ON THE AVERAGE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION

M. KAC

1. **Introduction.** Consider the algebraic equation

$$X_0 + X_1 x + X_2 x^2 + \cdots + X_{n-1} x^{n-1} = 0, \tag{1}$$

where the X's are independent random variables assuming real values only, and denote by $N_n = N(X_0, \cdots, X_{n-1})$ the number of real roots of (1). We want to determine the mean value (mathematical expectation = m.e.) of N_n when all X's have the same normal distribution with density

$$e^{-u^2}/\pi^{1/2}. \tag{2}$$

$$\text{m.e. } \{N_n\} = \frac{4}{\pi}\int_0^1 \frac{[1 - n^2[x^2(1-x^2)/(1-x^{2n})]^2]^{1/2}}{1-x^2}\,dx$$

HOW MANY ZEROS OF A RANDOM POLYNOMIAL ARE REAL?

ALAN EDELMAN AND ERIC KOSTLAN

Definition 2.1. If $P \in S^n$ is any point, the *associated equator* $P_\perp$ is the set of points of S^n on the plane perpendicular to the line from the origin to P.

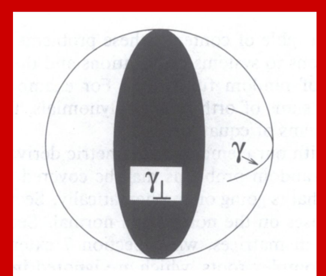
Definition 2.2. Let $\gamma_\perp$, the *equators of a curve*, be the set $\{P_\perp | P \in \gamma\}$.

Definition 2.3. The *multiplicity* of a point $Q \in \cup\gamma_\perp$ is the number of equators in $\gamma_\perp$ that contain Q, i.e., the cardinality of $\{t \in \mathbb{R} | Q \in \gamma(t)_\perp\}$.

Definition 2.4. We define $|\gamma_\perp|$ to be the area of $\cup\gamma_\perp$ counting multiplicity. More precisely, we define $|\gamma_\perp|$ to be the integral of the multiplicity over $\cup\gamma_\perp$.

Lemma 2.1. *If γ is a rectifiable curve, then*

$$\frac{|\gamma_\perp|}{\text{area of } S^n} = \frac{|\gamma|}{\pi}. \tag{1}$$



The curve in $\mathbb{R}^{n+1}$ traced out by $v(t)$ as t runs over the real line is called the *moment curve.*

The condition that $x = t$ is a zero of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ is precisely the condition that a is perpendicular to $v(t)$. Another way of saying this is that $v(t)_\perp$ is the set of polynomials which have t as a zero.

2.2. **The expected number of real zeros of a random polynomial.** What does the geometric argument in the previous section and formula (1) in particular have to do with the number of real zeros of a random polynomial? Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

be a non-zero polynomial. Define the two vectors

$$a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad v(t) = \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix}.$$

Theorem 2.1 (Kac formula). *The expected number of real zeros of a degree n polynomial with independent standard normal coefficients is*

$$E_n = \frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)^2 t^{2n}}{(t^{2n+2}-1)^2}}\,dt = \frac{4}{\pi}\int_0^1\sqrt{\frac{1}{(1-t^2)^2} - \frac{(n+1)^2 t^{2n}}{(1-t^{2n+2})^2}}\,dt. \tag{2}$$

$$E_n = \frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{\frac{\partial^2}{\partial x \partial y}\log\frac{1-(xy)^{n+1}}{1-xy}\Big|_{y=x=t}}\,dt.$$

Theorem 3.1. *Let $v(t) = (f_0(t), \ldots, f_n(t))^T$ be any collection of differentiable functions and $a_0, \ldots, a_n$ be the elements of a multivariate normal distribution with mean zero and covariance matrix C. The expected number of real zeros on an interval (or measurable set) I of the equation*

$$a_0 f_0(t) + a_1 f_1(t) + \cdots + a_n f_n(t) = 0$$

is

$$\int_I \frac{1}{\pi}\|\mathbf{w}'(t)\|\,dt,$$

where $\mathbf{w}$ is defined by Equations (7). In logarithmic derivative notation this is

$$\frac{1}{\pi}\int_I\left(\frac{\partial^2}{\partial x \partial y}\left(\log v(x)^T C v(y)\right)\Big|_{y=x=t}\right)^{1/2} dt.$$

