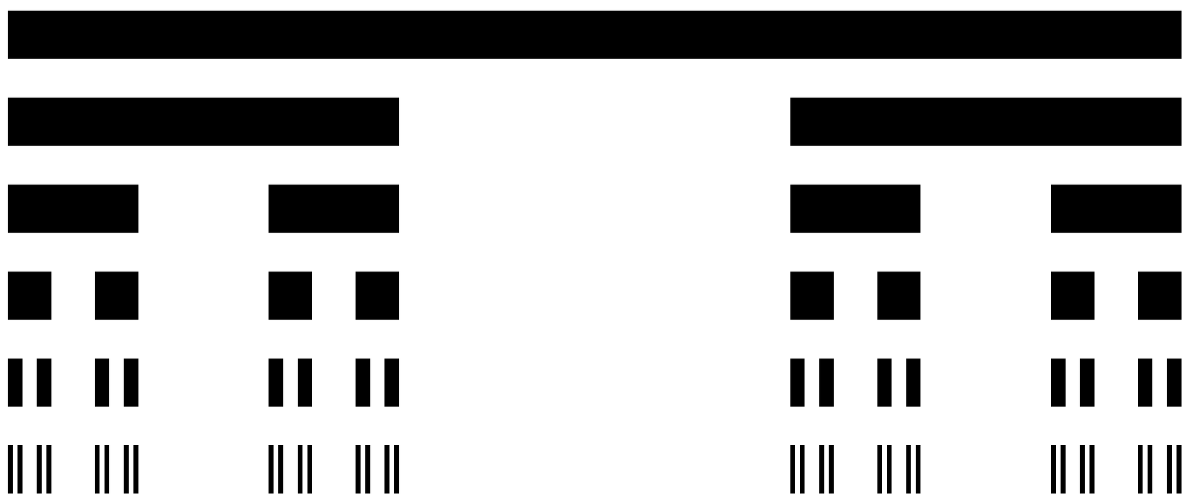
Metric Diophantine Approximation on Fractals



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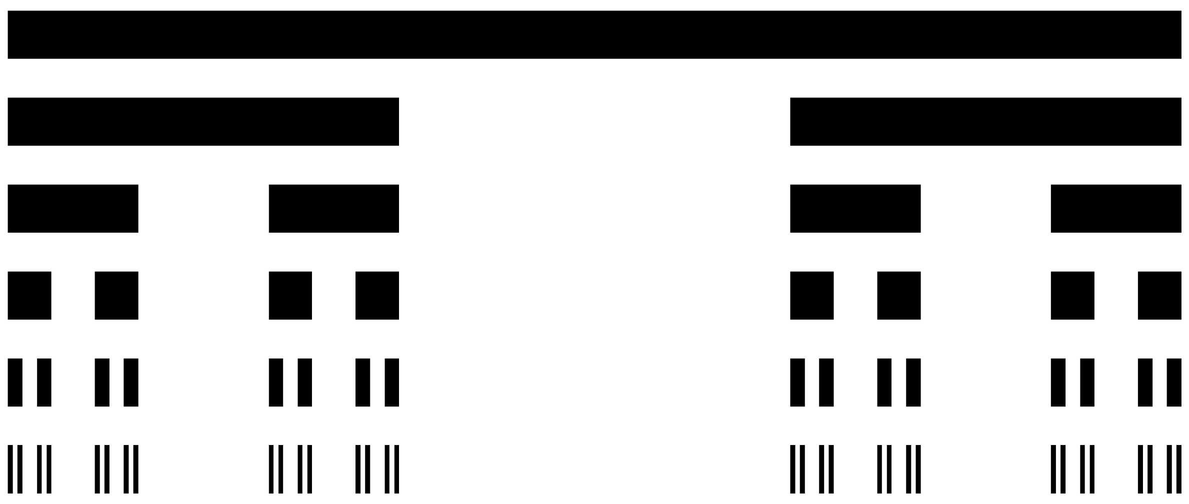

Introduction

Named after Diophantus of Alexandria, Diophantine approximation is a topic in number theory that studies how well real numbers can be approximated by rational numbers. In 1984, Mahler presented 8 problems in transcendental number theory and Diophantine approximation. One of these problems posed the questions,

- How well can irrationals in the middle third Cantor set be approximated by rationals in the middle third Cantor set?
- How well can irrationals in the middle third Cantor set be approximated by rationals not in the middle third Cantor set?

This problem serves as the inspiration for this research and we will address the following related question, 'How well can irrationals in a missing digit set be approximated by rationals with polynomial denominators?' and prove some related results.

