Cross-ratio degree problems for 7 and 8 points in projective space

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Abstract

This document is a report/log of my 2024 Undergraduate Research Support Scheme (URSS) project with the University of Warwick. This project was completed under the supervision of Helena Verrill, with the Warwick Mathematics Institute. The project has synergies with projective geometry, graph theory and combinatorics.

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1 Introduction and mathematical context

1.1 A brief introduction to projective geometry

The starting point for this project is a problem highlighted by Silversmith in [7, 8]. To understand the problem, a level of background knowledge in projective geometry is required. The purpose of this section is to illustrate the core concepts needed to grasp the essence of the problem. I use [10] as a basis for this section.

Definition 1.1.1 (**Projective space**)**.** Let V be a vector space of dimension $n+1$ over some field K and define $V^* := V \backslash \{0\}$. We define the *projective space of V* to be the set of equivalence classes of V^* under the following equivalence relation on $x, y \in V^*$:

$$x \sim y \Leftrightarrow Kx = Ky$$

where $Kx = \{k \cdot x : k \in K\}$. We denote the projective space of V as $\mathbb{P}(V)$, which can equivalently be written as $\mathbb{P}^n(K)$. This is often shortened to $\mathbb{P}^n$. The equivalence class of $x \in V^*$ is denoted $[x]$.

Example 1.1.2. $\mathbb{P}^1(\mathbb{R})$ is often referred to as a *projective line*. Almost every point $(x, y) \in (\mathbb{R}^2)^*$ falls on the line L, given by $y = \lambda x$, where $\lambda = y/x$. L is equivalent to the set $\mathbb{R}(x, y)$. Hence, we can represent the equivalence classes of $(\mathbb{R}^2)^*$, under $\sim$, as ratios of the form $(1 : \lambda)$, for $\lambda \in \mathbb{R}$.

The only remaining case to consider is when the point is of the form $(0, y) \in [(0, 1)]$, which can be denoted in ratio form as $(0 : 1)$, and is often referred to as the *point at infinity*.

We can choose any line in $\mathbb{R}^2$, and find that it intersects once with each of our equivalence classes, excluding the point at infinity. Hence, we can see that $\mathbb{P}^1(\mathbb{R})$ is actually a line.

Remark 1.1.3. Note that the choice of line in $\mathbb{R}^2$ will influence which element of $\mathbb{P}^1(\mathbb{R})$ is the point at infinity. The point at infinity is whichever equivalence class does not intersect with the chosen line. This notion becomes useful later, when changing the choice of line can make sense of dealing with infinities.

Definition 1.1.4 (**Projective linear group**)**.** For $A, B \in GL(n)$, the general linear group of dimension n, over some field K, we define the following equivalence relation:

$$A \sim B \Leftrightarrow A = \lambda B \; for \; some \; \lambda \in K^*$$

The *projective linear group* is defined as the set of equivalence classes of $GL(n)$ under $\sim$, denoted $PGL(n) := GL(n) \backslash \sim$.

Definition 1.1.5 (**Projective linear map**)**.** Let $A \in PGL(n+1)$ and $\mathbf{v} \in \mathbb{P}^n(K)$. A *projective linear map* is a map

$$T_A : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$$

$$[\mathbf{v}] \mapsto [A\mathbf{v}]$$

It is elementary to show that such a map is well-defined.

Example 1.1.6 (Möbius transformations). A classical Möbius transformation, as stated in [4, p. 11] , is a map

$$f : \mathbb{C} \to \mathbb{C}$$

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $\quad ad - bc \neq 0$.

The concept of a Möbius transformation can easily be written in the language of projective geometry as follows. Let $A \in PGL(2)$, with $\mathbb{C}$ as the field.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; a, b, c, d \in \mathbb{C} \; and \; ad - bc \neq 0.$$

Then for $(x, y) \in (\mathbb{C}^2)^*$, the projective linear map

$$T_A : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$$

$$[(x, y)] \mapsto [(ax + by, cx + dy)]$$

has the equivalent action to a standard Möbius transformation.

