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Lectures on

Orbits of Minimal Action for

Area-Preserving Maps

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May 1985

corrected Dec 1989

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Preface

These notes are intended to form part of a book on area-preserving maps. This is the reason why they begin at §1.4 and occasionally refer to non-existent sections. The reader is asked to excuse this inconvenience. We believe that they are essentially self-contained and of independent interest and for this reason we make them available now. We would be grateful for any comments and criticism.

Chapter 1 is a review of basic notions which the well prepared reader may prefer to skip. Chapter 2 then presents the theory of orbits of minimal action for area-preserving twist maps.

Contents

Chapter 1 Basic Notions

Chapter 2 Minimizing Orbits

§1.4 Homeomorphisms of the Circle

Invariant circles play a very important role in the theory of area preserving maps. If Γ is an invariant circle for T then T induces a homeomorphism on Γ. Furthermore we will prove the existence of many orbits of area-preserving maps with properties similar to those of orbits of homeomorphisms of the circle. Thus in this section we recall some of the basic ideas in the theory of the dynamics of homeomorphisms of the circle. For a more detailed study see [Arnold, 1965, 1983], [Herman,1980] or [Nitecki,1971].

1.4.1 Definition Let $f : \mathbb{T}^1 \to \mathbb{T}^1$ be a homeomorphism, and let $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{T}^1$ be the covering map $\pi(x) = x \pmod 1$. Then a *lift* $F : \mathbb{R} \to \mathbb{R}$ of f is a continuous map s.t. $f \circ \pi = \pi \circ F$.

1.4.2 Remark Different lifts differ only by a constant integer. If f is an orientation preserving homeomorphism of $\mathbb{T}^1$ then F is an orientation preserving homeomorphism of $\mathbb{R}$ (and hence is order preserving) which commutes with unit translation, i.e. if $R(x) = x-1$ then $F \circ R = R \circ F$.

1.4.3 Theorem [Poincare, 1885] Let $F : \mathbb{R} \to \mathbb{R}$ be the lift of an orientation preserving homeomorphism of $\mathbb{T}^1$. Then

$$\rho(x) = \lim_{n \to \pm\infty} \frac{F^n(x) - x}{n}$$

exists and is independent of $x \in \mathbb{T}^1$.

1.4.4 Definition Since the above limit is independent of x, we may write $\rho(F)$ and call this the *rotation number* of F. By 1.4.2 if F_1 and F_2 are both lifts of the same $f : \mathbb{T}^1 \to \mathbb{T}^1$ then their rotation numbers differ only by an integer. We thus define $\rho(f)$, the *rotation number* of f to be $\rho(F) \pmod 1$ for any lift F of f.

We give a somewhat unusual proof of 1.4.3 : we will deduce it as a corollary of the following lemma. The idea is due to Mather. It has the virtue that it generalizes naturally to commuting pairs of homeomorphisms of $\mathbb{R}$. For a more standard and direct proof of 1.4.3 see either [Arnold, 1983] or [Nitecki,1971].

1.4.5 Lemma Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of an orientation preserving homeomorphism of $\mathbb{T}^1$. Then $\exists\, \rho \in \mathbb{R}$, s.t. $\forall\, x \in \mathbb{R}$, $\forall\, (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$

$$n\rho > m \;\Rightarrow\; F^n(x) - x > m$$
$$n\rho < m \;\Rightarrow\; F^n(x) - x < m$$

This ρ is clearly unique.

1.4.6 Corollary If F and ρ are as above then $\forall\, x \in \mathbb{R}$, $n \in \mathbb{Z}$ and

$\lfloor n\rho \rfloor$ = greatest integer strictly less than $n\rho$

$\lceil n\rho \rceil$ = least integer strictly greater than $n\rho$

then

$$\lfloor n\rho \rfloor < F^n(x) - x < \lceil n\rho \rceil$$

Theorem 1.4.3 is then an obvious consequence of this. To prove 1.4.5 we use the following three lemmas:

1.4.7 Lemma $\forall p,q \in \mathbb{Z}$, $\forall n \in N$, $\forall x \in \mathbb{R}$,

$$F^qR^p(x) > x \iff F^{nq}R^{np}(x) > x$$
$$F^qR^p(x) = x \iff F^{nq}R^{np}(x) = x$$
$$F^qR^p(x) < x \iff F^{nq}R^{np}(x) < x$$

Proof $\Rightarrow$ obvious.

$\Leftarrow$ by elimination.

1.4.8 Lemma Given $x \in \mathbb{R}$, $p,r \in \mathbb{Z}$, $q,s \in N$, with $p/q < r/s$ s.t.

$$F^q(x) - x > p$$
$$F^s(x) - x > r$$

Then $\forall m \in \mathbb{Z}$, $n \in N$, s.t. $p/q \leq m/n \leq r/s$ (see Fig 1.4.1)

$$F^n(x) - x > m$$

Proof Define

$$N = rq - sp > 0$$
$$a = rn - ms \geq 0$$
$$b = mq - pn \geq 0$$

Then $aq + bs = Nn$ and $ap + br = Nm$. So by 1.4.7 :

$$F^{Nn}R^{Nm}(x) - x > 0$$
$$\Rightarrow \quad F^nR^m(x) - x > 0 \qquad \text{as required.}$$

□

1.4.9 Lemma Suppose $\exists\, x,y \in \mathbb{R}$, $p,r \in \mathbb{Z}$, $q,s \in N$ s.t. $F^qR^p(x) \geq x$ and $F^sR^r(y) \leq y$. Then $p/q \leq r/s$.

Proof Otherwise the orbits of x and y would get out of order.

Wlog $y \leq x < y+1$, hence $\forall k \in \mathbb{Z}$, $F^k(y) \leq F^k(x) < F^k(y) + 1$.

So suppose $p/q > r/s$, choose $n \in N$ s.t. $nps - nrq > 2$.

Then by 1.4.7 , $F^{nqs}(x) \geq x + nps$ and $F^{nqs}(y) \leq y + nrq$.

Thus $|F^{nqs}(y) - F^{nqs}(x)| \geq n|p/q - r/s| - |x - y| > 1$ ※

□

Proof of 1.4.5 For any $p \in \mathbb{Z}$, $q \in N$, define $\alpha_{pq} : \mathbb{R} \to \mathbb{R}$ by

$$\alpha_{pq}(x) = F^qR^p(x) - x .$$

Then by 1.4.7 $\forall n > 0$:

$$\alpha_{pq}(x) > 0 \ \ \forall x \in \mathbb{R} \iff \alpha_{np,nq}(x) > 0 \ \ \forall x \in \mathbb{R}$$
$$\alpha_{pq}(x) < 0 \ \ \forall x \in \mathbb{R} \iff \alpha_{np,nq}(x) < 0 \ \ \forall x \in \mathbb{R}$$

So we can define

$$A^+ = \{ p/q \in \mathbb{Q} : \alpha_{pq}(x) > 0 \ \ \forall x \in \mathbb{R} \}$$
$$A^- = \{ p/q \in \mathbb{Q} : \alpha_{pq}(x) < 0 \ \ \forall x \in \mathbb{R} \}$$

Now, for $p_0 \in \mathbb{Z}$ sufficiently large, $(p_0/1) \in A^-$ and $(-p_0/1) \in A^+$. Thus both A^+ and A^- are non-empty. By 1.4.8 if $a,b \in A^+$ with say $a < b$, then $[a,b] \subset A^+$ (here $[a,b] = \{c \in \mathbb{Q} : a \leq c \leq b\}$) and similarly for A^- . Thus $(p_0/1)$ is an upper bound for A^+ and $(-p_0/1)$ a lower bound for A^- . Let $\rho = \sup A^+$. By 1.4.9 if $p/q \in A^+$ and $r/s \in A^-$ then $p/q < r/s$. Thus $\inf A^- \geq \sup A^+$. Now, by 1.4.9 there is at most one

$p/q \in Q$ s.t. α_{pq} has a zero or changes sign in $\mathbb{R}$, thus there is at most one $p/q \in Q$ s.t $p/q \notin A^+ \cup A^-$. So $[\sup A^+, \inf A^-] \subset \mathbb{R}$ does not contain an open interval and hence $\inf A^- = \sup A^+ = \rho$. Thus from 1.4.8 :

$$\{ p/q \in Q : p/q > \rho \} \subset A^-$$

This proves 1.4.5 for the case $n \neq 0$, but for $n=0$ the lemma is trivial.

1.4.10 Remark Periodic orbits of f correspond precisely to zeros of α_{pq}. Note that since α_{pq} is continuous, if it has a change of sign, it has a zero. In other words f has a periodic orbit of rotation number p/q iff $p/q \in Q \backslash (A^- \cup A^+)$. Note that by 1.4.9 there is at most one such p/q. Since $\rho = \inf A^- = \sup A^+$ but $A^- \cap A^+ = \emptyset$ we have

<u>either</u> $\rho \notin Q$, $Q = A^- \cup A^+$, so there is no p,q s.t. α_{pq} has a zero in $\mathbb{R}$, thus f has no periodic orbits.

<u>or</u> $\rho = p/q \in Q$, so $p/q \notin A^- \cup A^+$. Thus α_{pq} has at least one zero, hence f has at least one periodic point of rotation number p/q .

We now look at the possible dynamics in each case; we will see a very similar classification in Chapter 2.

1.4.11 Rational Rotation Number

Let $F : \mathbb{R} \to \mathbb{R}$ be the lift of a homeomorphism of $\mathbb{T}^1$ with $\rho(F) = p/q$. Using 1.4.7 we can classify orbits of $x \in \mathbb{R}$ under F into three categories:

periodic	$\forall m \in \mathbb{Z}$, $n \in N$ s.t. $n\rho = m$:	$F^n R^m(x) = x$
advancing	$\forall m \in \mathbb{Z}$, $n \in N$ s.t. $n\rho = m$:	$F^n R^m(x) > x$
retreating	$\forall m \in \mathbb{Z}$, $n \in N$ s.t. $n\rho = m$:	$F^n R^m(x) < x$

As we remarked above in 1.4.10 there is at least one periodic orbit. If we take p/q in lowest terms with $q>0$ then all periodic orbits have period q. In fact they are all of so called *type (p,q)* (see §1.6.9) , that is $F^q R^p(x) = x$. If $x \in \mathbb{R}$ is periodic then so is $x + n$, $\forall n \in \mathbb{Z}$.
Then if $x^- < x^+$ are two periodic points s.t. there are no periodic points in (x^-, x^+) :

<u>either</u> all orbits in (x^-, x^+) are advancing ; then all these orbits are asymptotic to the orbit of x^+ as $n \to \infty$, and to the orbit of x^- as $n \to -\infty$ i.e. $\forall y \in (x^-, x^+)$, $F^n(y) \to F^n(x^+)$ as $n \to \infty$ and $F^n(y) \to F^n(x^-)$ as $n \to -\infty$ (see Fig. 1.4.2).

<u>or</u> all orbits in (x^-, x^+) are retreating and are asymptotic to the orbits of x^+ and x^- as $n \to -\infty$ and $n \to \infty$ respectively.

1.4.12 Irrational Rotation Number

This case was studied by Poincare [1885] and Denjoy [1932]. It is described by the following theorem:

1.4.13 Theorem Let $f : \mathbb{T}^1 \to \mathbb{T}^1$ be a homeomorphism with $\rho(f) = \alpha \in \mathbb{R}\backslash\mathbb{Q}$. For any given $x \in \mathbb{T}^1$ let $P(x) = \{\text{limit points of } \{f^n(x) : n \in \mathbb{Z}\}\}$. Then:

i) P is independent of x, it is called the *derived set* of f.

ii) P is f-invariant.

iii) P is either the whole circle or a Cantor set.

1.4.14 Lemma Let f be as above. Given $x \in \mathbb{T}^1$ and $k,l \in \mathbb{Z}$ with $k \neq l$ let Δ be the interval $[f^k x, f^l x]$. Then every orbit of f passes through Δ.

Proof Note that for $m \in N$, $f^{m(l-k)}\Delta$ is adjacent to $f^{(m-1)(l-k)}\Delta$. Thus consider the collection of intervals $\{f^{m(l-k)}\Delta : m \in N\}$. If these do not cover $\mathbb{T}^1$ the endpoints of $f^{m(l-k)}\Delta$ must converge to some $y \in \mathbb{T}^1$ as $m \to \infty$. Then $f^{(l-k)}y = y$, so y is a periodic point which contradicts $\rho(f)$ irrational.

□

Proof of 1.4.13

i) Let $y \in P(x)$, so $\exists\ n(k) \to \infty$ s.t. $f^{n(k)}x \to y$ as $k \to \infty$. Wlog $f^{n(k)}x \to y$ monotonically from the left. Then given $x' \in \mathbb{T}^1$ by the above lemma $\exists\ l(k) \in \mathbb{Z}$ s.t. $f^{l(k)}x' \in [f^{n(k)}x, f^{n(k+1)}x]$. So $f^{l(k)}x' \to y$.

ii) If $y = \lim f^{n(k)}x \in P$ then $f(y) = \lim f^{n(k)+1}x \in P$.

iii) P is closed and non-empty. It has no isolated points because if $x \in P$ then $x = \lim f^{n(k)}x$ for some sequence $n(k)$, but $f^{n(k)}x \in P$, $\forall k \in N$. If P contains an open set, then given any $x \in \mathbb{T}^1$, since the orbit of x is dense in P, $\exists\ k,l \in \mathbb{Z}$ s.t. $\Delta = [f^k x, f^l x] \subset P$. Then by the lemma above the images of Δ under f cover $\mathbb{T}^1$ and P is f-invariant, thus $P = \mathbb{T}^1$.

