improving this known approximation scheme

We picked a specific φ_0 as our preconditioner that we used to start the iteration process which gives the terms for our series. Naturally, this gives rise to an interesting question - *can we find a better preconditioner than the given* φ_0? And then start the same iteration process, but with this different preconditioner? The chosen preconditioner could have significant implications for the approximation obtained and hence is worth investigating.

When considering this, we are essentially asking 'can I find something that looks like the problem at hand and satisfies all boundary conditions, for which an analytic solution is known but which is not the φ_0 that we already have?'.

As we shall see, the main difficulty comes from our boundary conditions. When searching for different preconditioners we use ψ instead of φ for notational clarity. Looking at

$$(\nabla^2 + k^2)\psi(a) = k^2(1 - |\tan(a)|)\psi(a) + \delta(a - a_0) + \delta(a - \overline{a_0})$$

an alternative approach that comes to mind is if we can use another preconditioner that will turn this into a different, but still familiar PDE for which we have a known analytic solution.

One such candidate could be the Poisson equation. Consider the preconditioner

$$(\nabla^2 + k^2)\psi_0(a) = k^2\psi_0(a) + \delta(a - a_0) + \delta(a - \overline{a_0})$$

basically asking 'what if we start for values where $\tan(a)$ is small?'. Re-arranging,

$$\nabla^2\psi_0(a) = \delta(a - a_0) + \delta(a - \overline{a_0})$$

which is a Poisson equation in $\mathbb{H}$ with point forcing terms. This has a known analytic solution in $n = 2$,

$$\psi_0(a) = \frac{\log|a - a_0|}{|\partial B_1(a_0)|} + \frac{\log|a - \overline{a_0}|}{|\partial B_1(\overline{a_0})|}$$

which clearly does not decay and does not satisfy the Sommerfield radiation condition, so is not a good candidate for a preconditioner with which to start our scheme.

Finding a better candidate for an alternative preconditioner and then comparing their performance could be one direction of further work on this. However, examples of inhomogenous Helmholtz equations with known *analytic* solutions in the literature were rare.

It could well be that this is the best starting candidate we will have for the given approximation and hence we will explore some other directions.

Exploring alternative improvements

Accelerating convergence of the series

Another thing to explore is to see if we can 'get more of the series for less effort'. Something potentially useful here is a paper, [4], concerning a change of variables method to make a convergent Born series 'converge faster'. The change of variables 'reshuffles' the series, so any truncation for a given n better approximates the final convergent than the original series truncated for the same n, making approximating our solution less computationally expensive. Solving the inhomogenous Helmholtz equation can be reformulated in terms of linear operators (explored in [3]). This linear operator reformulation allows us to re-write the Born series on the left hand side as below (where T is the specific linear operator pertaining to our problem), with the form on the right being called a specific case of what is in general called a Neumann Series.

$$\sum_{n=0}^{\infty} \varphi_n = \sum_{n=0}^{\infty} T^n \varphi_0.$$

[4] defines and discusses a 'convergence factor' for a generalised Neumann series and then introduces a novel change of variables method which (provided suitable conditions are satisfied) decreases the convergence factor, increasing the speed of convergence. Further studying and attempting to apply this method (or modify it to be applied) to our problem could be another direction for future work on this project.

Alternative computation of each iteration

As previously discussed, for the nth step of our iteration we have to solve numerically

$$(\nabla^2 + k^2)\varphi_n(a) = k^2(1 - |\tan(a)|)\varphi_{n-1}(a)$$

in $\mathbb{H}$, for known φ_{n-1}, with the same boundary conditions as the original problem. Being able to do this quickly will speed up our iteration. When solving this for φ_n, we can think of this as trying to invert the Helmholtz differential operator which is applied to $\varphi_n(a)$ on the left hand side.

A recent paper, [1], published while I was undertaking this project provides a novel, faster method for inverting this differential operator, under conditions with similar (but not exactly the same) boundary conditions to the problem we have - the major difference being that the domain in this formulation is bounded, whereas ours is not in theory - but must be in practice for any numerics that are run. The method works by computing two sequential 'inverse square-root pseudodifferential operators', instead of directly inverting. These can be represented in a form they refer to as the 'operator Fourier transform' and then computed by more standard numerical methods. Again, further studying and then trying to modify and implement this method for the problem at hand, then comparing against previous numerics, could be another direction for future work on this project.

References

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