3.1. **Random polynomials.**
3.1.1. *The Kac formula.* If the coefficients of random polynomials are independent standard normal random variables, we saw in the previous section that from

$$v(x)^T C v(y) = \frac{1-(xy)^{n+1}}{1-xy}, \tag{8}$$

we can derive the Kac formula.

3.2. **Random infinite series.**
3.2.1. *Power series with uncorrelated coefficients.* Consider a random power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots,$$

where a_k are independent standard normal random variables. This has radius of convergence one with probability one. Thus we will assume that $-1 < x < 1$. In this case,

$$v(x)^T C v(y) = \frac{1}{1-xy}.$$

The logarithmic derivative reveals a density of zeros of the form

$$\rho(t) = \frac{1}{\pi(1-t^2)}.$$

3.1.2. *A random polynomial with a simple answer.* Consider random polynomials

$$a_0 + a_1 x + \cdots + a_n x^n,$$

where the a_i are independent normals with variances $\binom{n}{i}$. Such random polynomials have been studied because of their mathematical properties [31, 46] and because of their relationship to quantum physics [4].

By the binomial theorem,

$$v(x)^T C v(y) = \sum_{k=0}^{n}\binom{n}{k}x^k y^k = (1+xy)^n.$$

We see that the density of zeros is given by

$$\rho(t) = \frac{\sqrt{n}}{\pi(1+t^2)}.$$

Random Polynomials

Nikita Brykov Supervised by Roger Tribe

University of Warwick

The Complex Zeros of Random Polynomials

Larry A. Shepp
University of Pennsylvania

Robert J. Vanderbei

Theorem 1.1. *For each region $\Omega \in \mathbf{C}$,*

$$\mathbf{E}\nu_n(\Omega) = \int_\Omega h_n(x,y)\,dxdy + \int_{\Omega\cap\mathbf{R}} g_n(x)\,dx, \tag{1.3}$$

where

$$h_n = \frac{B_2 D_0^2 - B_0(B_1^2 + |A_1|^2) + B_1(A_0\bar{A}_1 + \bar{A}_0 A_1)}{\pi|z|^2 D_0^3},$$

and

$$g_n = \frac{(B_0 B_2 - B_1^2)^{1/2}}{\pi|z|B_0}.$$

ABSTRACT. Mark Kac gave an explicit formula for the expectation of the number, $\nu_n(\Omega)$, of zeros of a random polynomial,

$$P_n(z) = \sum_{j=0}^{n-1}\eta_j z^j,$$

in any measurable subset Ω of the *reals*. Here, $\eta_0, \ldots, \eta_{n-1}$ are independent standard normal random variables. In fact, for each $n > 1$, he obtained an explicit intensity function g_n for which

$$\mathbf{E}\nu_n(\Omega) = \int_\Omega g_n(x)\,dx.$$

Here, we extend this formula to obtain an explicit formula for the expected number of zeros in any measurable subset Ω of the complex plane $\mathbf{C}$. Namely, we show that

$$\mathbf{E}\nu_n(\Omega) = \int_\Omega h_n(x,y)\,dxdy + \int_{\Omega\cap\mathbf{R}} g_n(x)\,dx,$$

where h_n is an explicit intensity function. We also study the asymptotics of h_n showing that for large n its mass lies close to, and is uniformly distributed around, the unit circle.

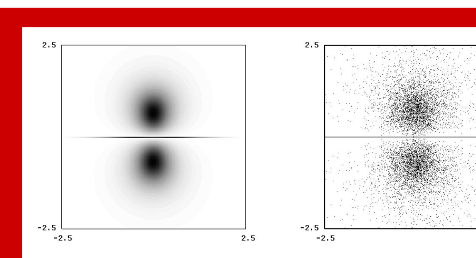


FIGURE 1. *Random quadratics* ($n = 3$). In this figure and the next several, the left-hand plot is a grey-scale image of the intensity functions h_n and g_n (which is concentrated on the x-axis). The right-hand plot shows 20,000 roots from randomly generated polynomials. Subsequent captions give more information pertaining to each of these figures.