Otherwise P contains no open set, thus it is nowhere dense and hence a Cantor set. □

Thus two possible cases arise:

Transitive P is the whole of $\mathbb{T}^1$ and so every orbit is dense on $\mathbb{T}^1$ under f. In this case F is topologically conjugate to the uniform rotation $R_\rho(x) = x + \rho$ (where $\rho = \rho(F)$) i.e. there exists a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ s.t $h \circ F = R_\rho \circ h$. To construct h choose any $x \in \mathbb{R}$ and define $h(F^n(x) - m) = n\rho - m$, then extend by continuity. If F is sufficiently smooth or analytic and ρ is sufficiently irrational then this conjugacy turns out to have some degree of smoothness (see [Arnold, 1965], [Herman, 1980] and [Yoccoz, 1984]).

Intransitive If P is a Cantor set there is no orbit dense in $\mathbb{T}^1$. However every orbit in P is dense in P, while $\mathbb{T}^1 \setminus P$ is a union of open intervals (called *gaps*). The orbit of any point in one of these intervals is homoclinic to P. This is because the total length of the gaps is at most 1, so the length of any gap goes to zero under iteration. Thus the orbit of any point in a gap is asymptotic to the endpoints of the gap as $n \to \pm\infty$. In this case F is only semiconjugate to the rotation R_ρ i.e. the h defined above is a continuous surjection but is not 1-1. It collapses each gap plus its endpoints to a single point. $f|_P$ is called a *Denjoy Minimal System.* This intransitive case cannot occur if F is C^1 and log DF has bounded variation (hence for instance if F is C^2). See [Denjoy, 1932] and [Nitecki, 1971].

§1.5 The Action Principle for Twist Maps

Recall that in many cases the dynamics of a Hamiltonian system can be expressed in terms of a variational principle. In a similar fashion the orbits of a large class of area-preserving maps can be obtained as the critical (or stationary) points of an action. This approach turns out to be very fruitful. In Chapters 2 and 3 it enables us to prove the existence of many different types of orbits and to investigate their properties.

1.5.1 Definition We say that a C^1 diffeomorphism $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a (monotone) *twist map* if there are coordinates (x,y) and a constant $K > 0$ s.t.

$$\partial x'/\partial y \geq K > 0 \qquad \text{where } (x',y') = T(x,y) \qquad 1.5.2$$

for all $(x,y) \in \mathbb{R}^2$.

1.5.2 is usually referred to as the twist condition. It is absolutely crucial to the development of much of the theory of area-preserving maps, and in particular when satisfied allows us to give a variational formulation of their dynamics. Relatively little is known about the dynamics of area-preserving maps which do not satisfy the twist condition, at least locally. Note however, that locally, twist is a very common property. Thus for instance near a typical elliptic fixed point ∃ coordinates (θ,r) s.t. the map is given by

$$\theta' = \theta + \Omega(r) + o(r)$$
$$r' = r + o(r^2)$$

with $d\Omega/dr \neq 0$ at $r = 0$ (Birkhoff Normal Form, see §X.X). Then $\partial\theta'/\partial r$ has constant sign in a neighbourhood of $r = 0$. In §X.X we will show how the results for twist maps extend to local twist maps. It is possible to give weaker notions of twist than above; for example, we may only require $\partial x'/\partial y$ to have constant sign on $\mathbb{R}^2$ rather than actually being bounded away from zero, or give a topological definition, so that T need not necessarily be differentiable. Many results remain true under these conditions, but there are often extra technical complications in the proofs. Thus for clarity of exposition in the remainder of these notes we shall almost always only consider maps which satisfy 1.5.2 globally.

For an example of an area-preserving twist map recall the generalised standard map of §1.1 given by

$$y' = y + f(x)$$
$$x' = x + y'$$

where $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function.

1.5.3 Exercise Show that if T is a twist map then so is T^{-1}, but T^2 need not be. Note that T^{-1} 'twists' in the opposite direction to T, i.e. in the same coordinates we obtain 1.5.2 with the inequalities reversed.

1.5.4 Remark If T is a twist map then the image of each vertical {x = constant} cuts each other vertical precisely once. This means that the map $(x,y) \to (x,x')$ is invertible (e.g. see the proof of 1.5.5) and (x,x') make perfectly good coordinates for $\mathbb{R}^2$. We shall make frequent use of this fact, both explicitly and implicitly.

1.5.5 Proposition Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^r area preserving twist map, with $(x',y') = T(x,y)$. Then there exists a C^{r+1} function $h : \mathbb{R}^2 \to \mathbb{R}$ s.t. if we denote $h_1(x,x') = \partial h(x,x')/\partial x$, $h_2(x,x') = \partial h(x,x')/\partial x'$ and $h_{12}(x,x') = \partial^2 h(x,x')/\partial x \partial x'$, then

$$\begin{aligned} y &= -h_1(x,x') \\ y' &= h_2(x,x') \end{aligned} \qquad 1.5.6$$

and $h_{12}(x,x') \leq C < 0$, $\forall (x,x') \in \mathbb{R}^2$, for some constant $C < 0$.

1.5.7 Definition The function h above is called the *generating function* for the map T.

Proof of 1.5.5 Let $\pi_i : \mathbb{R}^2 \to \mathbb{R}$, $i=1,2$, be the projections given by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$. Given $(\xi,\xi') \in \mathbb{R}^2$ let V,V' be the verticals $V = \{x = \xi\}$ and $V' = \{x = \xi'\}$. Then by 1.5.3 and 1.5.4 $TV \cap V'$ and $V \cap T^{-1}V'$ each consist of a single point, so define functions $Y'(\xi,\xi') = \pi_2(TV \cap V')$ and $Y(\xi,\xi') = \pi_2(V \cap T^{-1}V')$. Thus $y' = Y'(x,x')$ and $y = Y(x,x')$ (see Fig. 1.5.1). Fix an arbitrary $(x_0,x'_0) \in \mathbb{R}^2$, and given $(x,x') \in \mathbb{R}^2$ let $\gamma : [0,1] \to \mathbb{R}^2$ be a (say piecewise smooth) path from (x_0,x'_0) to (x,x'). Then define $h(x,x') = \int_\gamma Y'(\xi,\xi')d\xi' - Y(\xi,\xi')d\xi$. This is path independent: use Stokes' Theorem and note that since T is area preserving $dY' \wedge d\xi' = dY \wedge d\xi$. Then $h_1(x,x') = -Y(x,x') = -y$ and $h_2(x,x') = Y'(x,x') = y'$ as claimed. Finally note that $h_{12}(x,x') = -1/(\partial x'/\partial y) \leq -1/K < 0$.

□

Conversely:

1.5.8 Proposition Given a C^{r+1} function $h : \mathbb{R}^2 \to \mathbb{R}$ s.t. $h_{12} \leq C < 0$ for some constant C, then h generates a twist map by 1.5.6.

Proof Fix $x \in \mathbb{R}$, then since $h_{12} < 0$ we can invert $y = -h_1(x,x')$ (as a function of x') to give $x' = \alpha(x,y)$. Then define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x,y) = (\alpha(x,y),\ h_2(x,\alpha(x,y))) \qquad 1.5.9$$

Then

$$DT_{(x,y)} = \begin{bmatrix} -h_{11}/h_{12} & -1/h_{12} \\ h_{21} - h_{22}h_{11}/h_{12} & -h_{22}/h_{12} \end{bmatrix} \qquad 1.5.10$$

where the second derivatives of h are all evaluated at $(x,\alpha(x,y))$. Thus $\det DT = 1$ and hence T is area-preserving, and $\partial x'/\partial y = \partial\alpha/\partial y = -1/h_{12} \geq -1/C > 0$, and so T is a twist map .

□

Thus for example the generalised standard map (1.1.X) given by $y' = y + f(x)$, $x' = x + y'$ has generating function $h(x,x') = (x - x')^2/2 + F(x)$, where F is any indefinite integral of f.

1.5.11 Exercise Show that the generating function is unique up to addition of a constant.

Using the generating function we can characterise orbits of T in terms of sequences which make an action functional stationary.

1.5.12 Definition A *state* $\underline{x}$ is an bi-infinite sequence $\{x_i \in \mathbb{R} : i \in \mathbb{Z}\}$, and a *segment* $\underline{x}_{mn}$ is a finite sub-sequence $\{x_i \in \mathbb{R} : m \le i \le n\}$. Given $m,n \in \mathbf{Z}$, we define the *action* W_{mn} of a segment $\underline{x}_{mn}$ by

$$W_{mn}(\underline{x}_{mn}) = \sum_{i=m}^{n-1} h(x_i, x_{i+1})$$

We often write $W_{mn}(\underline{x})$ for $W_{mn}(\underline{x}_{mn})$. We say that a segment $\underline{x}_{mn}$ has *stationary action* if W_{mn} is stationary at $\underline{x}_{mn}$ wrt variations fixing the endpoints x_m and x_n. Thus $\partial W_{mn}(\underline{x})/\partial x_i = 0$, $\forall\, m<i<n$. We call a state *stationary* if all finite sub-sequences are stationary segments.

Segments with stationary action correspond precisely to finite orbit segments under the map generated by h. Thus:

1.5.13 Proposition Let T be an area-preserving twist map of $\mathbb{R}^2$ and h its action generating function. If $\{(x_i,y_i) : m \le i \le n\}$ is an orbit segment of T, i.e. $(x_{i+1}, y_{i+1}) = T(x_i,y_i)$, then $\{x_i : m \le i \le n\}$ has stationary action.

Proof $\quad h_2(x_{i-1},x_i) = y_i = -h_1(x_i,x_{i+1}) \qquad \forall m<i<n$

Thus

$$\partial W_{mn}(\underline{x})/\partial x_i = h_2(x_{i-1},x_i) + h_1(x_i,x_{i+1}) = 0 \qquad \forall m<i<n$$

□

1.5.14 Proposition Suppose that $\underline{x}_{mn}$ has stationary action. Define

$$y_i = -h_1(x_i,x_{i+1}) \qquad m \le i < n$$
$$y_n = h_2(x_{n-1},x_n)$$

Then $\{(x_i,y_i) : m \le i \le n\}$ is an orbit segment of T, i.e. $(x_{i+1}, y_{i+1}) = T(x_i,y_i)$ for $m \le i < n$.

Proof Recall from 1.5.9 that T is given by $T(x,y) = (\alpha(x,y), h_2(x,\alpha(x,y)))$ where α satisfies $\alpha(x,-h_1(x,x')) = x'$. Stationarity gives $h_2(x_{i-1},x_i) = -h_1(x_i,x_{i+1})$, $\forall m<i<n$. So for $m \le i < n$:

$$\begin{aligned} T(x_i,y_i) &= (\alpha(x_i,y_i), h_2(x_i,\alpha(x_i,y_i))) \\ &= (\alpha(x_i,-h_1(x_i,x_{i+1})), h_2(x_i,\alpha(x_i,-h_1(x_i,x_{i+1})))) \\ &= (x_{i+1}, h_2(x_i,x_{i+1})) \\ &= \begin{cases} (x_{i+1}, -h_1(x_{i+1},x_{i+2})) & m \le i < n-1 \\ (x_n,y_n) & i = n-1 \end{cases} \\ &= (x_{i+1}, y_{i+1}) \qquad m \le i < n \end{aligned}$$

□

Thus stationary states correspond precisely to orbits under the map T. From this we deduce that for a stationary state x_{n-1}, x_n determine x_{n+1} and x_{n-2}, and hence x_i, $\forall\, i \in \mathbf{Z}$: x_{n-1}, x_n determine y_n by $y_n = h_2(x_{n-1},x_n)$, and then (x_n,y_n) uniquely determines an orbit (x_i,y_i) under T. This is of course just a re-interpretation of 1.5.4 . Another way to see this is to note that since $h_{12} < 0$, given x_{n-1},x_n there is a unique x_{n+1} satisfying $h_2(x_{n-1},x_n) + h_1(x_n,x_{n+1}) = 0$.

1.5.15 Remark We thus see that if $\underline{u},\underline{v}$ are two stationary states, then $u_n = v_n$ and $u_{n+1} = v_{n+1} \Rightarrow \underline{u} = \underline{v}$. On the other hand if $\underline{u},\underline{v}$ are distinct and $u_n = v_n$ then $(u_{n-1} - v_{n-1})$ and $(u_{n+1} - v_{n+1})$ must have opposite signs. Again we can deduce this in two ways:

a) recall that T and T^{-1} twist in opposite directions, let $y_i = h_2(u_{i-1}, u_i)$ and $z_i = h_2(v_{i-1}, v_i)$, so that (u_i, y_i), (v_i, z_i) are the orbits corresponding to $\underline{u},\underline{v}$ respectively. Then by twist $y_n < z_n \Rightarrow u_{n+1} < v_{n+1}$, $u_{n-1} > v_{n-1}$, and vice versa if $y_n > z_n$.

b) From $(x_{n+1}, y_{n+1}) = T(x_n, h_2(x_{n-1}, x_n))$ we can evaluate $\partial x_{n+1}/\partial x_{n-1}$ directly using 1.5.10 . Thus

$$\left.\frac{\partial x_{n+1}}{\partial x_{n-1}}\right|_{x_n} = -\frac{h_{12}(x_{n-1}, x_n)}{h_{12}(x_n, x_{n+1})} < 0$$

§1.6 Twist Maps of the Cylinder

In the rest of these notes we shall mainly be concerned with area-preserving twist maps of either the cylinder $\mathbb{T}^1 \times \mathbb{R}$ or the annulus $\mathbb{T}^1 \times [0,1]$. As we remarked in 1.X , these are the area-preserving maps which typically arise in physical applications. We shall present the theory in terms of maps of $\mathbb{T}^1 \times \mathbb{R}$ since this avoids the technical difficulties associated with the boundaries of the annulus. With suitable modifications, however, the theory applies equally well to maps of $\mathbb{T}^1 \times [0,1]$. The precise class of maps we shall consider are:

1.6.1 Definition An *area-preserving twist map* of the cylinder is a C^1 diffeomorphism $T^*: \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$, which preserves area, orientation and the topological ends of $\mathbb{T}^1 \times \mathbb{R}$ and which satisfies the twist condition:

$$\partial\theta'/\partial y \geq K > 0 \qquad \text{where } (\theta', y') = T^*(\theta, y)$$

Given such a map it is often convenient to consider a lift $T: \mathbb{R}^2 \to \mathbb{R}^2$ to the universal cover (in exactly the same way as we lifted homeomorphisms of $\mathbb{T}^1$ to $\mathbb{R}$). Thus:

1.6.2 Definition Let $T^*: \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ be a homeomorphism, and let $p : \mathbb{R} \times \mathbb{R} \to (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} = \mathbb{T}^1 \times \mathbb{R}$ be the covering map $p(x,y) = ((x \pmod 1)), y)$. Then a *lift* $T: \mathbb{R}^2 \to \mathbb{R}^2$ of T^* is a continuous map s.t. $T^* \circ p = p \circ T$.