Missing Digit Sets

The fractals we will be looking at in this paper are missing digit sets. The most famous and recognisable missing digit set is the *middle-third Cantor set* and we can construct it as follows. We aim to construct a sequence of sets starting with $C_0 = [0,1]$. We remove the middle third of this giving $C_1 = [0, \frac{1}{3}) \cup [\frac{2}{3}, 1]$. Then we remove the middle thirds of both of these intervals giving

$$C_2 = \left[0, \frac{1}{9}\right) \cup \left[\frac{2}{9}, \frac{1}{3}\right) \cup \left[\frac{2}{3}, \frac{7}{9}\right) \cup \left[\frac{8}{9}, 1\right].$$

Repeating this process gives us a sequence C_k. We define the *middle-third Cantor set* C as,

$$C = \bigcap_{k=0}^{\infty} C_k$$

Let $b \in \mathbb{N}$, $b \geq 2$ and $D \subset \{0, 1, ...b-1\}$. We define a missing digit set $K_{b,D}$ to be the set of real numbers in the interval $[0,1]$ with at least one base expansion with digits only from the set D. This means that we are able to write the middle third Cantor set as $K_{3,\{0,2\}}$.

For a missing digit set $K_{b,D}$, let D_i, $\imath \geq 1$, be a collection of independently and identically distributed random variables with elements in D. We define the *missing digit measure* [3] $\mu_{b,D}$ supported on $K_{b,D}$ to be the distribution of the random number

$$\sum_{i=1}^{\infty} D_i b^{-i}.$$

Khintchine's Theorem

We define $||x||$ as the distance between x and the nearest integer.

$$||x|| = min\{|x-n| : n \in \mathbb{N}\}.$$

For a function $\psi : \mathbb{N} \to [0, +\infty)$, x is called *ψ-approximable* if for infinitely many $q \in \mathbb{N}$,

$$||qx|| < \psi(q).$$

Let $W(\psi) = \{x \in [0,1] : ||qx|| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}$

Khintchine's Theorem. *[2] Let $\psi : \mathbb{N} \to [0, +\infty)$ be a monotonically decreasing function and let λ denote the Lebesgue measure. Then,*

$$\lambda(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty. \end{cases}$$

Fourier Dimension

We use Vinogradov notation, primarily $\ll$. $f \ll g$ if there exists some constant $C > 0$ such that $|f| \leq C|g|$ pointwise.

Introduced by Yu in [3], the Fourier l^t dimension will be an essential part of the main proofs of this project. Let ν be a Borel probability measure on $[0,1]$. For $\xi \in \mathbb{Z}$, we define the ξ^{th} Fourier coefficient of ν as,

$$\hat{\nu}(\xi) = \int_0^1 e^{-2\pi i \xi x} d\nu(x)$$

[1] We define the *Fourier l^t dimension* as

$$\hat{\kappa}_t(\nu) = \sup\left\{ s \geq 0 : \sum_{\xi=0}^{Q} |\hat{\nu}(\xi)|^t \ll Q^{1-s} \right\}$$

The main proofs

To conclude the project, we were able to show a convergence Khintchine-like property holds for rationals with polynomial denominators on a missing digit set K. For a polynomial P let

$$W_{P(q)}(\psi) = \{x \in K : ||P(q)x|| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}.$$

Theorem. *Let ψ be a monotonically decreasing function. For a missing digit measure m on a missing digit set on $[0,1]$ with a large base and only one missing digit,*

$$m(W_{P(q)}(\psi)) = 0, \text{ if } \sum_{q=1}^{\infty} \psi(q) < \infty$$

To prove this we also had to prove the following lemma.

Let n be the order of polynomial P and define $A(q, \delta) = \{x \in K : ||P(q)x|| < \delta\}$, inspired by [1]. By choosing the base sufficiently large we can ensure that $1 - \frac{1}{n} < \hat{\kappa}_1(m)$, which we can deduce from [1, Theorem 2.6].

Lemma. *Let v be a number in the range $1 - \frac{1}{n} < v < \hat{\kappa}_1(m)$. Then*

$$\sum_{q=Q}^{2Q} m(A(q,\delta)) \ll \delta Q$$

for $Q \in \mathbb{N}$ and $\delta \geq Q^{-(2v-1)/(1-v)}$.

References

[1] Sam Chow, Peter Varju, and Han Yu. Counting rationals and diophantine approximation in missing-digit cantor sets, 2024.

[2] Aleksandr Khintchine. Über einen satz der wahrscheinlichkeitsrechnung. *Fundamenta Mathematicae*, 6:9–20, 1924.

[3] Han Yu. Rational points near self-similar sets, 2021.

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