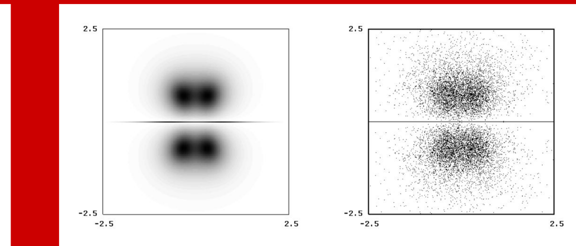


FIGURE 2. *Random cubics* ($n = 4$). Note that, for the left-hand plots, the grey-scales for h_n and g_n are produced separately and in such a way that both use the full range from white to black.

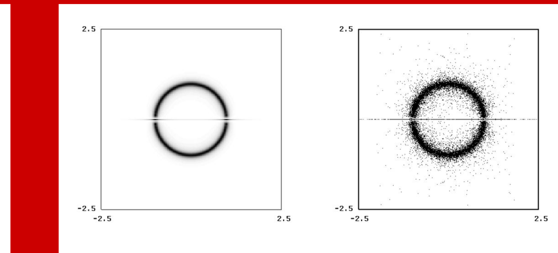


FIGURE 5. $n = 36$.

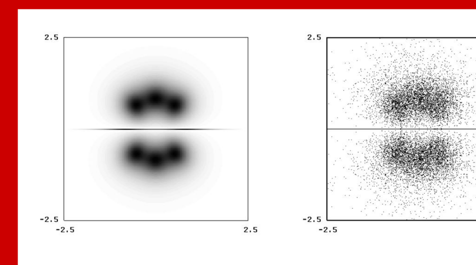


FIGURE 3. *Random quartics* ($n = 5$). Note that, for the right-hand plots, most if not all of the "pixels" on the x-axis have been hit by at least one root. A more accurate image for the x-axis would have been obtained had we used a grey-scale to indicate how many times each pixel on the x-axis was hit.

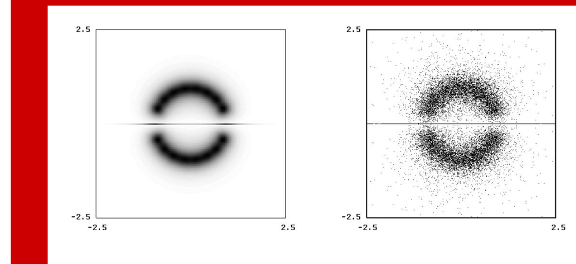


FIGURE 4. $n = 10$.

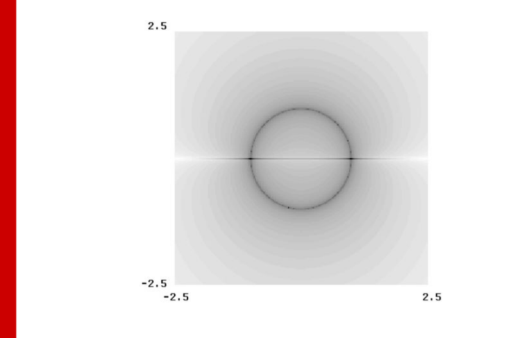


FIGURE 6. *Limiting intensity* (using a logarithmic grey-scale).

Zeros of Gaussian Analytic Functions and Determinantal Point Processes

John Ben Hough
Manjunath Krishnapur
Yuval Peres
Bálint Virág

Preliminaries

Joint intensities are akin to densities: Assume that $\mathscr{X}$ is simple. Then, the joint intensity functions may be interpreted as follows.