1.6.3 Remark Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the unit translation $R(x,y) = (x-1,y)$, then T commutes with R i.e. $T \circ R = R \circ T$. Different lifts differ only by a power of R, i.e. if T, T' are both lifts of T^* then $T' = R^n \circ T$ for some $n \in \mathbf{Z}$. If T^* is an orientation, end and area preserving twist map of $\mathbb{T}^1 \times \mathbb{R}$ then T is an orientation and area preserving twist map of $\mathbb{R}^2$. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be the generating function of T given by 1.5.5, we shall often also call it the generating function of T^*. Since T commutes with R, if h is a generating function then so is h' given by $h'(x,x') = h(x-1,x'-1)$. So by 1.5.11 $h(x,x') - h(x-1,x'-1)$ is a constant. We leave the reader to verify that this constant is independent of the choice of lift of T^* and of the function h, and define:

1.6.4 Definition If $h : \mathbb{R}^2 \to \mathbb{R}$ generates the area-preserving twist map $T^* : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$, the *net flux* or *Calabi invariant* of T^* is the constant $\mathrm{Flux}(T^*) = h(x,x') - h(x-1,x'-1)$.

This has the following geometrical interpretation (see Fig 1.6.1): let Γ_w be the circle $\mathbb{T}^1 \times \{w\} \subset \mathbb{T}^1 \times \mathbb{R}$, and let $V_w = \{ (\theta,y) : y > w \}$. If Γ is a homotopically non-trivial curve in V_w, let V be the component of the complement of Γ containing Γ_w, and denote $A_w(\Gamma) = \mu(V \cap V_w) =$ "area between Γ and Γ_w" (where μ is the measure given by $d\theta \wedge dy$ which is preserved by T^*). Then:

1.6.5 Proposition Let Γ be any homotopically non-trivial closed curve in $\mathbb{T}^1 \times \mathbb{R}$ and choose any $w \in \mathbb{R}$ s.t. both $\Gamma \subset V_w$ and $T^*(\Gamma) \subset V_w$. Then $\mathrm{Flux}(T^*) = A_w(T^*(\Gamma)) - A_w(\Gamma) =$ "net area crossing Γ in one iteration of T^*".

Proof The fact that $A_w(T^*(\Gamma)) - A_w(\Gamma)$ is independent of both w and Γ is an obvious consequence of the fact that T preserves μ. So wlog assume Γ and hence $T^*(\Gamma)$ are smooth sub-manifolds. Fix x_0 and consider the lift of Γ, this is a smooth periodic curve from $x = -\infty$ to $x = \infty$, let γ be one period of this from x_0 to x_0+1. As usual if $(\theta,y) \in \Gamma$, denote $(\theta',y') = T^*(\theta,y) \in T^*(\Gamma)$. Then $A_w(\Gamma) = \oint_\Gamma (y-w)d\theta = \int_\gamma (y-w)dx$ and $A_w(T^*(\Gamma)) = \oint_\Gamma (y'-w)dx' = \int_\gamma (y'-w)dx'$. Thus

$$\begin{aligned} A_w(T^*(\Gamma)) - A_w(\Gamma) &= \int_\gamma (h_2(x,x') - w)dx' + \int_\gamma (h_1(x,x') + w)dx \\ &= \int_\gamma (h_2(x,x')dx' + h_1(x,x')dx) \\ &= h(x_0+1,x'_0+1) - h(x_0,x'_0) \\ &= \mathrm{Flux}(T^*) \end{aligned}$$

□

1.6.6 Exercise Show that for the generalised standard map of §1.1 given by:

$$y' = y + f(\theta) \qquad \text{where } f(\theta+1) = f(\theta)$$
$$\theta' = \theta + y'$$

the net flux is given by $\oint f(\theta)d\theta$.

Most of the area-preserving twist maps that we shall study will have zero net flux, e.g. maps of the annulus or punctured plane. Thus for instance recall from §1.5 how locally in the neighbourhood of an elliptic fixed point we obtain a twist map of the cylinder; this obviously has zero net flux.

1.6.7 Example The motion of billiards (or equivalently light rays) in a smooth convex plane domain can be reduced to the dynamics of an area-preserving twist map of the cylinder $\mathbb{T}^1 \times (-1,1)$ (with zero net flux). This approach was first taken by [Birkhoff1917,1927]; much of his work on area-preserving twist maps was motivated by this example which clearly illustrates the interplay between the dynamical and variational viewpoints. The motion can be given by two alternative definitions :

a) angle of incidence = angle of reflection.

b) Fermat's principle: light rays take paths with stationary length subject to fixed endpoints.

We claim that the relation between these points of view is precisely that given by 1.5.13 and 1.5.14. A trajectory is clearly determined by where and at what angle it hits the boundary. We specify these by the arc length s along the boundary and by α the angle of incidence to the normal to the boundary (Fig 1.6.2). This gives a map $T : \mathbb{T}^1 \times (-1,1) \to \mathbb{T}^1 \times (-1,1)$ by $T(s, \sin\alpha) = (s', \sin\alpha')$. The reader should check that this preserves the form $ds \wedge d(\sin\alpha)$ and that $\partial(\sin\alpha')/\partial s > 0$. The generating function $h(s,s')$ for T is then given by the straight line distance between s and s'. Note that in this case the twist goes to zero as we approach $y = \pm 1$ and so h is not defined on the whole of $\mathbb{R}^2$. This is the sort of technical difficulty that we avoid by requiring the twist to be bounded away from zero on the whole of the cylinder.

Finally we collect together a number of useful definitions appropriate to maps of the cylinder. First recall the definition of rotation number for a homeomorphism of $\mathbb{T}^1$ from 1.4.3 and 1.4.4 . In the case of maps of $\mathbb{T}^1 \times \mathbb{R}$ the analogous limit does not necessarily exist, and will certainly not be the same for all orbits, but nevertheless the concept of rotation number of an orbit turns out to be very useful:

1.6.8 Definition Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the lift of an area preserving twist map of $\mathbb{T}^1 \times \mathbb{R}$, and let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection $\pi(x,y) = x$. If for a given $\underline{x} \in \mathbb{R}^2$ the limit ,

$$\rho(\underline{x}) = \lim_{n \to \pm\infty} \frac{\pi(T^n(\underline{x})) - \pi(\underline{x})}{n}$$

exists, then it is called the *rotation number* of $\underline{x}$ for T. This limit is independent of the choice of point on an orbit and thus we can talk of a rotation number of an orbit. Note that $[\rho(\underline{x}) \pmod 1]$ is independent of the choice of lift of T^* and we can thus also talk of rotation numbers for orbits under T^*.

Periodic orbits of T^* always have rotation numbers, it is useful to classify them as follows:

1.6.9 Definition Given $q \in \mathbb{N}$, $p \in \mathbb{Z}$ (not necessarily co-prime) , an orbit $\{ (x_n, y_n) : n \in \mathbb{Z} \}$ of T is said to be of *type p,q* if $x_{n+q} = x_n + p$, $\forall n \in \mathbb{Z}$. Such an orbit is then the lift of a periodic orbit of T^* which has rotation number p/q . The term 'type p,q' is then also applied to this periodic orbit, given the choice of lift.

It is a trivial observation that if (x_n, y_n) is an orbit of type p,q , then the stationary state $\underline{x} = \{x_n : n \in \mathbb{Z}\}$ corresponding to this orbit satisfies $x_{n+q} = x_n + p$, $\forall n \in \mathbb{Z}$. We shall refer to such states as *states of type p,q*, or even as *periodic states* . For these states, rather than considering the action W_{mn} of 1.5.12 it makes sense to define a 'periodic' version:

1.6.10 Definition Let $X_{p,q} = \{ \underline{x} : x_{n+q} = x_n + p \ , \forall n \in \mathbb{Z} \}$.

The map $\underline{x} \to (x_0, \dots, x_{q-1})$ then naturally identifies $X_{p,q} \simeq \mathbb{R}^q$.

Define the action $W_{p/q} : X_{p,q} \to \mathbb{R}$ by

$$W_{p/q}(x_0, \dots, x_{q-1}) = \sum_{n=0}^{q-1} h(x_i, x_{i+1}) \qquad (x_q = x_0 + p)$$

1.6.11 Remark If we define $R^* : X_{p,q} \to X_{p,q}$ by $[R^*(\underline{x})]_i = x_i + 1$, then if T has zero net flux then $W_{p/q} \circ R^* = W_{p/q}$, so $W_{p/q}$ is a function on $X_{p,q}/R^*$. This latter space can be identified in the obvious fashion with $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^{q-1}$, in other words wlog we can take $0 \le x_0 < 1$.

1.6.12 Lemma Stationary states of $W_{p/q}$ (w.r.t. variations preserving $x_q = x_0 + p$) correspond to orbits of type p,q.

Proof Follow the proof of 1.5.12 and 1.5.13 and use $x_{n+q} = x_n + p$.

More generally we want to consider recurrent, and in particular quasi-periodic states. Recall from 1.X.X the general definition of a recurrent orbit. Translated into the framework of stationary states this reads as follows:

1.6.13 Definition A stationary state $\underline{u}$ is *recurrent* if given $\varepsilon > 0$ and $k \in \mathbb{Z}$, $\exists$ $(m,n) \in \mathbb{Z}^2 \backslash (0,0)$ s.t.

$$|(u_{k+n} + m) - u_k| < \varepsilon$$
$$|(u_{k+n+1} + m) - u_{k+1}| < \varepsilon$$

Such a state then corresponds to a recurrent orbit under T^* . Then we define:

1.6.14 Definition A *quasi-periodic orbit* for T^* is a recurrent orbit which has an irrational rotation number. We shall call the stationary state corresponding to such an orbit a *quasi-periodic state* .

Of particular interest are quasi-periodic orbits which lie on invariant circles. We remarked on the importance of invariant circles in §1.3. For maps of the cylinder we want to distinguish a particular class of such circles:

1.6.15 Definition A *rotational invariant circle* (r.i.c.) for T^* is an invariant circle which is homotopically non-trivial (i.e. it "winds once around the cylinder") (see Fig 1.6.3) .

Given any invariant circle Γ of a general area-preserving homeomorphism T, we see that T induces a homeomorphism on Γ. From 1.4.4 we then obtain a rotation number for this homeomorphism. For rotational invariant circles this rotation number agrees with that given by definition 1.6.8, provided we choose compatible orientations for the two lifts involved. Also note that for an arbitrary Γ, if we take an annular neighbourhood $A \simeq \mathbb{T}^1 \times (0,1)$ of Γ, then Γ is a r.i.c in A (A may not be invariant under T, but this is mainly a technical problem). Typically we can find an annular neighbourhood A such that T has twist in A, thus the study of invariant circles reduces to a large extent to the study of r.i.c's for twist maps. We show the existence of r.i.c's under suitable conditions in Chapter 4 and discuss their properties, and give criteria for their non-existence in Chapter 5. See also §2.4 .

