Solving the Helmholtz Equation Over An Infinite Staircase

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WARWICK

UNDERGRADUATE RESEARCH SUPPORT SCHEME

1) MOTIVATION

In Chichen Itza, there is a temple called the Kukulkan temple. If you stand at the base of its stairs and clap, the sound waves scatter off the staircase in such a way that the noise you hear mimics the call of the local quetzal. We wish to be able to accurately model this.

$$\nabla^2\varphi(\mathbf{x}) + k^2\varphi(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}_0) - \delta(\mathbf{x}-\bar{\mathbf{x}}_0) \Longrightarrow \nabla^2\varphi(\mathbf{a}) + |\tan(\mathbf{a})|k^2\varphi(\mathbf{a}) = \delta(\mathbf{a}-\mathbf{a}_0) - \delta(\mathbf{a}-\bar{\mathbf{a}}_0)$$

$$f(z) = \int_0^z \sqrt{\tan(x)}\,dx$$

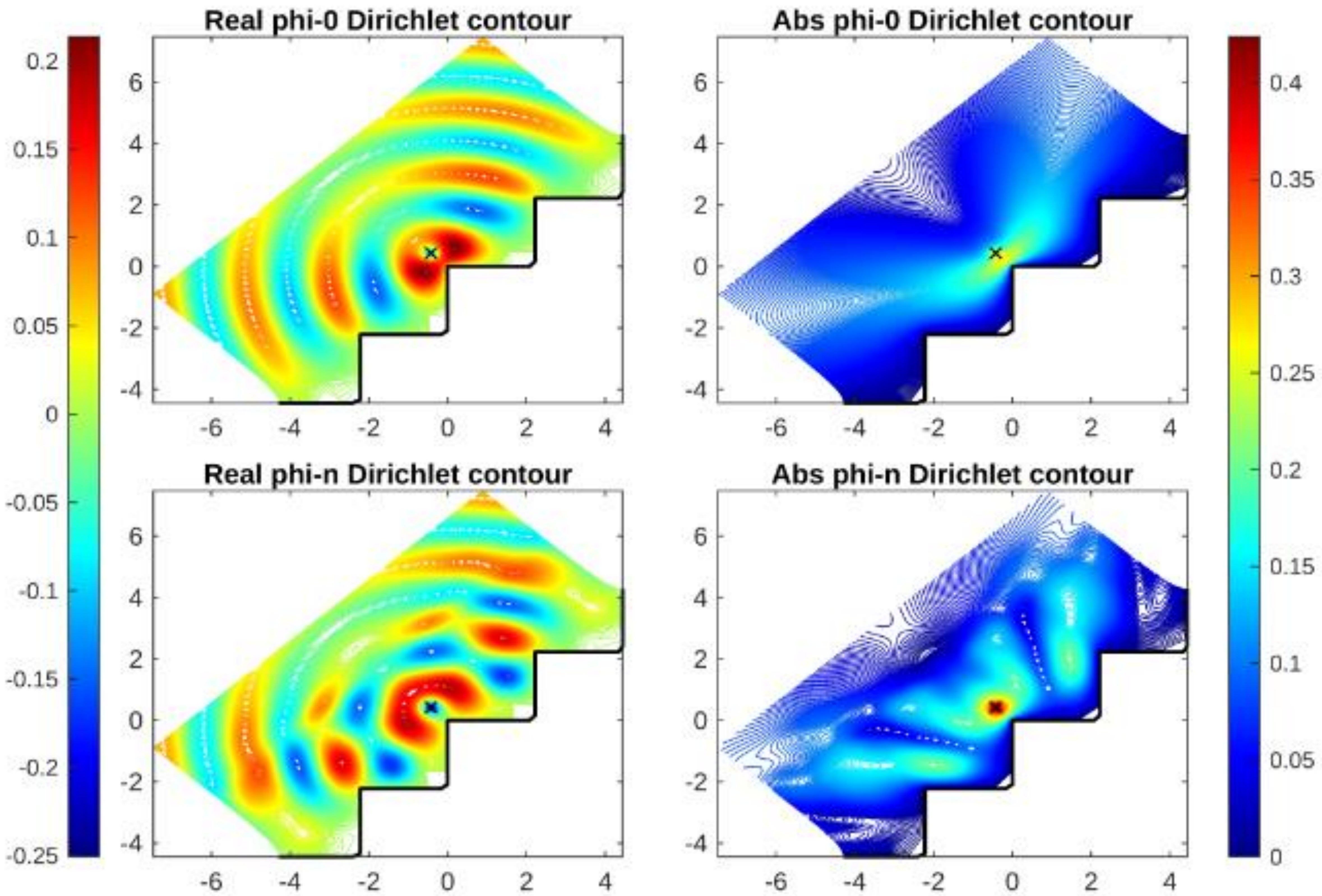
2) SETUP & EQUATIONS

The equation we wish to solve is a modified version of the Helmholtz equation obtained via a change of coordinates, seen above, and then use a conformal map, f(z), to map this solution to the staircase. **a, a'** and **a0** represent points before the mapping and **x, x'** and **x0** represent points after the mapping. **a0** and **x0** represent the location of the point source above the boundary before and after the conformal map is applied. k represents the wave number. G(**a**,**a'**) represents the Green's function, which is the sum of two Hankel functions, for the Helmholtz equation in the half-space $\mathbb{H} = \{(x,y)|x,y \in \mathbb{R}, y \geq 0\}$

$$\varphi_0(\mathbf{a}) = G(\mathbf{a},\mathbf{a}_0) = -\frac{i}{4}H_0^{(1)}(k|\mathbf{a}-\mathbf{a}_0|) + \frac{i}{4}H_0^{(1)}(k|\mathbf{a}-\bar{\mathbf{a}}_0|)$$