Definition 1.1.7 (Projective frame of reference). For $\mathbb{P}^n$, a *projective frame of reference* is a set of $n + 2$ points which are linearly independent, and hence span $\mathbb{P}^n$. Those readers less inclined to mathematical rigour can think of it as the projective equivalent of a basis.

One must be careful when using familiar mathematical terms within new contexts. In projective geometry, we use the phrase 'linearly independent' to refer to when the span of $k + 1$ points $p_0, p_1, \ldots, p_k$ in $\mathbb{P}^n$ has dimension k.

Definition 1.1.8 (Standard projective frame of reference). Let $e_1, \ldots, e_{n+1}$ be the standard basis for $\mathbb{R}^{n+1}$. It can easily be shown that the projections $[e_1], \ldots, [e_{n+1}]$ form a projective frame of reference for $\mathbb{P}^n$. This is called the *standard projective frame of reference.*

Definition 1.1.9 (Cross-ratio). The cross-ratio is a function of 4 distinct points in projective space. There are multiple ways to define it, based on the context of the problem. I will state it here as it is given in [7, p. 2].

For 4 distinct points a, b, c, d on the projective line $\mathbb{P}^1$, we define the *cross ratio* of these points, $CR(a, b, c, d)$, as follows:

$$CR(a, b, c, d) = \frac{(c - a)(d - b)}{(c - b)(d - a)} \in \mathbb{C} \backslash \{0, 1\} \qquad (*)$$

Remark 1.1.10 (Simplification). To make it easier to understand what the cross-ratio actually represents, consider that $\{\infty, 0, 1\}$ is a projective frame of reference for $\mathbb{P}^1$. For any 4 distinct points a, b, c, d on $\mathbb{P}^1$, we can find the unique projective linear map T such that $T(a) = \infty$, $T(b) = 0$ and $T(c) = 1$. It follows that $CR(a, b, c, d)$ is given by $T(d)$. This is stated in more detail in [3, p. 141].

Remark 1.1.11 (Dealing with infinities). It is necessary to note that we may end up in a situation where one of a, b, c, d is ∞, which causes problems when substituting the points into $(*)$. Since the cross-ratio is well-defined (as we will see below in Lemma 1.1.12) we can adopt the context of Remark 1.1.3 and chose a different line in $\mathbb{R}^2$ with which to define $\mathbb{P}^1$. This will change the original point at infinity to a different point, and then we can substitute into $(*)$ without issue. Although the notation is somewhat unorthodox, this framework essentially allows us to do the following:

$$CR(\infty, b, c, d) = \frac{\cancel{(c-\infty)}(d-b)}{(c-b)\cancel{(d-\infty)}} = \frac{(d-b)}{(c-b)}$$

Lemma 1.1.12 (Invariance of cross-ratio under projective linear transformations). Let T be a projective linear transformation acting on $\mathbb{P}^1$. Then for $p_1, p_2, p_3, p_4 \in \mathbb{P}^1$, $CR(p_1, p_2, p_3, p_4) = CR(T(p_1), T(p_2), T(p_3), T(p_4))$.

Outline of proof. Let T be the projective linear transformation given by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2)$$

and let the images of p_1, p_2, p_3, p_4 under T be given by

$$q_i = T(p_i) = [(ax_i + by_i, cx_i + dy_i)]$$

for $i \in \{1, 2, 3, 4\}$, where $p_i = [(x_i, y_i)] \in \mathbb{P}^1$.

We can calculate each of $(q_3 - q_1), (q_3 - q_2), (q_4 - q_2), (q_4 - q_1)$ in terms of the p_is and use these to find that $CR(q_1, q_2, q_3, q_4) = CR(p_1, p_2, p_3, p_4)$, as required. For a full proof of this form, see [5, p. 411]. □

1.2 My research question

My URSS is concerned with counting the *cross-ratio degrees* of groups of 7 and 8 points in the projective plane. In this section I will provide some further definitions and material required to understand this problem, however these will be more informal, with the intention of giving the reader a feel for the project, and to keep this report as accessible as possible.

