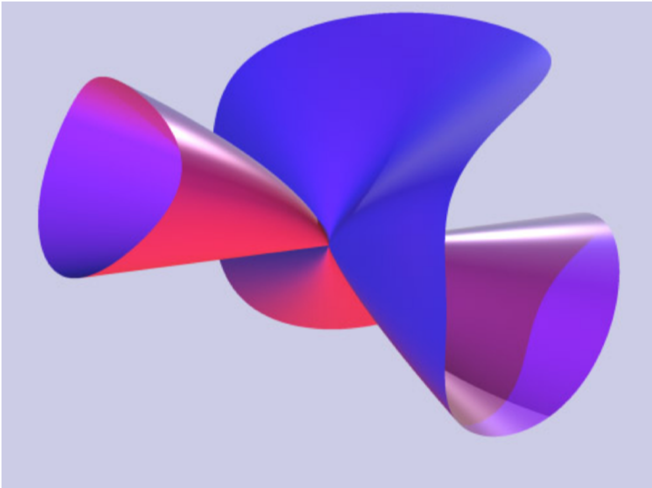
Lines on Non-Singular Cubic Surfaces

By Jack Lucas

Ever since the eighteenth century, mathematicians have been interested in finding lines on surfaces. The lines we are referring to here are straight lines on curved surfaces like the example below.



In algebraic geometry we often come across many surfaces containing infinitely many lines. A good example is a plane, where we can draw lines at any angle or through any point. However, sufficiently nice cubic surfaces are interesting since they always only contain a finite number, and this is made even more strange by the fact they have exactly 27.

This does beg the question about what is a cubic surface? To understand this, we consider an example. Most people will have seen the equation $x^2 + y^2 = 1$ for a circle. This is one dimensional since if we were standing on the surface of the circle there is only one axis we can move on. We can extend this to the equation $x^2 + y^2 + z^2 = 1$ which describes a sphere in three-dimensional space. Now if we were standing on top of the sphere we could move in 2 different axes, therefore making the surface two-dimensional itself.

Both a circle and a sphere are degree two surfaces since the highest power of the any element (where we add together the powers for example $x^2y^3 \ + z = 1$ has degree 5) another example would be surface defined by the equation $xy + yz + xz = 1$. A cubic is a degree three surface, for example $x^2y + 2z^3 + 33xz + y = 127$ or $z^3 = 1$.

When looking at lines on these surfaces we must look at solutions (x, y, z) to these equations in the complex numbers. If we only look at the solutions in the real numbers, there can be either 3, 7, 15 or 27 lines.

The way we come to this number requires some smart geometric thinking as well as a combinatorics-based approach. The general direction is to prove that there exists one line on the surface and then show that there are rules for how many lines intersect this, and then much like a sudoku puzzle fill in what the remaining lines must be.

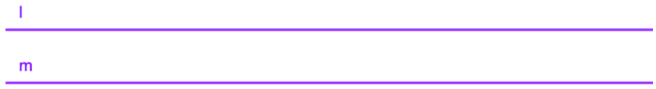
To work through this proof, we must first make some rather random and contrived assumptions, however the reasons why we can make these assumptions is heavily algebraic and beyond the scope of this poster. We assume the following:

1. There is at least one line on the surface
2. Each line intersects exactly 10 other lines in 5 planes (we call these tritangent planes and these are explored in greater depth below).
3. There are 2 lines on the surface that don't intersect
4. Every line intersects every plane

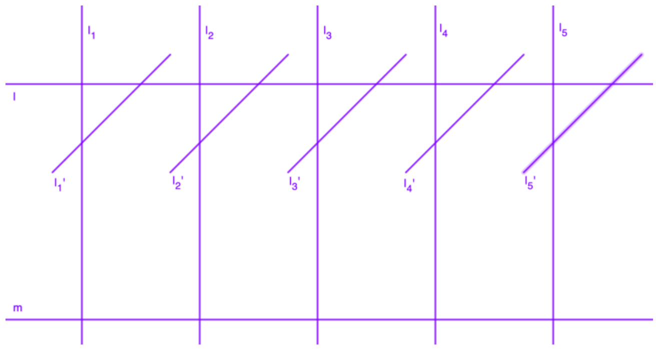
We first try to unpack the terminology used in the second assumption. A tangent to a surface is a plane which itself is tangent to all curves on the surface. If a tangent plane exists at every point, we call the surface non-singular, and these are the group of surfaces we are trying to show have this property of containing 27 lines. If we look at the intersection of the tangent plane with the surface, we either get a cubic curve (defined much the same line as a cubic surface), a conic (degree 2 curve) and a line, or three lines as shown at the top of the next column.

If there is a line on the surface, this line will be tangent to the surface and will therefore be in any tangent plane to the surface at any point along the line. From this we just now only draw our attention to the planes containing exactly three lines which we call tritangent planes. From the second assumption we know there are each line is contained in exactly 5 tritangent planes.

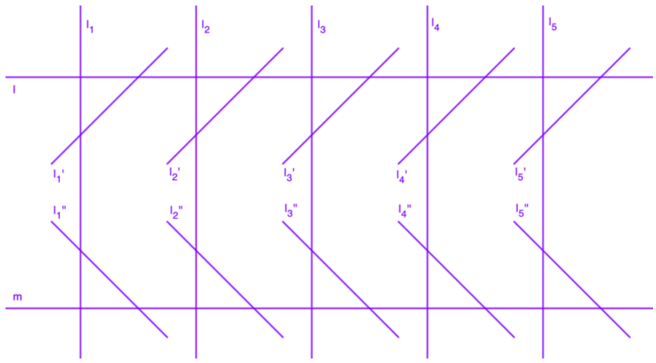
We now consider two disjoint lines (this means they don't intersect) which we call l and m. We know they exist from the third assumption. We display them on the diagram below



There are 5 tritangent planes containing the line l, we call these planes $\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5$. There are three lines in each of these planes one of which will always be l. We call the others l_i and $l_i{}'$ for each Π_i. From the fourth assumption every line intersects every plane, so m must intersect all the Π_i, so we choose it to intersect l_1, l_2, l_3, l_4, l_5. This is demonstrated in the diagram below



From the second assumption we know that these lines intersect in groups of three in a plane, so we know that there must be 5 more lines which we label $l_1'', l_2'', l_3'', l_4'', l_5''$.



So far, we have fifteen lines, meaning that there are twelve lines on the surface which we are still yet to find. To count all these lines is quite messy and repetitive, however we can use some combinatorics to come up with the following result (which we state as an assumption since we omit the proof):

Every remaining line we have not counted intersects exactly three of l_1, l_2, l_3, l_4, l_5, and we denote each of these lines l_{ijk} for the lines they intersect, for example l_{135} would intersect l_1, l_3, l_5.

There are twelve possible combinations of three different numbers 'ijk' from 1 to 5, so this gives us our twelve remaining lines l_{ijk} and after using similar techniques we can show that all these lines intersect exactly ten other lines, and we are done.