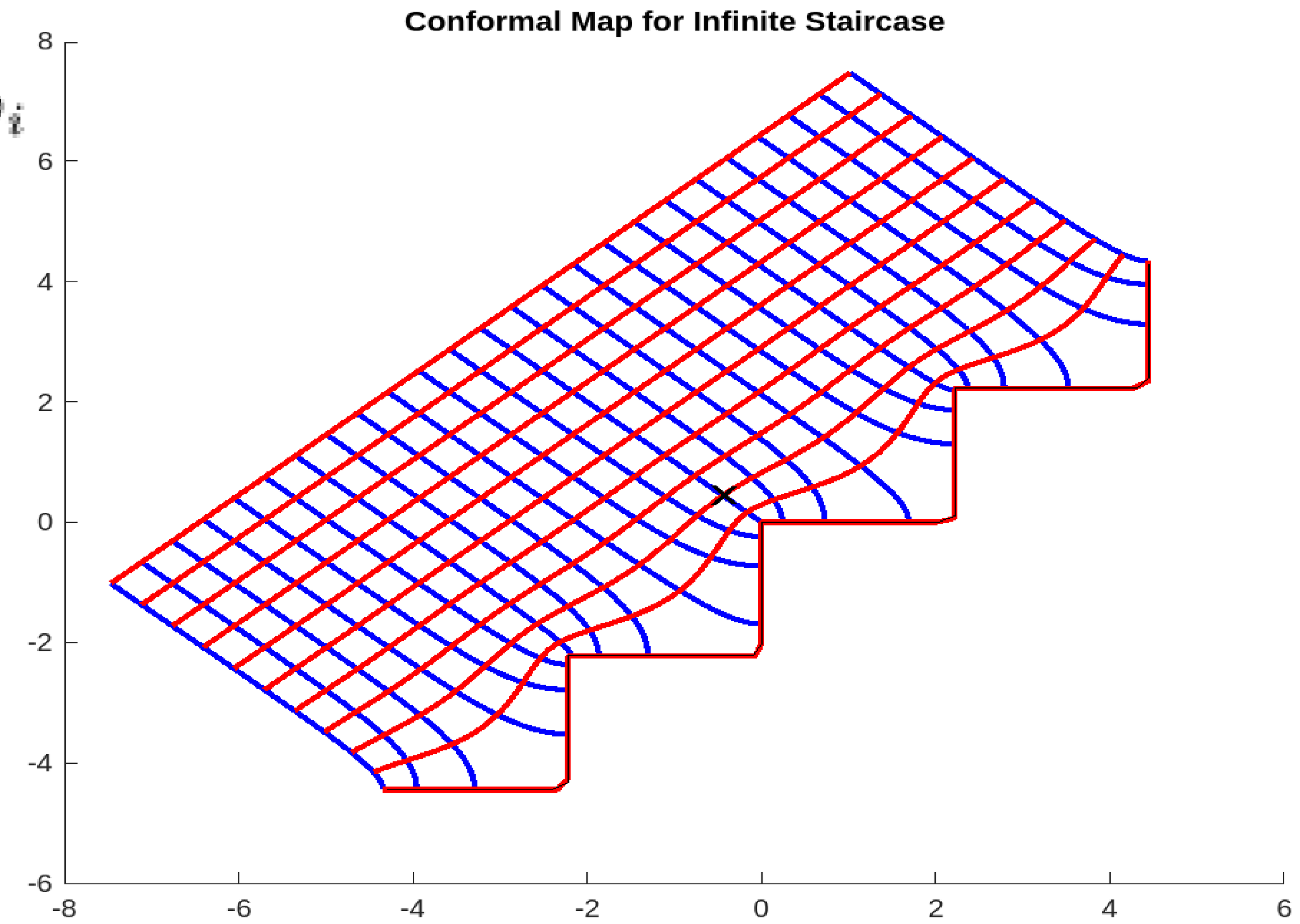
$$\varphi_n(\mathbf{a}) = \int_{\mathbb{H}} G(\mathbf{a},\mathbf{a}')k^2(1-|\tan(\mathbf{a}')|)\varphi_{n-1}(\mathbf{a}')d\mathbf{a}'$$

$$\varphi_{n+1}(\mathbf{a}) = \varphi(\mathbf{a}) - \sum_{i=0}^{n}\varphi_i(\mathbf{a}) \qquad \varphi \approx \sum_{i=0}^{n}\varphi_i$$



3) PLOTS

On the left are some examples of the results obtained. Plots were taken of the first and final iterations of the scheme and for each the real component and the absolute value at each point was compared. On the right is a showcase of how the conformal map transforms a grid of red horizontal lines and blue vertical lines.



4) RESULTS

A comparison was drawn between the Dirichlet and Neumann boundary. The Neumann boundary solutions seemed to diverge for lower wave numbers than the Dirichlet boundary solutions, suggesting that the Dirichlet boundary might be a more appropriate model. Additionally, the solutions for the Dirichlet boundary were found to be quite stable around k=1 and lower. Moving the point source to be twice as close to the stairs had a minimal impact on the wave's behaviour. However, moving the point source along parallel to the line boundary caused noticeably different behaviour of the solutions where the values seemed to bunch up near the closest inner corner of the staircase.

5) CONCLUSION

The problem of modelling the Helmholtz equation across an infinite staircase is still very much an open one. There are many possible avenues to go down to continue investigating this problem, some of which include: optimising the code, testing the impact of different inputs, finding the wave numbers for the convergence/divergence threshold, considering the half-infinite staircase and many more. Finding a solution could have widespread impact to things like modelling WiFi coverage and modelling airflow off serrated airplane wings.

REFERENCES

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