- If Λ is finite and μ is the *counting measure* on Λ, i.e., the measure that assigns unit mass to each element of Λ, then for distinct $x_1, \ldots, x_k$, the quantity $\rho_k(x_1, \ldots, x_k)$ is just the probability that $x_1, \ldots, x_k \in \mathscr{X}$.

$$\rho_k(x_1, \ldots, x_k) = \lim_{\epsilon\to 0}\frac{\mathbf{P}(\mathscr{X} \text{ has a point in } B_\epsilon(x_j) \text{ for each } j \le k)}{m(B_\epsilon)^k}$$

A **standard complex Gaussian** is a complex-valued random variable with probability density $\frac{1}{\pi}e^{-|z|^2}$ w.r.t the Lebesgue measure on the complex plane. Equivalently, one may define it as $X + iY$, where X and Y are i.i.d. $N(0, \frac{1}{2})$ random variables.

Let a_k, $1 \le k \le n$ be i.i.d. standard complex Gaussians. Then we say that $\mathbf{a} := (a_1, \ldots, a_n)^t$ is a standard complex Gaussian vector. Then if B is a (complex) $m \times n$ matrix, $B\mathbf{a} + \mu$ is said to be an m-dimensional complex Gaussian vector with mean μ (an $m \times 1$ vector) and covariance $\Sigma = BB^*$ (an $m \times m$ matrix). We denote its distribution by $N_{\mathbb{C}}^m(\mu, \Sigma)$.

- If $\mathbf{a}$ has $N_{\mathbb{C}}^m(\mu, \Sigma)$ distribution, then for every $j, k \le n$ (not necessarily distinct), we have
$$\mathbf{E}\left[(a_k - \mu_k)(a_j - \mu_j)\right] = 0 \text{ and } \mathbf{E}\left[(a_j - \mu_j)\overline{(a_k - \mu_k)}\right] = \Sigma_{j,k}.$$
- If a is a standard complex Gaussian, then $|a|^2$ and $\frac{a}{|a|}$ are independent, and have exponential distribution with mean 1 and uniform distribution on the circle $\{z : |z| = 1\}$, respectively.
- Suppose $\mathbf{a}$ and $\mathbf{b}$ are m and n-dimensional random vectors such that
$$\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} \sim N_{\mathbb{C}}^{m+n}\left(\begin{bmatrix}\mu\\ \nu\end{bmatrix}, \begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}\right),$$
where the mean vector and covariance matrices are partitioned in the obvious way. Then Σ_{11} and Σ_{22} are Hermitian, while $\Sigma_{12}^* = \Sigma_{21}$. Assume that Σ_{11} is non-singular. Then the distribution of $\mathbf{a}$ is $N_{\mathbb{C}}^m(\mu, \Sigma_{11})$ and the conditional distribution of $\mathbf{b}$ given $\mathbf{a}$ is
$$N_{\mathbb{C}}^n\left(\nu + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{a} - \mu), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right).$$

LEMMA 2.1.7 (Wick formula). *Let $(\mathbf{a}, \mathbf{b}) = (a_1, \ldots, a_n, b_1, \ldots b_n)^t$ have $N_{\mathbb{C}}(\mathbf{0}, \Sigma)$ distribution, where*

$$\Sigma = \begin{bmatrix}\Sigma_{1,1} & \Sigma_{1,2}\\ \Sigma_{2,1} & \Sigma_{2,2}\end{bmatrix}.$$

Then,

$$\mathbf{E}\left[a_1 \cdots a_n \bar{b}_1 \ldots \bar{b}_n\right] = \text{per}(\Sigma_{1,2}).$$

In particular

$$\mathbf{E}\left[|a_1 \cdots a_n|^2\right] = \text{per}(\Sigma_{1,1}).$$

Gaussian analytic functions

DEFINITION 2.2.1. Let $\mathbf{f}$ be a random variable on a probability space taking values in the space of analytic functions on a region $\Lambda \subset \mathbb{C}$. We say $\mathbf{f}$ is a Gaussian analytic function (GAF) on Λ if $(\mathbf{f}(z_1), \ldots, \mathbf{f}(z_n))$ has a mean zero complex Gaussian distribution for every $n \ge 1$ and every $z_1, \ldots, z_n \in \Lambda$.