Definition 1.2.1 (Cross-ratio degree). Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of n distinct points on $\mathbb{P}^1$, with $n \geq 4$. We then choose a collection of $(n-3)$ 4-tuples such that at least every point in P appears in at least one of these 4-tuples. Let us call this set of 4-tuples $\mathcal{T}$.

We can then define a new set $CR_{\mathcal{T}}$ to be the set of cross-ratios for each of the 4-tuples in $\mathcal{T}$. Omitting some of the algebraic geometry Silversmith includes in his definition of cross-ratio degree in [7, pp. 1–2], we essentially set each element of $CR_{\mathcal{T}}$ equal to a number $a_1, \ldots, a_{n-3}$. Depending on the tuples we chose for $\mathcal{T}$, this may uniquely determine the values of the points in P (in terms of the a_is), or the resulting equations may have multiple solutions. Informally, the *cross-ratio degree*, $d_{\mathcal{T}}$, is the number of possible solutions.

Definition 1.2.2 (**Hypergraph**)**.** I take the definition of hypergraph from [1, p. 1]. A *hypergraph* $H = (V, E)$ is given by V, a finite set of vertices, and E, family of subsets of V. We call each of the subsets in E a *hyperedge*. The *order* of H is $|V|$ and the *size* of H is $|E|$.

Definition 1.2.3 (**k-uniform hypergraph**)**.** We define the *rank* of H, $r(H)$ to be the maximum cardinality of a hyperedge, i.e the number of elements in the largest subset in E. Similarly, the *co-rank* of H, $cr(H)$, is the minimum cardinality of a hyperedge. we say H is *k-uniform* if $r(H) = cr(H) = k$. In other words, a hypergraph is k-uniform when all of its edges have k elements.

Remark 1.2.4. We can use the above two definitions to frame the cross-ratio degree problem in terms of graph theory. The set P in definition 1.2.1 is our V, and E is given by $\mathcal{T}$. Every element of $\mathcal{T}$ is a 4-tuple, so we are working in the context of a 4-uniform hypergraph, of order n and size $(n-3)$. There is the extra condition from the context of the problem that every vertex in V must appear in at least one element of E.

Definition 1.2.5 (**Biadjaceny matrix**)**.** We can represent all the necessary information for a cross-ratio degree problem with an $(n-3) \times n$ matrix, called a *biadjaceny matrix*. Each row of the matrix gives an element of $\mathcal{T}$. For any given row, a 1 in the ith column indicates that p_i is contained in the corresponding hyperedge, and similarly a 0 in the ith column indicates that p_i is not contained in that hyperedge.

Example 1.2.6. I will work through an example in the case of $n = 7$ whereby we begin with a biadjacency matrix and compute the cross-ratio degree in that instance. For $n = 7$, our matrix will be a 4×7, which we will call A:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

From this, we determine that the elements of $\mathcal{T}$ are:

$$\{p_1, p_2, p_3, p_4\}, \{p_2, p_3, p_4, p_5\}, \{p_1, p_2, p_6, p_7\}, \{p_4, p_5, p_6, p_7\}$$

and we obtain the following system of equations:

$$\begin{cases} CR(p_1,p_2,p_3,p_4) = a_1, & (1)\\ CR(p_2,p_3,p_4,p_5) = a_2, & (2)\\ CR(p_1,p_2,p_6,p_7) = a_3, & (3)\\ CR(p_4,p_5,p_6,p_7) = a_4 & (4) \end{cases}$$

Since cross-ratio is invariant under projective linear transformations (see Lemma 1.1.12), we can choose to work in a projective frame of reference where $p_1 = \infty$, $p_2 = 0$ and $p_3 = 1$. From remark 1.1.10, we know (1) yields that $p_4 = a_1$. For (2), we get

$$\frac{(p_4 - p_2)(p_5 - p_3)}{(p_4 - p_3)(p_5 - p_2)} = a_2$$

Using $p_2 = 0$, $p_3 = 1$ and $p_4 = a_1$, this simplifies to

$$\frac{a_1}{a_1 - 1} \cdot \frac{p_5}{p_5 - 1} = a_2$$

So

$$p_5 = \frac{\alpha}{\alpha - 1}, \text{where } \alpha = \frac{a_2(a_1 - 1)}{a_1}$$

