Integer Solutions on Cubic Surfaces

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The study of cubic surfaces is an active area of research within the realms of Number Theory, although they by and large remain unknown due to their complexity and because of the difficulty in proving general theorems. As is often the case in maths, in order to try and understand the general case, it's worth principally studying 'easy' examples of cubic surfaces to see if there is any general phenomena we can pick up lurking within these surfaces.

What is a cubic surface?

First and foremost, a cubic surface is, well, a surface just like that of the surface of the earth - locally it appears flat and therefore we can think of it being a two dimensional object.

In the same way that in two dimensions, the points (x, y) which satisfy the equation $y = mx + c$ form a line, a one dimensional space, points in three dimensions, (x, y, z), which satisfy a cubic equation form a cubic surface, a two dimensional space (so long as our cubic equation contains all our coordinate terms).

A cubic equation then is a special type of equation, involving a **polynomial** in three variables, which we'll label $f(x, y, z)$, which is of degree 3, that is all of the terms are of the form $cx^iy^jz^k$ with the coefficient c, a number, and $i + j + k \leq 3$, with at least one of the terms we have $i + j + k = 3$. This polynomial is then equal to some constant, which we'll label k.This gives us the cubic equation $f(x, y, z) = k$.

In this context, we'll be mainly interested in **integral** solutions to the equation, that is points for which all coordinates are integers, so typically we'll want the coefficients to be integers, and k also an integer.

We can study the cubic equations in and of themselves e.g. looking at the algebraic properties of such an equation, seeing if they're any factorisations to exploit etc. We can also study the cubic surfaces which the cubic equations characterise, which is studies in the branch **algebraic geometry.**

Examples of Cubic Surfaces

As we can see, it's an easy task to come up with all different types of cubic equations, there are infinitely many after all. What's a bit harder is to figure out which ones are most worth studying. Typically these are the simplest one which have symmetrical properties. Such examples include:

- Sum of three cubes: $x^3 + y^3 + z^3 = k$
- Markov Surfaces: $x^2 + y^2 + z^2 - xyz = k$
- Surfaces: $(ax + 1)(bx + 1) + (cy + 1)(dy + 1) = xyz + 1$ with a, b, c, d various integers.
- Surfaces: $a(x^3 + y^3) + z^3 = 1$ with a an integer.

Sum of three cubes

One particular class of cubic surface that are of interest are the sum of three cubes, $x^3 + y^3 + z^3 = k$. One question that we can ask is: what integers can be written as a sum of three integer cubes?

The easier question of what integers can be written as a sum of three cubed rational numbers (fractions) has already been answered, since 1825 it has been known that every integer can be written as a sum of three rational cubes in infinitely many different ways, the infinitely many solutions obtained by varying n in the formula below:

$$k = \left(\frac{27k^3 - n^9}{27k^2n^2 + 9kn^5 + 3n^8}\right)^3 + \left(\frac{-27k^3 + 9kn^6 + n^9}{27k^2n^2 + 9kn^5 + 3n^8}\right)^3 + \left(\frac{27k^2n^3 + 9kn^6}{27k^2n^2 + 9kn^5 + 3n^8}\right)^3$$

The original question is therefore a much harder task. Ruling out positive integers of the form $9t + 4$ or $9t + 5$,we call the remaining admissible, then it's still an open problem as to whether all remaining positive integers have an integral solution or not.

There have been very few theoretical tools developed in this area of study, so in order to test this hypothesis we must use a computer to search for solutions. Only as recently as 2019, the last admissible $k \leq 100$ for which no solution had yet been found, 42, a solution was found, involving 17+ (!) digit values:

$$(-80538738812075974)^3 + (80435758145817515 3)^3 + (12602123297335631 3)^3 = 42.$$

Polynomial Parametrisations

For a cubic surface with lots of known points, $x^3 + y^3 + z^3 = 1$ say, then it's natural to ask if any of these points are related to each other in some particular way. One way in which points could be related is if they lie on the same curve that runs along the surface.

The type of curves that we're interested in are **integer polynomial parametrisations,** in essence these are curves that lie on a cubic surface such that each coordinate is a polynomial with integer coefficients with the same parameter t.

An example for the surface $x^3 + y^3 + z^3 = 1$ being $c(t) = (9t^4, 3t - 9t^4, 9t^3 + 1)$, varying t for different integers we're able to generate infinitely many integral points. Similarly we have $c(t) = (1 + 6t^3, 1 - 6t^3, -6t^2)$ for the surface $x^3 + y^3 + z^3 = 2$.

Multiplying both side of these equations by m^3 gives us similar parametrisations for k equal to m^3 or $2m^3$. Whether or not there exists parametrisations for k not of the above form remains an open problem.