- $\{\mathbf{f}^{(k)}\}$ are jointly Gaussian, i.e., the joint distribution of $\mathbf{f}$ and finitely many derivatives of $\mathbf{f}$ at finitely many points,
$$\left\{\mathbf{f}^{(k)}(z_j) : 0 \le k \le n, 1 \le j \le m\right\},$$
has a (mean zero) complex Gaussian distribution. (Hint: Weak limits of Gaussians are Gaussians and derivatives are limits of difference coefficients).
- For any $n \ge 1$ and any $z_1, \ldots, z_n \in \Lambda$, the random vector $(\mathbf{f}(z_1), \ldots, \mathbf{f}(z_n))$ has a complex Gaussian distribution with mean zero and covariance matrix $\left(K(z_i, z_j)\right)_{i,j\le n}$. By Exercise 2.1.1 it follows that the covariance kernel K determines all the finite dimensional marginals of $\mathbf{f}$. Since $\mathbf{f}$ is almost surely continuous, it follows that the distribution of $\mathbf{f}$ is determined by K.
- Analytic extensions of GAFs are GAFs.

EXERCISE 2.1.4. If $\mathbf{a}_n$ has $N_{\mathbb{C}}(\mu_n, \Sigma_n)$ distribution and $\mathbf{a}_n \xrightarrow{d} \mathbf{a}$, then $\{\mu_n\}$ and $\{\Sigma_n\}$ must converge, say to μ and Σ, and $\mathbf{a}$ must have $N_{\mathbb{C}}(\mu, \Sigma)$ distribution.

DEFINITION 2.1.5. For an $n \times n$ matrix M, its permanent, denoted per(M) is defined by

$$\text{per}(M) = \sum_{\pi\in S_n}\prod_{k=1}^{n} M_{k\pi_k}.$$

The sum is over all permutations of $\{1, 2, \ldots, n\}$.

REMARK 2.1.6. The analogy with the determinant is clear - the signs of the permutations have been omitted in the definition. But note that this makes a huge difference in that per$(A^{-1}MA)$ is not in general equal to per(M). This means that the permanent is a basis-dependent notion and thus has no geometric meaning unlike the determinant. As such, it can be expected to occur only in those contexts where the entries of the matrices themselves are important, as often happens in combinatorics and also in probability.

The following lemma gives a general recipe to construct Gaussian analytic functions.

LEMMA 2.2.3. *Let ψ_n be holomorphic functions on Λ. Assume that $\sum_n |\psi_n(z)|^2$ converges uniformly on compact sets in Λ. Let a_n be i.i.d. random variables with zero mean and unit variance. Then, almost surely, $\sum_n a_n\psi_n(z)$ converges uniformly on compact subsets of Λ and hence defines a random analytic function.*

In particular, if a_n has standard complex Gaussian distribution, then $\mathbf{f}(z) := \sum_n a_n\psi_n(z)$ is a GAF with covariance kernel $K(z,w) = \sum_n \psi_n(z)\overline{\psi_n}(w)$.

Distribution of zeros - The first intensity

2.4.2. First intensity by linearization. This is a more probabilistic approach. Let $z \in \Lambda$. We want to estimate the probability that $\mathbf{f}(w) = 0$ for some $w \in D(z, \epsilon)$, up to order ϵ^2. Expand $\mathbf{f}$ as a power series around z:

$$\mathbf{f}(w) = \mathbf{f}(z) + \mathbf{f}'(z)(w-z) + \mathbf{f}''(z)\frac{(w-z)^2}{2!} + \ldots$$