Through similar manipulation, (3) yields that $p_7 = a_3 p_6$. For (4), we can substitute $a_3 p_6$ for p_7, and a_2 for p_4 to get

$$\frac{p_6 - a_2}{p_6 - p_5} \cdot \frac{a_3 p_6 - p_5}{a_3 p_6 - a_2} = a_4 \qquad (*)$$

Note that here we can treat p_5 as a constant, since we found its value from (2). Cross-multiplying in $(*)$ gives a quadratic equation in the variable p_6. Since we are working under the assumption that a_1, a_2, a_3, a_4 can have any values, there will be cases in which p_6 has two allowed values. It is also important to note that whilst this means p_7 also has two allowed values ($p_7 = a_3 p_6$), p_7 is determined by the values of p_6, hence there are two possible sets of solutions to this cross-ratio degree problem, rather than 4.

So, we obtain that for the given biadjacency matrix, the cross-ratio degree $d_{\mathcal{T}} = 2$.

My URSS project is concerned with finding the maximal cross-ratio degree over all possible sets $\mathcal{T}$. The main focus will be doing this for seven points ($n = 7$), with a short study of how the methods I use can be applied to $n = 8$.

2 n=7 case

As described in Definition 1.2.1, for the $n = 7$ case, we consider a set of 7 points in $\mathbb{P}^1$. The objects we are working with to find the maximal cross ratio degree can be considered as 4-uniform hypergraphs of order 7 and size 4, as stated in Remark 1.2.4. I give one possible visualisation of this in Fig. 1.

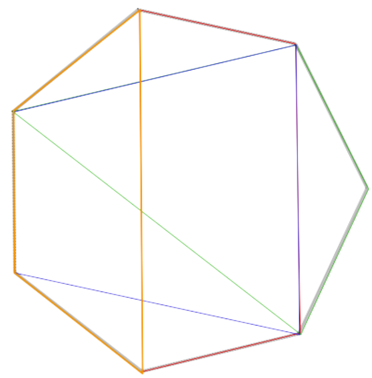

Figure 1: An example of 4 hyperedges on a 4-uniform hypergraph of order 7

2.1 Initial approach

In this section I will outline the initial approaches I took when beginning this project. In search of insights about the problem, I roughly followed the order set out below.

1. Calculations of examples by hand, such as in Example 1.2.6.
2. Consider an alternative visualisation of the problem, in terms of bigraphs. This will be explained below.
3. Move focus towards the biadjacency matrix and searching for a canonical form.
4. A combinatorial approach, utilising computational methods. This will be explored in Section 2.2.

2.1.1 Visualisation using bigraphs

We already have two different ways in which we can visualise the cross-ratio degree problem: hypergraphs and biadjacency matrices. Bigraphs provide a third, and very useful, way to visualise this set-up.

Definition 2.1.1 (r-partite graph). Let r be an integer, with $r \geq 2$, and let $G = (E, V)$ be a graph. G is an *r-partite graph* if it can be partitioned into r distinct classes such that every edge has its ends in different classes. A 2-partite graph is usually called a *bipartite* graph, or *bigraph*. This definition is taken from [2, p. 17].

Remark 2.1.2. We can consider our 4-uniform hypergraph from 1.1.10 as a bigraph of order 11, with one class containing 4 vertices and a second class containing 7 vertices. Each of the 7 vertices in the second class correspond to one of $p_1, \ldots, p_7$. Each of the 4 vertices in the first class represent an element of $\mathcal{T}$. We can draw 4 edges connecting each of the first 4 vertices to 4 distinct elements of the second class, such that overall every element of the second class has an edge to at least one of the first 4 vertices. An example of this can be seen below in Fig. 2, where $E1, E2, E3, E4$ represent the hyperedges.