§1.7 Monotone Sets and Orbits

Recall from §1.4 that for a lift of a homeomorphism of $\mathbb{T}^1$, orbits must preserve the order on $\mathbb{R}$. The natural generalisation to the context of maps of the cylinder is:

1.7.1 **Definition** (Mather, Katok) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the lift of a twist homeomorphism of the cylinder, and $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the translation $R(x,y) = (x-1,y)$, so that T and R commute. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection $\pi(x,y) = x$. A non-empty set M which is invariant under both T and R is *monotone* if $\pi|_M$ is injective and:

$$\forall \underline{x},\underline{x}' \in M, \quad \pi(\underline{x}) < \pi(\underline{x}') \Rightarrow \pi(T(\underline{x})) < \pi(T(\underline{x}')) .$$

1.7.2 **Remark** If M is monotone then

$$\forall \underline{x},\underline{x}' \in M, n \in \mathbf{Z}, \quad \pi(\underline{x}) < \pi(\underline{x}') \Rightarrow \pi(T^n(\underline{x})) < \pi(T^n(\underline{x}')) .$$

We shall mainly be interested in the theory of monotone orbits and states for area-preserving twist maps. We will use this extensively in Chapters 2 and 3. However in this section we do not need to suppose that T preserves area, thus by a stationary state we will simply mean a sequence $\underline{x} = \{x_n : n \in \mathbf{Z}\}$ s.t. $(x_n, y_n) = T^n(x_0, y_0)$. This of course agrees with definition 1.5.12 when T is an area-preserving twist map. Note that to define monotone sets we need not suppose that T is a twist map. However for arbitrary maps of the cylinder several of the following results do not hold and a number of technical difficulties arise. Thus for simplicity of exposition, in this section we shall always assume that T is a twist map. We also wish to define:

1.7.3 **Definition** A stationary state $\underline{u}$ is *monotone* if $\forall r,s,p,q \in \mathbf{Z}$,

$$u_s + r = u_q + p \Rightarrow u_{s+1} + r = u_{q+1} + p$$
$$u_s + r < u_q + p \Rightarrow u_{s+1} + r < u_{q+1} + p$$

If $\underline{u}$ is a stationary state then the corresponding orbit of T plus all its translates under R form an invariant set for (T,R). It is clear that this set is monotone iff $\underline{u}$ is a monotone state. In other words the lift of the orbit on the cylinder corresponding to a monotone state is a monotone set.

We often want to take the closure of a monotone set or the limit of monotone states.

1.7.4 **Lemma** If M is a monotone set for a twist map T then so is Cl(M) (the closure of M).

Proof Suppose that $\underline{x},\underline{x}' \in Cl(M)$. Using 1.7.2, since M is monotone and T is continuous

$$\pi(\underline{x}) < \pi(\underline{x}') \Rightarrow \pi(T^n(\underline{x})) \le \pi(T^n(\underline{x}')), \forall n \in \mathbf{Z} . \qquad 1.7.5$$

By twist (recall 1.5.15 and note that it does not depend on T being area- preserving) if $\pi(T(\underline{x})) = \pi(T(\underline{x}'))$ then $\pi(T^2(\underline{x})) > \pi(T^2(\underline{x}'))$ which contradicts 1.7.5. Hence

$$\pi(\underline{x}) < \pi(\underline{x}') \Rightarrow \pi(T(\underline{x})) < \pi(T(\underline{x}')) . \qquad 1.7.6$$

This also implies that $\pi|_M$ is injective : suppose that $\pi(\underline{x}) = \pi(\underline{x}')$ but $\underline{x} \neq \underline{x}'$. Then by twist $\pi(T^{-1}(\underline{x})) \neq \pi(T^{-1}(\underline{x}'))$, which contradicts 1.7.6.

1.7.7 Lemma The limit of monotone states is a monotone state , i.e. if $\underline{u}(n)$ are monotone and $u(n)_i \to u_i$ as $n \to \infty$ then $\underline{u}$ is monotone.

Proof Similar to 1.7.4 .

As the next lemma shows the dynamics on monotone sets are essentially the same as the dynamics of homeomorphisms of the circle. In particular appropriate versions of the results in §1.4 hold for monotone sets. There are two approaches to proving these: either we can directly copy the proofs of §1.4 , or apply the results of §1.4 to the homeomorphism given by 1.7.9 .

1.7.8 Remark If M is monotone then since $\pi|_M$ is injective we can define a homeomorphism $f : \pi(M) \to \pi(M)$ by $f = \pi \circ T \circ \pi^{-1}$.

1.7.9 Lemma [Katok, 1982] If T is a twist map, $f : \pi(M) \to \pi(M)$ can be extended to a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(x+1) = f(x) + 1$ and hence gives a homeomorphism $g : \mathbb{T}^1 \to \mathbb{T}^1$.

Proof By 1.7.4 , Cl(M) is monotone, and we can extend f to the closure of M by continuity , thus wlog we can take M to be closed. Then $\mathbb{R}\backslash M$ is a disjoint union of open intervals. We extend f to these linearly in [0,1) and then periodically to $\mathbb{R}$. It is easy to check that this gives a homeomorphism of $\mathbb{R}$ with the required properties.

1.7.10 Lemma If M is a monotone set for (T,R) , then $\exists\, \omega \in \mathbb{R}$, s.t. $\forall n,m \in \mathbb{Z}$, $\forall\, \underline{x} \in M$, if we denote $(x_n, y_n) = T^n(\underline{x})$

$$n\omega < m \quad \Rightarrow \quad x_n - x_0 < m$$
$$n\omega > m \quad \Rightarrow \quad x_n - x_0 > m$$

1.7.11 Corollary If M is a monotone set for (T,R) , $\exists\, \omega \in \mathbb{R}$, s.t.

if $\lfloor n\omega \rfloor$ = greatest integer less than $n\omega$

and $\lceil n\omega \rceil$ = least integer greater than $n\omega$

then $\forall\, n \in \mathbb{Z}$, $\forall x \in M$

$$\lfloor n\omega \rfloor < x_n - x_0 < \lceil n\omega \rceil$$

1.7.12 Corollary If M is a monotone set for (T,R) , let ω be as above, then every orbit in M has rotation number ω (recall definition 1.6.8). We will denote this ω as $\omega(M)$ and call it the *rotation number* of M.

1.7.13 Remark Although the $g : \mathbb{T}^1 \to \mathbb{T}^1$ given by 1.7.9 is not uniquely defined it is clear that the rotation number of such a g is independent of the choice of extension of $f : \pi(M) \to \pi(M)$. By comparing the definitions we see that it is precisely $\omega(M)$.

Lemma 1.7.10 applies to monotone states in the obvious way. In fact together with the classification of 1.4.11 for rational ω, it characterizes monotone orbits:

1.7.14 Lemma $\underline{u}$ is a monotone state $\Leftrightarrow$ $\exists\, \omega \in \mathbb{R}$ s.t. $\forall k,n,m \in \mathbf{Z}$

$$n\omega < m \Rightarrow u_{n+k} - u_k < m$$
$$n\omega > m \Rightarrow u_{n+k} - u_k > m$$

and (if ω rational), $\forall\, k \in \mathbf{Z}$

<u>either</u> $\forall\, m \in \mathbf{Z}, n \in \mathbb{N}$, s.t. $n\omega = m$, $u_{n+k} - u_k = m$
<u>or</u> $\forall\, m \in \mathbf{Z}, n \in \mathbb{N}$, s.t. $n\omega = m$, $u_{n+k} - u_k > m$
<u>or</u> $\forall\, m \in \mathbf{Z}, n \in \mathbb{N}$, s.t. $n\omega = m$, $u_{n+k} - u_k < m$

We will denote this ω as $\omega(\underline{u})$ and call it the *rotation number* of $\underline{u}$.

Proof $\Rightarrow$ follows directly from 1.7.10 and 1.4.11 .
$\Leftarrow$ exercise.

By 1.7.7 the limit of monotone states is also monotone. One might hope that the rotation number behaves continuously under this limiting process, and this indeed turns out to be the case. We then use this to show that the set of rotation numbers of monotone states is closed. This will be crucial in Chapter 2 in showing the existence of minimizing states of arbitrary rotation number.

1.7.15 Proposition Let $\underline{u}(n)$ be a sequence of monotone states with $u(n)_i \to u_i$ as $n \to \infty$. Then $\lim_{n\to\infty} \omega(\underline{u}(n))$ exists and is equal to $\omega(\underline{u})$.

1.7.16 Lemma Let $\underline{u},\underline{v}$ be two monotone states and suppose that for some $M \in \mathbb{N}$ and $\delta > 0$ we have $|u_i - v_i| \le \delta$, $\forall\, 0 \le i \le M$.

Then

$$|\omega(\underline{u}) - \omega(\underline{v})| \le 2(1+\delta)/M$$

Proof 1.7.11 $\Rightarrow |u_i - u_0 - i\omega(\underline{u})| \le 1$ and similarly for $\underline{v}$. Thus

$$|u_i - v_i - (u_0 - v_0) - i[\omega(\underline{u}) - \omega(\underline{v})]| \le 2$$
$$\Rightarrow \quad |u_i - v_i| \ge i|\omega(\underline{u}) - \omega(\underline{v})| - |u_0 - v_0| - 2$$
$$\Rightarrow \quad 2(1+\delta) \ge M|\omega(\underline{u}) - \omega(\underline{v})|$$

□

Proof of 1.7.15 Given $\varepsilon > 0$, choose $M \in \mathbb{N}$ s.t. $M > 3/\varepsilon$ and $N \in \mathbb{N}$ s.t. $|u_i - u(n)_i| < 1/2$, $\forall\, 0 \le i \le M$, $\forall\, n \ge N$. Then by 1.7.16

$$|\omega(\underline{u}) - \omega(\underline{u}(n))| \le 2(1+(1/2))/M \le 3/M < \varepsilon \qquad \forall\, n \ge N$$

□

1.7.17 Proposition Let $\underline{u}(n)$ be a sequence of monotone states such that $\omega = \lim_{n\to\infty} \omega(\underline{u}(n))$ exists. Then there exists a monotone state $\underline{u}$ such that $\omega(u) = \omega$.

Proof Wlog $0 \le u(n)_0 < 1$. By 1.7.16 $|u(n)_0 - u(n)_1| \le 2 + \omega(\underline{u}(n))$. Since $\omega(\underline{u}(n))$ is bounded so is $u(n)_1$. So the sequence $(u(n)_0, u(n)_1) \in \mathbb{R}^2$ is bounded, so wlog (take a convergent subsequence) $(u(n)_0, u(n)_1) \to (u_0, u_1)$ as $n \to \infty$. Now generate a stationary state $\underline{u}$ from (u_0,u_1) using 1.5.14 . It is easy to see that $u(n)_i \to u_i$ as $n \to \infty$: define $x_i(x,x') = \pi(T^i(x,(-h_1(x,x')))$, thus $u_i = x_i(u_0,u_1)$ and $u(n)_i = x_i(u(n)_0, u(n)_1)$. But x_i is continuous hence

$$\lim_{n\to\infty} u(n)_i = \lim_{n\to\infty} x_i(u(n)_0, u(n)_1)$$
$$= x_i(\lim_{n\to\infty} (u(n)_0, u(n)_1))$$
$$= x_i(u_0,u_1) = u_i$$

So by 1.7.15 $\underline{u}$ is a monotone state and $\omega(u) = \lim_{n\to\infty} \omega(\underline{u}(n))$.

□

Lastly we show that monotone sets (for a twist map) satisfy a Lipschitz condition. In Chapter 5 this is used to develop criteria for the non-existence of rotational invariant circles. To prove this we need:

1.7.18 Lemma If M is a monotone set and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is the projection $\pi_2(x,y) = y$ then $\pi_2(M)$ is bounded. Thus the image of M on the cylinder $\mathbf{T}^1 \times \mathbb{R}$ is bounded.

Proof Choose $(x_0,y_0)\in M$, let $(x'_0,y'_0) = T(x_0,y_0)$ and $K = |x_0 - x'_0| + 1$. Let $X = \{(x,x')\in\mathbb{R}^2 : |x - x_0| \le 1 \text{ and } |x - x'| \le K\}$, then $h_1(x,x')$ is bounded on X, say $|h_1| \le C$.
Now if $(x,y)\in M$ denote $(x',y') = T(x,y)$ and choose $m\in\mathbf{Z}$ s.t. $x_0 \le x - m \le x_0 + 1$. Then M monotone $\Rightarrow x'_0 \le x' - m \le x'_0 + 1$. So $|x - m - (x' - m)| \le |x_0 - x'_0| + 1 = K$. Thus $(x-m,x'-m)\in X$, and so $|y| = |h_1(x,x')| = |h_1(x-m,x'-m)| \le C$ as required.

1.7.19 Proposition ([Birkhoff, 1920], [Katok, 1982], [Chenciner,1984])
Let T be a twist map and M a monotone set for (T,R). Then there exists a constant L, s.t. if $(x_0,y_0),(x_1,y_1)\in M$, then

$$|y_0 - y_1| \le L|x_0 - x_1|$$

Proof By 1.7.18 choose C s.t. $M \subset \mathbb{R}\times[-C,C]$ and denote $A = \mathbb{R}\times[-C,C]$. As usual let $(x'_i,y'_i) = T(x_i,y_i)$, $i = 0,1$, and define $(x^*,y^*) = T(x_1,y_0)$

Define
$$B = \max_{(x,y)\in A} (\partial x'/\partial x)$$
$$D = \max_{(x,y)\in A} (\partial y'/\partial y)$$

and let K be the constant in 1.5.1, thus $(\partial x'/\partial y) \ge K > 0$ on $\mathbb{R}^2$.

Wlog $x_0 < x_1$, then

<u>either</u> $y_0 > y_1$:

$K(y_0 - y_1) \le (x^* - x'_1)$	by twist
$x'_0 < x'_1$	M monotone
$(x^* - x'_0) \le B(x_1 - x_0)$	definition of B

$\Rightarrow \quad K(y_0 - y_1) \le B(x_1 - x_0)$

<u>or</u> $y_0 < y_1$ in which case a similar calculation using T^{-1} gives

$$K(y_1 - y_0) \le D(x_1 - x_0)$$

So take
$$L = \max\{D/K, -B/K\}$$
□

§ 1.8 The Poincare-Birkhoff Theorem

Poincare [1912] was convinced that the key to understanding the dynamics of area-preserving maps lies in their periodic orbits. In the last year of his life he published a conjecture, that for any area-preserving map of the annulus satisfying a twist condition (in fact, weaker than our 1.6.1) there exist at least two fixed points. This was subsequently proved by Birkhoff [1913]. There have followed several proofs under slightly different conditions. We give the version most relevant to our purposes. Recall from §1.X that the residue R of a periodic orbit of period q of an area-preserving map T is defined by $R = (2 - \mathrm{Tr}\, DT^q_{\underline{x}})/4$, where $\underline{x}$ is any of the points of the orbit.