The idea is that up to an event of probability $o(\epsilon^2)$, $\mathbf{f}$ and its linear approximant,

$$\mathbf{g}(w) := \mathbf{f}(z) + (w-z)\mathbf{f}'(z),$$

$$\begin{aligned}\rho_1(z) &= \lim_{\epsilon\to 0}\frac{\mathbb{E}[f \text{ has a zero in } D(z,\epsilon)]}{V(D(z,\epsilon))}\\ &= \lim_{\epsilon\to 0}\frac{\mathbb{P}(\text{f has a zero in } D(z,\epsilon))}{\pi\epsilon^2}\\ &= \lim_{\epsilon\to 0}\frac{\mathbb{P}(\text{g has a zero in } D(z,\epsilon))}{\pi\epsilon^2}\\ &= \lim_{\epsilon\to 0}\frac{\mathbb{P}\left(\frac{-f(z)}{f'(z)} \in D(0,\epsilon)\right)}{\pi\epsilon^2}\\ &= \text{Probability density of } \frac{-f(z)}{f'(z)} \text{ at } 0.\end{aligned}$$

Definition 1 (First Intensity). *The first intensity is given by*

$$\rho_1(z) = \lim_{\epsilon\to 0}\frac{\mathbb{E}[f \text{ has a zero in } D(z,\epsilon)]}{V(D(z,\epsilon))}.$$

$$\begin{aligned}\mathbb{E}[\text{number of zeros of } f \text{ in } D(z,\epsilon)] = {} & 1\cdot\mathbb{P}(f \text{ has a zero in } D(z,\epsilon))\\ & + 2\cdot\mathbb{P}(f \text{ has 2 zeros in } D(z,\epsilon))\\ & + 3\cdot\mathbb{P}(f \text{ has 3 zeros in } D(z,\epsilon)) + \ldots\end{aligned}$$

And because the later terms are of order ϵ^4

$$\begin{aligned}\mathbb{E}[\text{number of zeros of } f \text{ in } D(z,\epsilon)] = {} & 1\cdot\mathbb{P}(f \text{ has a zero in } D(z,\epsilon))\\ & + 1\cdot\mathbb{P}(f \text{ has 2 zeros in } D(z,\epsilon))\\ & + 1\cdot\mathbb{P}(f \text{ has 3 zeros in } D(z,\epsilon)) + \ldots + O(\epsilon^4)\end{aligned}$$

have the same number of zeros in $D(z,\epsilon)$. Assuming this, it follows from (1.2.8) that

$$\begin{aligned}\rho_1(z) &= \lim_{\epsilon\to 0}\frac{\mathbf{P}[\mathbf{f} \text{ has a zero in } D(z,\epsilon)]}{\pi\epsilon^2}\\ &= \lim_{\epsilon\to 0}\frac{\mathbf{P}[\mathbf{g} \text{ has a zero in } D(z,\epsilon)]}{\pi\epsilon^2}\\ &= \lim_{\epsilon\to 0}\frac{\mathbf{P}\left[\frac{-\mathbf{f}(z)}{\mathbf{f}'(z)} \in D(0,\epsilon)\right]}{\pi\epsilon^2}\\ &= \text{Probability density of } \frac{-\mathbf{f}(z)}{\mathbf{f}'(z)} \text{ at } 0.\end{aligned}$$

If a, b are complex-valued random variables then, by an elementary change of variables, we see that the density of a/b at 0 is equal to $\chi_a(0)\mathbf{E}\left[|b|^2 \mid a = 0\right]$, where χ_a is the density of a at 0 (assuming the density a and the second moment of b given $a = 0$ do exist).

When $\mathbf{f}$ is Gaussian, $(\mathbf{f}(z), \mathbf{f}'(z))$ is jointly complex Gaussian with mean zero and covariance

$$\begin{bmatrix}K(z,z) & \frac{\partial}{\partial\bar{z}}K(z,z)\\ \frac{\partial}{\partial z}K(z,z) & \frac{\partial}{\partial z}\frac{\partial}{\partial\bar{z}}K(z,z)\end{bmatrix}.$$