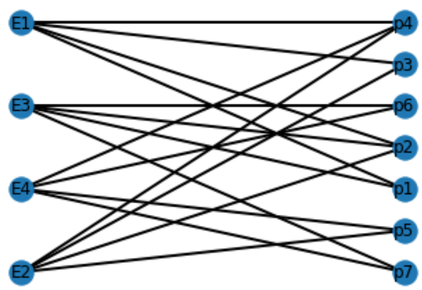

Figure 2: A bigraph representation of the hypergraph in Example 1.2.6

Throughout this section of the project, I was primarily calculating examples, using the method in Example 1.2.6, without use of a diagram or visualisation, however the bigraph visualisation provides a more compact way to conceptualise the problem.

2.1.2 Features of the biadjacency matrices

After calculating a reasonable amount of cross-ratio degrees by hand, I made a realisation that would help reduce the number of biadjacency matrices one needs to consider to find the maximal cross-ratio degree.

Example 2.1.3. Consider the biadjacency matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

As in Example 1.2.6, we take the cross-ratio of each of the 4 sets of points in $\mathbb{P}^1$ that are specified by B, and set them equal to the constants a_1, a_2, a_3, a_4. We obtain the following equations.

$$\begin{cases} CR(p_1, p_2, p_3, p_4) = a_1, & (5) \\ CR(p_1, p_2, p_3, p_6) = a_2, & (6) \\ CR(p_1, p_2, p_4, p_5) = a_3, & (7) \\ CR(p_1, p_3, p_5, p_7) = a_4 & (8) \end{cases}$$

Equations (5) and (6), are simply the definition of the cross-ratio, as in Remark 1.1.10. These give $p_4 - a_1$ and $p_6 = a_2$ respectively. For (7), we get

$$\frac{p_4 - p_1}{p_4 - p_2} \cdot \frac{p_5 - p_2}{p_5 - p_1} = a_3$$

We can then substitute $p_1 = \infty$, $p_2 = 0$, $p_4 - a_1$ and $p_6 = a_2$ to get the following.

$$\frac{\cancel{(a_4 - \infty)}(p_5 - 0)}{(a_1 - 0)\cancel{(p_5 - \infty)}} = a_3$$

So $p_5 = a_3 \cdot a_1$. We can do a similar substitution and cancellation in (8) to obtain

$$\frac{\cancel{(p_5 - \infty)}(p_7 - 1)}{(p_5 - 1)\cancel{(p_7 - \infty)}} = a_4$$

Substituting $p_5 = a_3 a_1$ and rearranging yields $p_7 = a_4(a_1 a_3 - 1) + 1$. Clearly there is only one possible solution set for this particular case, hence $d_{\mathcal{T}} = 1$.

Remark 2.1.4. Note that whenever p_1 appear in one of the cross-ratio equations, the cancellation of infinity, as in Remark 1.1.3, yields a linear equation. Hence, if p_1 appears in all 4 equations in our cross-ratio degree problem, as we solve the equations chronologically we will simply be solving a series of linear equations that each yield a single solution in terms of the previously solved variables. Thus in any of these cases, $d_{\mathcal{T}}$ will be 1. This occurrence corresponds to having the first column of the biadjacency matrix containing only 1s.

Understanding that certain features of the biadjacency matrix rule out the possibility of having a particular cross-ratio degree led me to considering the problem through studying the biadjaceny matrix. In particular, I shifted my focus to reducing the number of biadjacency matrices one needs to consider in order to find the maximal cross-ratio degree.

2.2 Reducing cases of possible biadjacency matrices

The problem we are considering on hypergraphs displays a level of symmetry. For example, rotating the hypergraph in Fig 1 will not change the value of $d_{\mathcal{T}}$, but it will change the biadjacency matrix. Hence, it is natural to consider how to classify all possible biadjacency matrices up to hypergraph (or bigraph) isomorphism. In particular, the aim of this part of my project was to find a canonical form for these biadjacency matrices.