1.8.1 Theorem (Poincare-Birkhoff) Let T be an area-preserving twist map of the cylinder $\mathbb{T}^1 \times \mathbb{R}$, with zero net flux. Then $\forall\, p/q \in \mathbb{Q}$ (p,q coprime) T has at least two periodic orbits of type p,q ; one of non-positive residue and one of non-negative residue.

Proof One orbit can be found as a minimum of the action $W_{p/q}$ which we defined in 1.6.10 (see §2.3). If there is a whole circle of such minimizing orbits they all have residue zero (see §X.X). If not then another orbit can be found as a "minimax" of $W_{p/q}$ (see §3.X). It then only remains to prove that orbits corresponding to minimizing states have non-positive residue, and those given by "minimax" states have non-negative residues. These facts come from the following relation between the residue of a periodic orbit and M, the second variation of $W_{p/q}$. It is clear that for a minimizing orbit $\det M \geq 0$; in §3.X we give a topological argument to show that for a "minimax" state $\det M \leq 0$.

1.8.2 Lemma ([MacKay and Meiss, 1984]) Given $\underline{x} \in X_{p,q}$ stationary, let M be the matrix of second variations of $W_{p/q}$ about $\underline{x}$, thus

$$M_{ij} = \frac{\partial^2 W_{p/q}(\underline{x})}{\partial x_i \, \partial x_j} \qquad (i,j = 0 \ldots q-1)$$

Then if $R_{p/q}$ is the residue of the corresponding periodic orbit

$$R_{p/q} = - \frac{\det M}{4 \cdot \prod_{i=0\ldots q-1} (-b_i)} \qquad (b_i = h_{12}(x_i, x_{i+1}))$$

Proof Recall that orbits are given by sequences satisfying

$$h_2(x_{i-1}, x_i) + h_1(x_i, x_{i+1}) = 0$$

Differentiating, we obtain the following equation for tangent orbits $\delta\underline{x}$:

$$b_{i-1}\, \delta x_{i-1} + a_i\, \delta x_i + b_i\, \delta x_{i+1} = 0 \qquad 1.8.3$$

where we have set

$$a_i = h_{22}(x_{i-1}, x_i) + h_{11}(x_i, x_{i+1})$$
$$b_i = h_{12}(x_i, x_{i+1}) < 0$$

Here as usual $x_{i+q} = x_i + p$. The eigenvalues $\lambda, 1/\lambda$ of DT^q are given by the existence of a tangent orbit $\delta\underline{x}$ satisfying $\delta x_q = \lambda \delta x_0$. For such an orbit $\delta x_{q+1} = \lambda \delta x_1$, and $\delta x_0 = (1/\lambda) \delta x_q$. Thus $\delta\underline{x}$ must satisfy

$$M(\lambda) \delta\underline{x} = 0$$

where $M(\lambda)$ is the cyclic tridiagonal matrix given by:

$$M(\lambda) = \begin{bmatrix} a_0 & b_0 & 0 & \dots & 0 & (1/\lambda)b_{q-1} \\ b_0 & a_1 & b_1 & 0 & \dots & 0 \\ 0 & b_1 & a_2 & b_2 & & \\ \vdots & & & & & \\ \vdots & & & & & \\ & & & & & \\ 0 & & & & a_{q-2} & b_{q-2} \\ \lambda b_{q-1} & 0 & \dots & & b_{q-2} & a_{q-1} \end{bmatrix}$$

There is then a non-zero solution for $\delta \underline{x}$ iff $D(\lambda) \equiv \det M(\lambda) = 0$.

Expanding $D(\lambda)$ by the top row and first column we get :

$$D(\lambda) = D(1) + (\lambda-1)(-1)^{q-1} \prod_{i=0 \dots q-1} (b_i) + ((1/\lambda)-1)(-1)^{q-1} \prod_{i=-1 \dots q-2} (b_i)$$

$$= D(1) - (\lambda + (1/\lambda) - 2) \prod_{i=0 \dots q-1} (-b_i)$$

Thus $R_{p/q} = (2 - \lambda - 1/\lambda)/4 = -D(1) / (4 \prod (-b_i))$,

but $D(1) = M$, hence giving the required formula. □

Alternative Proof It is possible to give an alternative proof using recurrence formulae, based on an idea in [MacKay and Percival, 1985].

From 1.8.3 we see that

$$\begin{bmatrix} \delta x_i \\ \delta x_{i+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b_{i-1}/b_i & -a_i/b_i \end{bmatrix} \begin{bmatrix} \delta x_{i-1} \\ \delta x_i \end{bmatrix}$$

Set

$$\begin{bmatrix} k_i & l_i \\ m_i & n_i \end{bmatrix} = \prod_{j=1}^{i} \begin{bmatrix} 0 & 1 \\ -b_{j-1}/b_j & -a_j/b_j \end{bmatrix} = DT^i_{(x_0,x_1)}$$

Thus Trace $DT^q = k_q + n_q$.

But $k_i = m_{i-1}$, $l_i = n_{i-1}$ so m_i and n_i satisfy the following recurrence relations:

$$m_i = -(a_i/b_i)\, m_{i-1} - (b_{i-1}/b_i) m_{i-2} \qquad 1.8.4$$
$$n_i = -(a_i/b_i)\, n_{i-1} - (b_{i-1}/b_i) n_{i-2}$$

with initial conditions $m_0 = 0$, $m_1 = -b_0/b_1$, $n_{-1} = 0$, $n_0 = 1$.

Now consider the following matrices:

$$A_i = \begin{bmatrix} a_1 & b_1 & \dots & 0 & 0 \\ b_1 & a_2 & & & 0 \\ \vdots & & & & \\ & & & & \\ 0 & & & a_{i-1} & b_{i-1} \\ 0 & 0 & \dots & b_{i-1} & a_i \end{bmatrix}$$

$$B_i = \begin{bmatrix} a_2 & b_2 & \dots & 0 & 0 \\ b_2 & a_3 & & & 0 \\ \vdots & & & & \\ & & & & \\ 0 & & & a_{i-1} & b_{i-1} \\ 0 & 0 & \dots & b_{i-1} & a_i \end{bmatrix}$$

Let N_i = det A_i and M_i = det B_i, then expanding by the bottom row we get :

$$M_{i+1} = a_{i+1} M_i - (b_i)^2 M_{i-1}$$
$$N_{i+1} = a_{i+1} N_i - (b_i)^2 N_{i-1}$$

with initial conditions $N_{-1} = 0$, $N_0 = 1$, $M_0 = 0$, $M_1 = 1$.
By comparison with 1.8.4 this implies that:

$$m_i = b_0 M_i / \prod_{j=1...i} (-b_j)$$
$$n_i = N_i / \prod_{j=1...i} (-b_j)$$

(c.f. [Aubry, Le Daeron and Andre, 1982, App. H]) Expanding det M by the bottom row and last column we get:

$$\begin{aligned} \det M &= a_q N_{q-1} - (b_{q-1})^2 N_{q-2} - 2\prod_{j=1...q}(-b_j) - (b_q)^2 M_{q-1} \\ &= N_q - (b_q)^2 M_{q-1} - 2\prod_{j=1...q}(-b_j) \\ &= n_q \prod_{j=1...q}(-b_j) - b_q m_{q-1} \prod_{j=1...q-1}(-b_j) - \prod_{j=1...q}(-b_j) \\ &= (n_q + m_{q-1} - 2)(\prod_{j=1...q}(-b_j)) \\ &= (\text{Trace } DF^q - 2)(\prod_{j=1...q}(-b_j)) \\ &= (-4R_{p/q})(\prod_{j=1...q}(-b_j)) \qquad \square \end{aligned}$$

[Mather,1984 b] develops these ideas further to obtain even more information about $DT^q_{\underline{x}}$ from the Morse Index of $\underline{x}$ as a critical point of $W_{p/q}$.

Chapter 2 : Minimizing Orbits

§2.1 Introduction

In §1.5 we showed that orbits of an area-preserving twist map could be regarded as stationary states of the action. It turns out that the states which actually minimize the action have a particularly important role to play. Firstly, they provide existence proofs of orbits of arbitrary rotation number, secondly their dynamical behaviour can be completely understood and we can give a classification of all such states, and lastly the concept of minimal action turns out to be important in the study of invariant circles and Converse KAM theory.

For the whole of this chapter let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of a twist map of the cylinder with zero net flux, and let $h : \mathbb{R}^2 \to \mathbb{R}$ be the associated action generating function.

2.1.1 Definition A *segment* $\{x_i : m \le i \le n\}$ is *minimizing* if $W_{mn}(\underline{x})$ is (globally) minimal w.r.t. variations fixing x_m, x_n.

2.1.2 Definition A *state* $\{x_i : i \in \mathbf{Z}\}$ is *minimizing* if every finite segment $\{x_i : m \le i \le n\}$ is minimizing .

Clearly every minimizing segment or state is stationary and thus corresponds to an orbit segment or a complete orbit of T respectively. We will thus use the terms minimizing orbit and minimizing state interchangeably. In this chapter we have three aims:

1) Show the existence of a minimizing orbit for every rotation number (§2.3 and §2.4).

2) Discuss the main properties of minimizing states:

   i) Two minimizing states cannot "cross" more than once (Aubry's Fundamental Lemma, 2.2.8 and 2.6.1).

   ii) Minimizing states lie "within a bounded distance of some straight line" (Hedlund's Lemma, 2.5.2). A corollary of this is that all minimizing orbits have a rotation number.

   iii) Minimizing orbits are all monotone (§2.7).

3) Classify the set Min_ω of all minimizing states of rotation number ω. There are significant differences between ω irrational (§2.8) and ω rational (§2.9).

The results in this chapter are essentially due to the work of Aubry and Mather within the last few years ([Aubry, Le Daeron and Andre, 1982], [Aubry and Le Daeron, 1983] and [Mather, 1982, 1984]). However 1), 2) and the rational case of 3) were also done within the framework of geodesics on surfaces by [Morse, 1924] and [Hedlund, 1932]. We have used a number of their ideas to simplify and clarify the work of Aubry and Mather. In particular §2.5 is due to Hedlund and §2.9 to Morse. We also give a slightly stronger version of Aubry's Fundamental Lemma in §2.6, and use this to give a (new) direct proof that minimizing states are always monotone (§2.7). See [Bangert] for an alternative presentation of the same theory.

§ 2.2 Minimizing Segments

2.2.1 Lemma $\forall\, b,c \in \mathbf{R}$, $m,n \in \mathbf{Z}$ with $m < n-1$, $\exists$ minimizing segment $\underline{x} = \{x_i : m \le i \le n\}$ with $x_m = b$ and $x_n = c$.

Proof $W_{mn} : \mathbf{R}^{m-n+1} \to \mathbf{R}$ is continuous. We will show that $\forall\, a,b,c \in \mathbf{R}$, the set $\{\underline{x} \in \mathbf{R}^{n-m+1} : W_{mn}(\underline{x}) \le a,\ x_m = b,\ x_n = c\}$ is compact, which immediately implies the required result. First we need the following:

2.2.2 Lemma [Mather, 1984] $\exists$ constant C s.t. $h(x,x') \ge C + |x - x'|$.

Proof Set $T(x,y) = (x',y')$ and define $R(x,y) = (x+1,y)$. Since T commutes with R, $|x - x'|$ is defined on the quotient space $\mathbf{R}/\mathbf{Z} \times \mathbf{R}$. Since $\mathbf{R}/\mathbf{Z} \times [-1,1]$ is compact we see that $|x - x'|$ is bounded on $\{|y| \le 1\}$, and similarly it is bounded on $\{|y'| \le 1\}$. Let K be the maximum of $|x - x'|$ over $\{|y| \le 1\} \cup \{|y'| \le 1\}$. Then on $\{|x - x'| \ge K\}$ we have $|y| \ge 1$ and $|y'| \ge 1$. By twist this gives:

$$x' - x \ge K \Rightarrow y \ge 1 \text{ and } y' \ge 1$$

$$x' - x \le -K \Rightarrow y \le 1 \text{ and } y' \le 1$$

Now $y' = h_2(x,x')$ and $y = -h_1(x,x')$ and thus:

$$x' - x \ge K \Rightarrow h_1(x,x') \le -1 \text{ and } h_2(x,x') \ge 1$$

$$x' - x \le -K \Rightarrow h_1(x,x') \ge 1 \text{ and } h_2(x,x') \le -1$$

Since $h(x+1,x'+1) = h(x,x')$ and h is continuous, h is bounded on $\{|x - x'| \le K\}$, so $\exists$ constant A s.t $h(x,x') \ge A$ on this set. Combining the above we get $h(x,x') \ge A - K + |x - x'|$ everywhere. □

2.2.3 Lemma $\forall\, a,b,c \in \mathbf{R}$, $\{\underline{x} \in \mathbf{R}^{n-m+1} : W_{mn}(\underline{x}) \le a,\ x_m = b,\ x_n = c\}$ is compact.