Here we use the standard notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The density of $\mathbf{f}(z)$ at 0 is $\frac{1}{\pi K(z,z)}$. Moreover, $\mathbf{f}'(z)\big|_{\mathbf{f}(z)=0}$ has

$$N_{\mathbb{C}}\left(0, \frac{\partial}{\partial z}\frac{\partial}{\partial\bar{z}}K(z,z) - \frac{1}{K(z,z)}\left(\frac{\partial}{\partial\bar{z}}K(z,z)\right)\left(\frac{\partial}{\partial z}K(z,z)\right)\right)$$

distribution. Thus we can write the first intensity as

$$\rho_1(z) = \frac{\frac{\partial}{\partial z}\frac{\partial}{\partial\bar{z}}K(z,z) - \frac{1}{K(z,z)}\frac{\partial}{\partial\bar{z}}K(z,z)\frac{\partial}{\partial z}K(z,z)}{\pi K(z,z)}.$$

This is equivalent to the Edelman-Kostlan formula (2.4.8) as can be seen by differentiating $\log K(z,z)$ (since $\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial\bar{z}}$).

LEMMA 2.4.2. *Let $\mathbf{f}(z) = a_0 + a_1 z + \ldots$ be a GAF. Assume that a_0 is not constant. Let A_ϵ denote the event that the number of zeros of $\mathbf{f}$ in the disk $D(0,\epsilon)$ differs from the number of zeros of $\mathbf{g}(z) := a_0 + a_1 z$ in the same disk. Then for any $\delta > 0$, there exists $c > 0$ so that for all $\epsilon > 0$ we have*

$$\mathbf{P}[A_\epsilon] \le c\epsilon^{3-2\delta}.$$

Joint Intensities

COROLLARY 3.4.2 (Density formula for Gaussian analytic functions). *Let $\mathbf{f}$ be a Gaussian analytic function on Λ with covariance kernel K. If $\det K(z_i, z_j)_{i,j\le k}$ does not vanish anywhere on Λ, then the k-point intensity function ρ_k exists and is given by*

$$\rho_k(z_1, \ldots, z_k) = \frac{\mathbf{E}\left[|\mathbf{f}'(z_1)\cdots\mathbf{f}'(z_k)|^2 \mid \mathbf{f}(z_1) = \ldots = \mathbf{f}(z_k) = 0\right]}{\pi^k \det K(z_i, z_j)_{i,j\le k}}. \tag{3.4.1}$$

Equivalently,

$$\rho_k(z_1, \ldots, z_k) = \frac{\text{per}(C - BA^{-1}B^*)}{\det(\pi A)}, \tag{3.4.2}$$

where A, B, C are $k \times k$ matrices defined by

$$\begin{aligned}A(i,j) &= \mathbf{E}\left[\mathbf{f}(z_i)\overline{\mathbf{f}}(z_j)\right]\\ B(i,j) &= \mathbf{E}\left[\mathbf{f}'(z_i)\overline{\mathbf{f}}(z_j)\right]\\ C(i,j) &= \mathbf{E}\left[\mathbf{f}'(z_i)\overline{\mathbf{f}'}(z_j)\right]\end{aligned}$$

PROOF. [Proof of Corollary 3.4.2] Exercise 3.4.1 yields (3.4.1). Exercise 2.1.3 tells us that given $\{\mathbf{f}(z_i) = 0, 1 \le i \le k\}$, the conditional distribution of $(\mathbf{f}'(z_1), \ldots, \mathbf{f}'(z_k))$ is again complex Gaussian with zero mean and covariance $C - BA^{-1}B^*$. Apply Wick formula (Lemma 2.1.7) to obtain (3.4.2). ∎

References

[1] Hough, J. B., Krishnapur, M., Peres, Y., & Virág, B. (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes.* American Mathematical Society.

[2] Edelman, A., & Kostlan, E. (1995). How Many Zeros of a Random Polynomial Are Real?. *Bulletin of the American Mathematical Society*, 32(1), 1-37. https://doi.org/10.1090/S0273-0979-1995-00571-4

[3] Shepp, L. A., & Vanderbei, R. J. (1995). The Complex Zeros of Random Polynomials. *Transactions of the American Mathematical Society*, 347(11), 4365-4384. https://repository.upenn.edu/statistics$_papers$/187

[4] Kac, M. (1943). On the Average Number of Real Roots of a Random Algebraic Equation. *American Journal of Mathematics*, 65(2), 609-620.