Definition 2.2.1 (Isomorphism of graphs). We say that two graphs, G_1 and G_2, are *isomorphic* if there is a bijection $f : G_1 \to G_2$ between the vertices of the two graphs such that for any two vertices a, b on G_1, the number of edges between them is equal to the number of edges between $f(a)$ and $f(b)$. This definition is taken from [11, p. 11].

2.2.1 Counting and reducing by sorting

This area of the project was devoted to studying the possible biadjacency matrices and using computational methods to restrict those needed to find the maximal cross-ratio degree.

Since the cross-ratio is invariant under projective linear transformations (Lemma 1.1.12), we can fix the first row to be the same for all biadjacency matrices we consider. For simplicity, let this row be

$$\begin{pmatrix}1 & 1 & 1 & 1 & 0 & 0 & 0\end{pmatrix}$$

There are ${}^{7}C_3 = 35$ possible choices for any given row of such a biadjacency matrix. This leaves us with ${}^{34}C_3 = 5984$ choices for the remaining rows.

However, every p_i for $i \in \{1, 2, 3, 4, 5, 6, 7\}$ must appear at least once in the system of cross-ratio equations, hence all columns must be non-zero. To filter these cases, I added a 'column sum row' to each biadjacency matrix, where the ith entry in the row contained the sum of the elements of the ith column of the original 4×7 matrix.

Example 2.2.2. This is a very trivial demonstration. Taking the matrix from Example 1.2.6 and adding a 'column sum row' results in the following matrix.

$$\begin{pmatrix}2 & 3 & 2 & 3 & 2 & 2 & 2\\ 1 & 1 & 1 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 & 1 & 1\end{pmatrix}$$

```
    import numpy as np

R7=[]       #list of all possible rows

for i in range(4):
   for j in range(i+1,5):
     for k in range(j+1,6):
       for l in range(k+1,7):
          s = [0,0,0,0,0,0,0]
          s[i]=1
          s[j]=1
          s[k]=1
          s[l]=1
          R7.append(s)

A7 = []
#list of all possible biadjacency matrices without col sum row
M7 = []
#list of all possible biadjacency matrices with col sum row

for i in range(1,35):
  for j in range(i+1,35):
    for k in range(j+1,35):
         su = [x + y + z + w for x, y, z, w in zip(R7[0],R7[i],R7[j],R7[k])] #su = sum of 1s in each column
```

```
        if (not(0 in su)):
          s=[su,R7[0],R7[i],R7[j],R7[k]]   # adding top row to the matrix to sum column sums
          M7.append(s)
          A7.append([R7[0],R7[i],R7[j],R7[k]])
```

Listing 1: Python code to generate biadjacency matrices and remove those with zero-columns

One can use the code in Listing 1 to show the number of matrices, after removing any with zero-columns, has been reduced to 4904 by enumerating the list $A7$.

The 'column sum row' also provides another way of sorting these matrices. Again, by Lemma 1.1.12, changing the order of rows or columns in our biadjacency matrix won't alter the value of $d_{\mathcal{T}}$ since the cross-ratio is invariant under projective linear transformations. Additionally, we know from Remark 2.1.4 that any case in which the first column is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

will have $d_{\mathcal{T}} = 1$. Such matrices can be disregarded, since we know from Example 1.2.6 that the maximal cross-ratio degree in $n = 7$ is at least 2. Hence we are motivated to sort the matrices in such a way that the first column has as many 1s as possible. This can be achieved writing the columns in order of decreasing column sum. In cases where multiple columns have the same sum, we treat each column as a binary string and sort by decreasing lexicographic order. After this sorting process, we can expect some of the biadjacency matrices that represent isomorphic graphs to have been put into an identical format.