Proof By 2.2.2

$$W_{mn}(x_m, \dots, x_n) \geq (n-m)C + \sum |x_i - x_{i+1}|$$

In particular $W_{mn} \geq (n-m)C$, thus it is enough to show that if $W_{mn}(x_m, \dots, x_n) \leq (n-m)C + a$, for some $a>0$, then the x_i lie in some bounded set in $\mathbb{R}^{n-m-1}$. So suppose that $W_{mn}(x_m, \dots, x_n) \leq (n-m)C + a$ .Then

$$\sum |x_i - x_{i+1}| \leq a$$
$$\Rightarrow |x_i - x_{i+1}| \leq a$$
$$\Rightarrow |x_m - x_i| \leq (n-m)a \qquad \forall m < i < n$$ □

2.2.4 Remark The minimizing segment is not necessarily unique.

The next result is a weak version of Aubry's Fundamental Lemma. We shall not be able to prove the strongest form of this until §2.6 . However for most applications the version we prove here is sufficient, and indeed its use is crucial throughout this chapter.

2.2.5 Definition We say that two distinct stationary segments $\underline{u}_{mn}, \underline{v}_{mn}$ *cross* if $u_i - v_i$ has a zero or change of sign in $[m,n]$. Two states $\underline{u}, \underline{v}$ *cross* in $[m,n]$ if the corresponding segments $\underline{u}_{mn}, \underline{v}_{mn}$ cross. In both cases we say they *cross once* if there is only one change of sign or zero in $[m,n]$ and *cross properly* if there is a change of sign but no zero in $[m,n]$.

2.2.6 Proposition Given $m < n-1$, let $\underline{u}_{mn}, \underline{v}_{mn}$ be two minimizing segments for W_{mn} with either $u_m \neq v_m$ or $u_n \neq v_n$ (or both) ; then $\underline{u}_{mn}, \underline{v}_{mn}$ cross at most once (see Fig. 2.2.1 and Fig. 2.2.2).

Proof of 2.2.6 By contradiction. Suppose we have two distinct minimizing segments $\underline{u}_{mn}, \underline{v}_{mn}$ which do not satisfy 2.2.6 . Then by reversing time if necessary we can find $[k,l] \subset [m,n]$ such that one of the following three cases holds (Fig. 2.2.3)

1) $u_k \neq v_k$, $u_l = v_l$, $u_{k+1} - v_{k+1}$ has opposite sign to $u_k - v_k$ and $u_i - v_i$ is non-zero and has constant sign for $i \in [k+1, l-1]$.
2) $u_i - v_i \neq 0 \ \forall i \in [k,l]$, and $u_i - v_i$ has precisely two changes of sign in $[k,l]$.
3) $u_k \neq v_k$, $u_{k+1} = v_{k+1}$, $u_l = v_l$ and $u_i - v_i$ is non-zero and has constant sign for $i \in [k+2, l-1]$.

We claim that each of these cases leads to a contradiction. Note that $\underline{u}, \underline{v}$ are still minimizing segments on $[k,l]$. Wlog $u_k < v_k$, so define segments $\underline{u}', \underline{v}'$ by (Fig. 2.2.3)

$$u'_i = \min(u_i, v_i)$$
$$v'_i = \max(u_i, v_i)$$

Thus $\underline{u}$ and $\underline{u}'$ have the same endpoints on $[k,l]$, and similarly for $\underline{v}$ and $\underline{v}'$.

Case 1)

$$\begin{aligned} \Delta W &= W_{kl}(\underline{u}') + W_{kl}(\underline{v}') - W_{kl}(\underline{u}) - W_{kl}(\underline{v}) \\ &= h(u_k, v_{k+1}) + h(v_k, u_{k+1}) - h(u_k, u_{k+1}) - h(v_k, v_{k+1}) \\ &= \int_{u_k}^{v_k} \int_{v_{k+1}}^{u_{k+1}} h_{12}(\xi, \xi')\, d\xi\, d\xi' \end{aligned}$$

But $h_{12}(\xi,\xi') < 0 \quad \forall (\xi,\xi') \in \mathbb{R}^2$ and hence $\Delta W < 0$. But this implies that either $W_{kl}(\underline{u}') < W_{kl}(\underline{u})$

or $W_{kl}(\underline{v}') < W_{kl}(\underline{v})$

This contradicts the minimality of $\underline{u}$ or $\underline{v}$. ※

Case 2) is dealt with similarly except that now ΔW has two terms, both negative.

Case 3) : now $W_{kl}(\underline{u}') + W_{kl}(\underline{v}') = W_{kl}(\underline{v}) + W_{kl}(\underline{u})$.
Note that by 1.5.14 stationarity determines u_{k+2} and v_{k+2} uniquely. However $u_k = u'_k$, $u_{k+1} = u'_{k+1}$, $\underline{u}$ is stationary and hence $\underline{u}'$ cannot be stationary. Similarly $\underline{v}'$ cannot be stationary. In particular neither $\underline{u}'$ nor $\underline{v}'$ are minimizing. Hence at least one of $\underline{u}$ or $\underline{v}$ is not minimizing. ✻

□

2.2.7 Corollary Two distinct minimizing states $\underline{u},\underline{v}$ can cross at most once in $\mathbf{Z}$.

2.2.8 Remark We shall see in §2.6 that in fact if they do cross then $u_i - v_i$ is bounded away from zero everywhere else, and in particular $\underline{u}$ and $\underline{v}$ cannot be asymptotic to each other either at $+\infty$ or $-\infty$.

§ 2.3 Periodic Minimizing States

In this section we first show the existence of states which minimize the action $W_{p/q}$ (defined in 1.6.10) and then prove that such states are in fact minimizing states. The existence of these periodic states is crucial to the development of the theory in the rest of this chapter. Also recall that such a minimizing state corresponds precisely to a type p,q periodic orbit of non-positive residue in the Poincare-Birkhoff Theorem (1.8.1) .

2.3.1 Proposition Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the lift of an area-preserving twist map of the cylinder $\mathbb{T}^1 \times \mathbb{R}$, with zero net flux. Then $\forall\, p/q \in \mathbb{Q}$, (p,q not necessarily co-prime) there exists a state $\underline{u} \in X_{p,q}$ which is a (global) minimum of $W_{p/q}$ on $X_{p,q}$.

Proof The proof is very similar to that of 2.2.1 .
Recall that $X_{p,q} = \{\, \underline{x} \in \mathbb{R}^{\mathbf{Z}} : x_{n+q} = x_n + p \,,\ \forall\, n \in \mathbf{Z} \,\}$.
The map $\underline{x} \to (x_0, \ldots, x_{q-1})$ then naturally identifies $X_{p,q} \simeq \mathbb{R}^q$.
Let $R^* : X_{p,q} \to X_{p,q}$ be defined by $[R^*(\underline{x})]_i = x_i + 1$.
Then $W_{p/q} \circ R^* = W_{p/q}$, so $W_{p/q}$ is a function on $X_{p,q}/R^*$. This latter space can be identified in the obvious fashion with $\mathbb{R}/\mathbf{Z} \times \mathbb{R}^{q-1}$, in other words wlog we can take $0 \le x_0 < 1$. We will show that $\forall\, a \in \mathbb{R}$, the set $\{\, \underline{x} \in X_{p,q}/R^* : W_{p/q}(\underline{x}) \le a \,\}$ is compact, which is enough to guarantee the existence of the required minimum of $W_{p/q}$.
As before, by 2.2.2 , $W_{p/q}(x_0, \ldots, x_{q-1}) \ge qC + \sum |x_i - x_{i+1}|$,
and thus $W_{p/q} \ge qC$. The only difficulty is that x_0 is no longer fixed, however by above we may take $x_0 \in [0,1)$ and thus
$W_{p/q}(x_0, \ldots, x_{q-1}) \le qC + a \quad \Rightarrow \quad -qa \le x_i \le qa + 1 \,,\ \forall\, i = 0 \ldots q-1$.

2.3.2 Remark If $\underline{x}$ minimizes $W_{p/q}$ then it minimizes W_{0q} and indeed $W_{n,n+q}$, $\forall n \in \mathbf{Z}$.

2.3.3 Proposition If $\underline{u} \in X_{p,q}$ minimizes $W_{p/q}$ then $\underline{u}$ is a monotone state.

Proof Define the states $\underline{v}(m,n) \in X_{p,q}$ by $v(m,n)_i = u_{i+m} + n$. If $\underline{u}$ is not monotone , then $\exists\, m,n \in \mathbf{Z}$ s.t. $\underline{v}(m,n)$ and $\underline{u}$ are distinct and cross in $[0,q]$. Wlog $u_0 \neq v(m,n)_0$ (by choice of origin in time). Then by periodicity $\underline{v}(m,n)$ and $\underline{u}$ must cross again in $[0,q]$ (see Fig. 2.3.1). But $\underline{u}$ and $\underline{v}(m,n)$ minimize W_{0q} .This contradicts 2.2.6 .

2.3.4 Proposition If $\underline{u} \in X_{kp,kq}$ minimizes $W_{kp/kq}$ for some $k>1$, then $\underline{u} \in X_{p,q}$, i.e. $\underline{u}$ is of type p,q.

Proof Suppose $\exists\, \underline{u} \in X_{kp,kq}$, $\underline{u} \notin X_{p,q}$ minimizing $W_{kp/kq}$. Then $u_q \neq u_0 + p$, so wlog $u_q > u_0 + p$. Define $\underline{v}$ by $v_i = u_{i+q} - p$. Then both $\underline{u}$ and $\underline{v}$ minimize $W_{kp/kq}$ and hence both minimize $W_{0,kq}$. But $v_0 = u_q - p > u_0$ and hence $v_{kq} > u_{kq}$. Thus by 2.2.6 $v_i > u_i \;\; \forall i \in \mathbf{Z}$. Hence

$$
\begin{aligned}
u_{kq} &= v_{(k-1)q} + p > u_{(k-1)q} + p = v_{(k-2)q} + 2p > u_{(k-2)q} + 2p \\
&\;\vdots \\
&> u_0 + kp \qquad\qquad ※ \qquad\qquad \text{since } \underline{u} \in X_{kp,kq} .
\end{aligned}
$$

□

2.3.5 Corollary $\underline{u}$ minimizes $W_{p/q} \Leftrightarrow \underline{u}$ minimizes $W_{kp/kq}$.

Proof By 2.3.1 $\exists\, \underline{v} \in X_{kp,kq}$ s.t. $\underline{v}$ minimizes $W_{kp/kq}$. By 2.3.4 , $\underline{v} \in X_{p,q}$. To prove the corollary it is thus sufficient to show that $W_{p/q}(\underline{u}) = W_{p/q}(\underline{v})$ and $W_{kp/kq}(\underline{u}) = W_{kp/kq}(\underline{v})$. But:

$\underline{v} \in X_{p,q}$ and $\underline{v}$ minimizes $W_{kp/kq}$

$$\Rightarrow \quad kW_{p/q}(\underline{v}) = W_{kp/kq}(\underline{v}) \leq W_{kp/kq}(\underline{u}) = kW_{p/q}(\underline{u})$$

and $\underline{u}$ minimizes $W_{p/q}$

$$\Rightarrow \quad [W_{kp/kq}(\underline{u})]/k = W_{p/q}(\underline{u}) \leq W_{p/q}(\underline{v}) = [W_{kp/kq}(\underline{v})]/k$$

Hence $W_{p/q}(\underline{u}) = W_{p/q}(\underline{v})$ and $W_{kp/kq}(\underline{u}) = W_{kp/kq}(\underline{v})$ as required.

□

2.3.7 Remark In higher dimensions Prop. 2.3.4 and Cors 2.3.5 and 2.3.6 are no longer true [Hedlund, 1932].

2.3.6 Corollary $\underline{u}$ minimizes $W_{p/q} \Rightarrow \underline{u}$ is a minimizing state.

Proof

$$
\begin{aligned}
& \underline{u} \text{ minimizes } W_{p/q} \\
\Rightarrow\; & \underline{u} \text{ minimizes } W_{kp/kq} && \forall k>1 \\
\Rightarrow\; & \underline{u} \text{ minimizes } W_{0,kq} && \forall k>1 \\
\Rightarrow\; & \underline{u} \text{ minimizes } W_{n,n+kq} && \forall k>1 ,\; n \in \mathbf{Z} \\
\Rightarrow\; & \underline{u} \text{ is a minimizing state}
\end{aligned}
$$

□

2.3.8 Proposition: All minimizing states satisfying $x_{i+q} = x_i + p$ are minima of $W_{p/q}$

Proof: Suppose not. Then we can decrease $W_{p/q}$ by $\epsilon > 0$ by a periodic perturbation. Suppose it costs energy E to push the end points x_0, x_q back. Choose $N > E/\epsilon$. Take the segment $x_0, \dots x_{Nq}$. Apply the periodic perturbation and then push x_0, x_{Nq} back.

§2.4 Existence of Quasi-Periodic Minimizing States

The main aim of this section is to prove the Aubry-Mather theorem which states that for any irrational ω, there is either an invariant circle or an invariant cantor set of rotation number ω. In contrast to homeomorphisms of the circle (§1.4) the case of a Cantor set is not a pathology for area-preserving maps. It can occur even for analytic area-preserving maps (see 2.4.12). First we prove the existence of minimizing orbits with rotation number ω as a limit of minimizing periodic orbits and then show that there is a cantor set's or circle's worth of such orbits. Our approach to this is based on [Aubry and Le Daeron, 1983] and [Katok, 1982, 1983]: the underlying philosophy is to use the monotonicity of periodic minimizing states. Indeed the results and proofs in this section all remain valid if we replace 'minimizing' by 'monotone'. It is precisely this monotonicity which allows us to take limits of states, and conclude that the limiting states have the correct rotation number (recall 1.7.15, 1.7.16 and 1.7.17). Mather's proof [1982] uses a different approach, based on an idea of Percival [1980]. We outline this in 2.4.13 .