Example 2.2.3 (**Sorting by column sum**)**.** Consider the following matrix.

$$\begin{pmatrix} 4 & 2 & 2 & 3 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

After sorting, this becomes

$$\begin{pmatrix} 4 & 3 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Note that I have included the 'column sum row' here to clearly demonstrate the sorting process.

```
M7a=[]

for i in range(len(M7)):
  x=(np.transpose(M7[i])).tolist() #makes the array into a list of lists, which we call x
  x.sort(reverse=True) #lists the list elements in descending order by first element
  y=np.transpose(x).tolist() #y is the list of rows of matrices in M7
  y.sort(reverse=True)  #sorts rows in the same way as above
  x=(np.transpose(y)).tolist()
  x.sort(reverse=True)
  y=np.transpose(x).tolist()
  y.sort(reverse=True)
  M7a.append(y)

M7a.sort(reverse=True)

M7b=[M7a[0]] #filtering sorted matrices that are the same

for i in range(1,len(M7a)):
  if (not(M7a[i]==M7a[i-1])):
     M7b.append(M7a[i])
```

Listing 2: Sorting and filtering using Python

It is important to note that Listing 2 uses $M7$ as it appears in Listing 1. Additionally, the list of interest here is $M7b$, which is the list of possible matrices after filtering those that are identical after sorting. Asking Python to return the length of $M7b$ gives the number of remaining matrices we need to consider to find the maximal cross-ratio degree. This number is 66.

2.2.2 Reducing by graph isomorphisms

A reduction from 5984 to 66 matrices is significant, although not enough to begin considering $d_{\mathcal{T}}$ for individual cases. At this point I started to exploit the fact that each biadjacency matrix represents bigraph. Rather than coding in Python, I used a coding language called Magma, which is better developed to deal with graph theory. Further information about Magma can be found in [9].

Hypergraph (and bigraph) isomorphisms are defined in the same way as in Definition 2.2.1. Programming Magma to identify the biadjacency matrices with their respective bigraphs, and remove any repeated graphs up to isomorphism, returns a list of 29 unique bigraphs that fit the conditions of the problem, in the form of biadjacency matrices. After examining this list and removing those matrices containing

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

in any column, we obtain 14 matrices with a $d_{\mathcal{T}}$ possibly greater than 1. These are displayed in Table 1.

$$\begin{pmatrix}1&1&1&0&0&0&1\\1&1&0&1&1&0&0\\1&0&1&1&0&1&0\\0&1&1&0&1&1&0\end{pmatrix}\quad\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&1&0&0\\1&1&0&1&0&1&0\\0&0&1&1&1&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&1&0&0\\1&1&0&0&0&1&1\\0&0&1&1&1&1&0\end{pmatrix}\quad\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&1&0&0\\1&1&0&0&1&1&0\\0&0&1&1&0&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&0&0&1\\1&1&0&0&1&1&0\\0&0&1&1&1&1&0\end{pmatrix}\quad\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&1&0&0\\1&1&1&0&0&1&0\\0&0&0&1&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&0&0&0\\1&1&0&1&1&0&0\\1&0&1&0&1&1&0\\0&1&1&0&0&1&1\end{pmatrix}\quad\begin{pmatrix}1&1&1&1&0&0&0\\1&1&0&0&1&1&0\\1&0&1&1&0&0&1\\0&1&0&0&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&0&0&0\\1&1&0&0&1&1&0\\1&0&1&0&1&0&1\\0&1&0&1&0&1&1\end{pmatrix}\quad\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&1&0&0\\1&0&0&1&0&1&1\\0&1&0&0&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&1&0&0\\1&1&0&0&0&1&1\\0&0&0&1&1&1&1\end{pmatrix}\quad\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&1&0&0\\1&1&0&1&0&1&0\\0&0&1&0&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&0&1&0&0\\1&1&0&1&1&0&0\\1&0&1&1&0&1&0\\0&1&1&1&0&0&1\end{pmatrix}\quad\begin{pmatrix}1&1&1&1&0&0&0\\1&1&0&1&0&0&1\\1&0&1&0&1&1&0\\0&1&1&0&1&1&0\end{pmatrix}$$

Table 1: The 14 matrices returned from Magma, after filtering those with a column of 1s.