2.4.1 Theorem $\forall\, \omega \in \mathbb{R}\backslash\mathbb{Q}$, $\exists$ minimizing monotone state of rotation number ω .

Proof By 2.3.2, 2.3.6 and 2.3.10 we have the existence of monotone minimizing states of rotation number p/q for all $p/q \in \mathbb{Q}$. So take a sequence $p_n/q_n \in \mathbb{Q}$ s.t. $p_n/q_n \to \omega$ as $n \to \infty$. By 1.7.17 we conclude that there exists a monotone state of rotation number ω. It is also minimizing as a consequence of the following lemma:

2.4.2 Lemma Let $\underline{u}(k)$ be a sequence of minimizing states with $u(k)_i \to u_i$ as $i \to \infty$. Then $\underline{u}$ is a minimizing state.

Proof Take any $m,n \in \mathbb{Z}$ with $m < n-1$. We wish to show that if $\underline{v} = \{ v_i : m \le i \le n \}$ is any segment with $v_m = u_m$ and $v_n = u_n$, then $W_{mn}(\underline{v}) \ge W_{mn}(\underline{u})$.

Define $\quad \varepsilon(k) = \max_{m \le i \le n} |u(k)_i - u_i|$

$$v(k)_i = \begin{cases} u(k)_i & i = m\ ,\ i = n \\ v_i & m < i < n \end{cases}$$

Since h is C^1 there exists a $K>0$ and $N \in \mathbb{N}$ s.t $\forall k \ge N$, $\forall\, m \le i < n$:

$$|h(u(k)_i, u(k)_{i+1}) - h(u_i, u_{i+1})| < K\varepsilon(k)$$

Thus $\quad |W_{mn}(\underline{u}(k)) - W_{mn}(\underline{u})| < K(n-m)\varepsilon(k) \qquad \forall k \ge N$

and $\quad |W_{mn}(\underline{v}(k)) - W_{mn}(\underline{v})| < 2K\varepsilon(k) \qquad \forall k \ge N$

Hence

$$W_{mn}(\underline{v}) - W_{mn}(\underline{u}) = W_{mn}(\underline{v}(k)) - W_{mn}(\underline{u}(k)) + [W_{mn}(\underline{u}(k)) - W_{mn}(\underline{u})] - [W_{mn}(\underline{v}(k)) - W_{mn}(\underline{v})]$$

So $\forall k \ge N$

$$W_{mn}(\underline{v}) - W_{mn}(\underline{u}) > W_{mn}(\underline{v}(k)) - W_{mn}(\underline{u}(k)) - K(n-m+2)\varepsilon(k)$$

But $W_{mn}(\underline{v}(k)) - W_{mn}(\underline{u}(k)) \ge 0$, $\forall k \in \mathbb{N}$ and $\varepsilon(k) \to 0$, as $k \to \infty$, thus

$$W_{mn}(\underline{v}) - W_{mn}(\underline{u}) \ge 0$$

□

2.4.3 Theorem ([Aubry and Le Daeron, 1983], [Mather 1982])

$\forall\, \omega \in \mathbb{R}\setminus\mathbb{Q}$, $\exists$ a monotone set M_ω of recurrent minimizing states of rotation number ω, whose image $M^\#_\omega$ on $\mathbb{T}^1\times\mathbb{R}$ is either an invariant circle or an invariant cantor set. Furthermore, every orbit in $M^\#_\omega$ is dense in $M^\#_\omega$ (i.e. $M^\#_\omega$ is minimal in the sense of topological dynamics).

Proof For the proof we follow [Aubry and Le Daeron, 1983]. We shall first construct M_ω and then show that it has the properties claimed. So, let $\underline{u}$ be a monotone minimizing state of rotation number ω. For $p,q \in \mathbb{Z}$, let $\beta = q\omega - p$ and define $u(\beta)_n = u_{n+q} - p$.

Since $\underline{u}$ is monotone, from 1.7.10 we get:

$$\beta > \beta' \Rightarrow u(\beta)_n > u(\beta')_n \;,\; \forall\, n \in \mathbb{Z}\,. \qquad 2.4.4$$

Thus we can define $f^\pm : \mathbb{R} \to \mathbb{R}$ by:

$$f^+(\theta) = \inf_{\beta>\theta} u(\beta)_0$$

$$f^-(\theta) = \sup_{\beta<\theta} u(\beta)_0$$

Next define states $\underline{u}^\pm(\theta)$, for $\theta \in \mathbb{R}$ by:

$$u^+(\theta)_n = \inf_{\beta>\theta} u(\beta)_n = f^+(n\omega + \theta) \qquad 2.4.5$$

$$u^-(\theta)_n = \sup_{\beta<\theta} u(\beta)_n = f^-(n\omega + \theta)$$

Then $\underline{u}^\pm(\theta)$ are limits of monotone minimizing states of rotation number ω, hence by 1.7.15 and 2.4.2 they are such states themselves.

Define

$$y^\pm(\theta)_n = -h_1(u^\pm(\theta)_n, u^\pm(\theta)_{n+1})$$
$$y^\pm(\theta) = y^\pm(\theta)_0 = -h_1(f^\pm(\theta), f^\pm(\theta+\omega))$$

$$M^\pm_\omega = \{ (u^\pm(\theta)_n, y^\pm(\theta)_n) : \theta \in \mathbb{R} \,,\, n \in \mathbb{Z} \}$$
$$= \{ (f^\pm(\theta), y^\pm(\theta)) : \theta \in \mathbb{R} \}$$

$$M_\omega = M^+_\omega \cup M^-_\omega$$

Note that the dynamics on M_ω is given very simply by:

$$T(f^\pm(\theta), y^\pm(\theta)) = (f^\pm(\theta+\omega), y^\pm(\theta+\omega)) \qquad 2.4.6$$

To prove the theorem we will show:

i) M_ω is invariant under T and R.
ii) M_ω is a closed monotone set.
iii) Every orbit in $M^\#_\omega$ is recurrent and dense in $M^\#_\omega$.
iv) $M^\#_\omega$ is either a circle or a cantor set.

First, some general properties of $f^\pm$:

2.4.7 Lemma

1) $f^\pm(\theta)$ are (strictly) increasing.
2) $f^\pm(\theta+1) = f^\pm(\theta) + 1$, $\forall\, \theta \in \mathbb{R}$.
3) f^- is left continuous.
4) f^+ is right continuous.
5) $f^-(\theta) \le f^+(\theta)$, $\forall\, \theta \in \mathbb{R}$.

Proof Exercise

2.4.8 Lemma $\theta_i \searrow \theta \;\Rightarrow\; f^{\pm}(\theta_i) \searrow f^{+}(\theta)$ and hence $y^{\pm}(\theta_i) \to y^{+}(\theta)$

$\theta_i \nearrow \theta \;\Rightarrow\; f^{\pm}(\theta_i) \nearrow f^{-}(\theta)$ and hence $y^{\pm}(\theta_i) \to y^{-}(\theta)$

Proof First we show that $\forall\, \theta,\theta' \in \mathbb{R}$, $\theta < \theta' \;\Rightarrow\; f^{+}(\theta) < f^{-}(\theta')$: choose $\beta = q\omega - p$ and $\beta' = q'\omega - p'$ with $\theta < \beta < \beta' < \theta'$, then by 2.4.5 $f^{+}(\theta) \le u(\beta)_0 < u(\beta')_0 \le f^{-}(\theta')$. Now suppose that $\theta_i \searrow \theta$; since f^{+} is right continuous $f^{+}(\theta_i) \searrow f^{+}(\theta)$, but $f^{+}(\theta_i) \ge f^{-}(\theta_i) > f^{+}(\theta)$ and hence $f^{-}(\theta_i) \searrow f^{+}(\theta)$. Thus also $f^{\pm}(\theta_i+\omega) \searrow f^{+}(\theta+\omega)$ and since $y^{\pm}(\theta)$ is a continuous function of $f^{\pm}(\theta)$ and $f^{\pm}(\theta+\omega)$ we have $y^{\pm}(\theta_i) \to y^{+}(\theta)$ as claimed. If $\theta_i \nearrow \theta$ the proof is similar.

□

i) That M_ω is invariant under T follows from 2.4.6 , whilst the invariance under R is an immediate consequence of $f^{\pm}(\theta+1) = f^{\pm}(\theta)+1$.

ii) First note that both M^{+}_{ω} and M^{-}_{ω} are monotone since f^{+} and f^{-} are strictly increasing. So by 1.7.4 to show that M_ω is monotone it is enough to prove that $M_\omega = \mathrm{Cl}(M^{+}_{\omega}) = \mathrm{Cl}(M^{-}_{\omega})$. Now by 2.4.8 $M^{-}_{\omega} \subset \mathrm{Cl}(M^{+}_{\omega})$ and thus $M_\omega \subset \mathrm{Cl}(M^{+}_{\omega})$ and similarly $M_\omega \subset \mathrm{Cl}(M^{-}_{\omega})$. It remains to prove that M_ω is closed. Suppose we have a sequence $\underline{x}_i \in M_\omega$ with $\underline{x}_i \to \underline{x} = (x,y)$, wlog $\underline{x}_i = (f^{\sigma}(\theta_i), y^{\sigma}(\theta_i))$ for some $\sigma \in \{+,-\}$, and wlog either $f^{\sigma}(\theta_i) \searrow x$ or $f^{\sigma}(\theta_i) \nearrow x$. In the former case by 1) and 2) of 2.4.7 , $\{\alpha : f^{+}(\alpha) > x\}$ is bounded below so let $\theta = \inf\{\alpha : f^{+}(\alpha) > x\}$. Then $f^{+}(\theta) \ge x$, but $f^{+}(\theta_i) > x$ and by 2.4.8 $f^{+}(\theta_i) \searrow x$, hence $f^{+}(\theta) = x$. Thus $(f^{\sigma}(\theta_i), y^{\sigma}(\theta_i)) \to (f^{+}(\theta), y^{+}(\theta))$ and hence $\underline{x} = (f^{+}(\theta), y^{+}(\theta)) \in M_\omega$. Similarly if $f^{\sigma}(\theta_i) \nearrow x$ then

$(f^{\sigma}(\theta_i), y^{\sigma}(\theta_i)) \to (f^{-}(\theta), y^{-}(\theta))$ and thus M_ω is closed.

iii) Suppose $\underline{x} \in M^{+}_{\omega}$, say $\underline{x} = (f^{+}(\alpha), y^{+}(\alpha))$. Given $\underline{x}' \in M_\omega$, we have $\underline{x}' = (f^{\sigma}(\theta), y^{\sigma}(\theta))$ for some $\theta \in \mathbb{R}$, $\sigma \in \{+,-\}$. Choose sequences $m_i, n_i \in \mathbb{Z}$ s.t. $(\theta + n_i\omega - m_i) \searrow \alpha$, then 2.4.8 ,

$[f^{\sigma}(\theta + n_i\omega) - m_i] \searrow f^{+}(\alpha)$ and $[y^{\sigma}(\theta + n_i\omega - m_i)] \to y^{+}(\alpha)$

Thus $T^{n_i}R^{m_i}(\underline{x}') = (f^{\sigma}(\theta + n_i\omega) - m_i \,,\, y^{\sigma}(\theta + n_i\omega - m_i))$

$\to (f^{+}(\alpha), y^{+}(\alpha)) = \underline{x}$ as required.

A similar argument works if $\underline{x} \in M^{-}_{\omega}$. To show that $\underline{x}$ is recurrent simply take $\theta = \alpha$ in the above. As a corollary M_ω has no isolated points.

iv) To see that M^{*}_{ω} is either a circle or a cantor set we can either follow the proof of 1.4.12 or consider the homeomorphism $g : \mathbb{T}^1 \to \mathbb{T}^1$ obtained by extending $\pi \circ T \circ \pi^{-1} : \pi(M_\omega) \to \pi(M_\omega)$ as in 1.7.9 . We do the latter here; the former argument is given in 2.4.10. Let $\pi' : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1$ be the projection $\pi'(\theta,y) = \theta$, note that this is a homeomorphism of M^{*}_{ω} onto $\pi'(M^{*}_{\omega})$. Then by i) and ii) $\pi'(M^{*}_{\omega})$ is a closed g-invariant subset of $\mathbb{T}^1$, and by iii) every orbit (under g) in $\pi'(M^{*}_{\omega})$ is dense. Hence $\pi'(M^{*}_{\omega})$ is the unique minimal set for g (in the sense of topological dynamics). As we proved in 1.4.13, this can either be the whole of $\mathbb{T}^1$ or a Cantor set. Thus $M^{*}_{\omega} \simeq \pi'(M^{*}_{\omega})$ is either a circle or a Cantor set.

□

2.4.9 Remark Since $f^{\pm}$ are strictly increasing they are continuous except possibly for a countable set of jumps. By lemma 2.4.8 these

correspond precisely to those points where $f^- \neq f^+$ i.e.
$f^-(\theta) = f^+(\theta) \Leftrightarrow f^+$ continuous at $\theta \Leftrightarrow f^-$ continuous at θ. Thus M_ω is a circle iff $f = f^- = f^+$ is continuous. In this case f is actually a homeomorphism by 1) and 2) of 2.4.7 and f^{-1} is precisely the conjugacy of T on M_ω to the uniform rotation R_ω. On the other hand if $f^- \neq f^+$ the discontinuities of $f^\pm$ correspond precisely to the 'gaps' in the cantor set M_ω, thus if $f^-(\theta) < f^+(\theta)$ we see that $f^-(\theta)$ and $f^+(\theta)$ correspond respectively to the left and right ends of a gap. In this case $f^\pm$ have a common left inverse $h(x) = \inf\{ \theta : f^+(\theta) > x\} = \sup\{ \theta : f^-(\theta) < x\}$. It is left as an exercise to show that this is a semi-conjugacy of T on M_ω to R_ω on $\mathbb{R}$.