2.3 Final insights for n=7

Since the biadjacency matrix in Example 1.2.6 has $d_{\mathcal{T}} = 2$, we know the maximal cross-ratio degree is at least 2, so can limit our search to a cross-ratio degree of 3 or more. With 14 matrices to consider, it is feasible (but arduous) to calculate $d_{\mathcal{T}}$ by hand for each case.

At this point in the project, I had a final realisation that greatly simplified this problem. Given that a column

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

means each of the cross-ratio equations are linear, if the biadjacency matrix contains a column

$$\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

then the first 3 equations solved must be linear, followed by a 4th equation that is at most quadratic, after substituting expressions from the previously solved linear equations. By Definition 1.2.1 this means the cross-ratio degree for such a matrix is at most 2. Note that this works if any column of the matrix is of this form, since it can be moved to the first column by a projective linear transformation, under which the cross-ratio is invariant (Lemma 1.1.12), then $p_1 = \infty$ will lead to a cancellation as in Remark 1.1.3.

All of the 14 matrices in Table 1 have

$$\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

as their first column, hence $d_{\mathcal{T}} \leq 2$ for $n = 7$. As stated already, Example 1.2.6 has $d_{\mathcal{T}} = 2$, hence we have found that the maximal cross-ratio degree for $n = 7$ is 2.

3 Reflections and additional ideas

3.1 n=8 case

Much of the work done on the $n = 7$ case can be generalised and applied to any n, however due to the cumulative complexity of the problem when n is increased, such ideas become considerably less decisive. I will state the two main results from Section 2 for a general n.

- If the $(n-3) \times n$ biadjaceny matrix contains a column of n 1s, then we get a system of $(n-3)$ linear cross-ratio equations resulting in $d_{\mathcal{T}} = 1$.
- If the $(n-3) \times n$ biadjaceny matrix contains a column of $(n-1)$ 1s and exactly one 0, then we get a system of $(n-4)$ linear cross-ratio equations and one equation that is at most quadratic. This results in $d_{\mathcal{T}} \leq 2$.

In the interest of formality, one may wish to prove these first for $n = 7$ and then proceed inductively, but I shall omit this. The results are intuitive enough for the lack of proof not to hinder the overall coherence of this report.

For $n = 8$, in addition to the above insights, I used Magma in the same way as Section 2.2.2 to return a list of biadjacency matrices up to graph isomorphism. This returned 741 unique bigraphs. Clearly this is not enough to approach the problem from a directly computational perspective.

In [7][p. 2], Silversmith provides a lower bound of 4 on the maximal cross-ratio degree for $n = 8$. Given further time on this project, my initial approach would be to aim for an upper bound, similar to my final approach in $n = 7$, through examination of the biadjacency matrix.

3.2 Hypergraph enumeration

Having taken a computational approach to finding the number of unique bigraphs for $n = 7$ and $n = 8$, it is natural to consider how one can rigourously prove this number is correct. As in Remark 1.2.4, this is equivalent to counting the number of 4-uniform hypergraphs, with order n and size $(n-3)$.

In [6, pp. 378–380], Palmer introduces a method of enumerating hypergraphs of this type, using the Pólya enumeration theorem. He introduces a counting polynomial

$$s_p^n(x) = \sum s_{p,k}^n x^k$$

for n-plexes of order p, where $s_{p,k}^n$ is the number of k-simplexes of dimension n.

For $n = 7$, computing the number of unique hypergraphs was equivalent to finding $s_{7,4}^3$. Palmer explains the method to calculate $s_p^n(x)$, after which we can examine the coefficients to find the desired result. Roughly speaking, the polynomial is constructed using ideas of group actions of the symmetric group of n elements on general subsets of a certain size. This computation becomes very complex very quickly, so given further time on the project, I would write some code aid the calculation. Additionally, with some further mathematical background, it would be valuable to explore a proof of the Pólya enumeration theorem and how it gives the polynomial $s_p^n(x)$.

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