2.4.10 Remark The discontinuities in $f^\pm$ come in orbits under $\theta \to \theta+\omega$ i.e.

$$\forall\, \theta\in\mathbb{R}\ ,\ f^-(\theta) = f^+(\theta) \Leftrightarrow f^-(\theta+\omega) = f^+(\theta+\omega) \quad \text{and hence}$$
$$\forall\, \theta\in\mathbb{R}\ ,\ f^-(\theta) < f^+(\theta) \Leftrightarrow f^-(\theta+\omega) < f^+(\theta+\omega)$$

To see this note that $f^-(\theta) = f^+(\theta) \Rightarrow u^-(\theta)_0 = u^+(\theta)_0$. By 2.4.7 $u^-(\theta)_{-1} \leq u^+(\theta)_{-1}$, if $u^-(\theta)_{-1} < u^+(\theta)_{-1}$ then by twist (1.5.15) $u^-(\theta)_1 > u^+(\theta)_1$ which contradicts 2.4.7. Hence $u^-(\theta)_{-1} = u^+(\theta)_{-1}$ and $u^-(\theta)_1 = u^+(\theta)_1$, in other words $f^-(\theta-\omega) = f^+(\theta-\omega)$ and $f^-(\theta+\omega) = f^+(\theta+\omega)$. This allows us to show that $M^{\#}_\omega$ is a Cantor set when $f^\pm$ are not continuous without appealing to the theory of homeomorphisms of the circle: if $f^\pm$ has a jump at θ, then it has a jump at $\theta + n\omega + m$, $\forall\, m,n\in\mathbb{Z}$. These points are dense in $\mathbb{R}$, hence there is a gap between any two points of $M^{\#}_\omega$, and thus $M^{\#}_\omega$ is totally disconnected. We have already shown that $M^{\#}_\omega$ is closed and has no isolated points, hence it must be a Cantor set. The reader should compare this argument to the proof of 1.4.13 .

2.4.11 Remark The original state $\underline{u}$, used to construct M_ω need not lie in M_ω, but does if it is recurrent. To show this, note that by 2.4.5 ,

$$u^-(\beta)_i \leq u(\beta)_i \leq u^+(\beta)_i \qquad \forall\, i\in\mathbb{Z}\ ,\ \beta = q\omega - p \qquad 2.4.12$$

Suppose that for some $k\in\mathbb{Z}$ and $\beta = q\omega - p$ we had $u^-(\beta)_k < u(\beta)_k < u^+(\beta)_k$. Then if $\underline{u}$ (and hence the $\underline{u}(\beta)$) are recurrent we can find $m,n\in\mathbb{Z}$ s.t. $u^-(\beta)_k < u(\beta)_{k+m} - n < u^+(\beta)_k$ i.e. $u^-(\beta)_k < u(\beta + m\omega - n)_k < u^+(\beta)_k$. Now if $m\omega - n > 0$, by definition $u^+(\beta)_k \leq u(\beta + m\omega - n)_k$ whilst if $m\omega - n < 0$, then $u(\beta + m\omega - n)_k \leq u^-(\beta)_k$. But ω is irrational so $m\omega - n \neq 0$. ※
Thus either $u(\beta)_k = u^+(\beta)_k$ or $u(\beta)_k = u^-(\beta)_k$ and hence either $\underline{u}(\beta) = \underline{u}^+(\beta)$, $\forall\, \beta = q\omega - p$ or $\underline{u}(\beta) = \underline{u}^-(\beta)$, $\forall\, \beta = q\omega - p$.

On the other hand if $\underline{u}$ is not recurrent, then by 2.4.3 it cannot belong to M_ω. Thus we have strict inequalities in 2.4.12 , $M^{\#}_\omega$ is a Cantor set and the orbit given by $\underline{u}$ lies in the orbit of some gap of this Cantor set (recall 1.4.12 and see §2.8) .

2.4.12 Example [Aubry, 1983] When $k > 2$ all the $M^{\#}_\omega$, $\omega\in\mathbb{R}\backslash\mathbb{Q}$, for the standard map

$$y' = y - (k/2\pi)\sin(2\pi x)$$
$$x' = x + y' \qquad (\text{mod } 1)$$

are Cantor sets.

Proof Let $\underline{u}$ be a stationary state. Consider the segment $\underline{u}_{-1,1}$; the second variation of the action with respect to u_0 and fixed end points is given by:

$$\frac{\partial^2 W_{-1,1}}{\partial u_0^2} = 2 - k\cos(2\pi u_0)$$

Thus if $k > 2$ then no minimizing orbit can have a point on $x = 0$. Thus M^*_ω cannot be a circle for any irrational ω.

In fact by considering longer orbit segments this result can be improved to $k > 1.23...$ [Aubry, 1983] ([Mather, 1984 c] gets $k > 4/3$ by a similar method). By doing rigorous computer estimates [Mackay and Percival, 1985] are able to show that M^*_ω is a Cantor set for all irrational ω for $k \geq 63/64$ (see Chapters 5 and 6).

2.4.13 The Percival-Mather Approach Mather's proof [1982] of 2.4.3 is based on the following idea of Percival [1980]. Given $\eta : \mathbb{R} \to \mathbb{R}$, with $\eta(x+1) = \eta(x) + 1$ and $\omega \in \mathbb{R}$ consider the functional:

$$F_\omega(\eta) = \int_0^1 h(\eta(t), \eta(t+\omega))\, dt$$

If η is continuous and makes F_ω stationary with respect to variations in the same class, then one easily sees that the set:

$$M^*_\omega = \{ (\eta(t), -h_1(\eta(t), \eta(t+\omega))) : t \in \mathbb{R} \} / R$$

is a rotational invariant circle of rotation number ω. Thus one might hope to prove existence of rotational invariant circles by proving existence of a minimum of F_ω. Unfortunately F_ω does not necessarily have a minimum in the space of continuous η. Percival [1980] suggested that it might however have a minimum if η is only required to be increasing. If η were discontinuous and ω irrational then instead

of a circle the closure of M^*_ω would give an invariant Cantor set which he called a *cantorus*. These conjectures were subsequently proved by Mather [1982]. If we require η to be right continuous and ω is irrational then up to a phase difference the minimizing η is precisely the function f^+ of our proof of 2.4.3, i.e. $\exists\, \delta \in \mathbb{R}$ s.t. $\eta(t) = f^+(t+\delta)$. This approach also works if $\omega = p/q$ is rational, in which case it gives the existence of a periodic orbit of type p,q which minimizes $W_{p/q}$ (recall 2.3.1).

§2.5 Hedlund's Lemma

Having proved the existence of periodic (§2.3) and quasiperiodic (§2.4) minimizing states we go on to derive some useful properties of minimizing states in general. These will be required in §2.8 and §2.9 where we will describe and classify the set of all minimizing states. Suppose that $\underline{u} \in X_{p,q}$ is a minimizing periodic state. By 2.3.3, it is a monotone state and hence by 1.7.11 u_n lies within a bounded distance of some straight line. The slope of this line clearly must be p/q, i.e.

$$| u_n - u_0 - np/q | \leq 1 .$$

In this section we will show that in fact any minimizing state is within a bounded distance of some straight line. The basic idea, due to Hedlund, is to bound an arbitrary minimizing state by nearby periodic states using 2.2.6 and 2.2.8.

2.5.1 Lemma If $\underline{u} = \{ u_i : m \leq i \leq n \}$ is a minimizing segment then $\underline{u}$ remains within 3 of the straight line joining u_m to u_n, i.e. if $\lambda = (u_n - u_m)/(n - m)$ then:

$$| u_i - u_m - (i - m)\lambda | \leq 3 .$$

Proof Let $p = [u_n - u_m]$ = greatest integer less than or equal to $u_n - u_m$, and let q = n-m. By 2.3.2 and 2.3.10 $\exists$ minimizing periodic states $\underline{v}^-$, $\underline{v}^+$ of types (p,q) and (p+1,q) respectively and s.t. (Fig. 2.5.1):

$$u_m - 1 < v^-_m \leq u_m$$
$$u_m \leq v^+_m < u_m + 1$$

then

$$v^-_n = v^-_m + p \leq u_m + p \leq u_m + u_n - u_m = u_n$$
$$v^+_n = v^+_m + p + 1 \geq u_m + p + 1 \geq u_m + u_n - u_m = u_n$$

thus

$$v^-_m \leq u_m \leq v^+_m$$
$$v^-_n \leq u_n \leq v^+_n$$

Hence by Aubry's Fundamental Lemma (2.2.6)

$$v^-_i \leq u_i \leq v^+_i \qquad \forall\ m \leq i \leq n$$

But by 1.7.17, both $\underline{v}^-$ and $\underline{v}^+$ remain within 1 of the straight line joining their ends:

$$| v^-_i - v^-_m - (i-m)p/q | \leq 1$$
$$| v^+_i - v^+_m - (i-m)(p+1)/q | \leq 1$$

Thus since $| p/q - \lambda | \leq 1/q$ and $| (p+1)/q - \lambda | \leq 1/q$

$$|v^-_i - v^-_m - (i-m)\lambda| \leq |v^-_i - v^-_m - (i-m)p/q| + |(i - m)(p/q - \lambda)|$$
$$\leq 2$$

and similarly

$$|v^+_i - v^+_m - (i-m)\lambda| \leq 2$$

thus

$$u_i \geq v^-_i \geq v^-_m + (i-m)\lambda - 2 \geq u_m + (i-m)\lambda - 3$$
$$u_i \leq v^+_i \leq v^+_m + (i-m)\lambda + 2 \leq u_m + (i-m)\lambda + 3$$

□

2.5.2 Corollary If $\underline{u}$ is a minimizing state then $\exists\ \omega, c \in \mathbb{R}$ s.t. $\underline{u}$ remains within 3 of the straight line given by $n\omega + c$, i.e. :

$$| u_n - (n\omega + c) | \leq 3 .$$

Proof Let $\omega_n = (u_n - u_{-n})/2n$

$$c_n = u_{-n}$$
$$L_n = n\omega_n + c_n$$
$$K_n = (n+1)\omega_n + c_n$$

Then

$$|u_0 - L_n| \leq 3 \quad \text{and} \quad |u_1 - K_n| \leq 3$$

Hence (L_n, K_n) is a bounded sequence in $\mathbb{R}^2$. Take a convergent subsequence $(L_{n(i)}, K_{n(i)}) \to (L,K)$. Then define :

$$\omega = K - L$$
$$c = L$$

Then given m, $\forall\, n \geq |m|$ we have:

$$| u_m - [(n+m)\omega_n + c_n] | \leq 3$$
$$| u_m - [(n+m)(K_n - L_n) + L_n - n(K_n - L_n)] | \leq 3$$

$$| u_m - [m(K_n - L_n) + L_n] | \leq 3$$

By taking the limit for the convergent subsequence we obtain:

$$| u_m - (m(K-L) + L) | \leq 3$$
$$| u_m - (m\omega + c) | \leq 3$$

□

2.5.3 Corollary Every minimizing orbit has a rotation number.

§2.6 Aubry's Fundamental Lemma

In 2.2.6 we showed that two distinct minimizing states can cross at most once in $\mathbb{Z}$. We now show that in fact being asymptotic at either $+\infty$ or $-\infty$ (or both) counts as a crossing, thus states which are asymptotic to each other cannot cross even once.

2.6.1 Lemma [Aubry and Le Daeron, 1983] Let $\underline{u},\underline{v}$ be distinct minimizing states. Then $\underline{u}$ and $\underline{v}$ cross at most once in $\mathbb{Z}$, and if they do cross, say in $[n,n+1]$, then $u_i - v_i$ is bounded away from zero outside of $[n,n+1]$, i.e. $\exists\ \delta>0$ s.t $\forall\ i\in\mathbb{Z}\setminus\{n,n+1\}$, $|u_i - v_i| > \delta > 0$.

Proof We showed that $\underline{u},\underline{v}$ can cross at most once in 2.2.6. So now suppose that $\underline{u},\underline{v}$ do cross in $[n,n+1]$. Let $\omega(\underline{u})$ and $\omega(\underline{v})$ be the rotation numbers of $\underline{u},\underline{v}$ respectively. By 2.5.2

$$|u_m - u_n - (m-n)\omega(\underline{u})| \leq 6$$
$$|v_m - v_n - (m-n)\omega(\underline{v})| \leq 6$$

So if $\omega(\underline{u}) \neq \omega(\underline{v})$

$$|u_m - v_m - (u_n - v_n) - (m-n)(\omega(\underline{u}) - \omega(\underline{v}))| \leq 12$$
$$|u_m - v_m| \geq |(m-n)(\omega(\underline{u}) - \omega(\underline{v}))| - |(u_n - v_n)| - 12$$

Hence $\exists\ N\in\mathbb{N}$ s.t

$$|u_i - v_i| > 1 \qquad \forall\ i\in\mathbb{Z} \text{ s.t. } |i-n| \geq N .$$

as required.

Now suppose that $\omega(\underline{u}) = \omega(\underline{v})$. Then

$$|u_i - v_i| \leq 12 + |u_n - v_n| \qquad \forall\ i\in\mathbb{Z} \qquad 2.6.2$$

We proceed by contradiction, so suppose there is no $\delta>0$ satisfying the conclusions of the lemma, thus $\liminf_{\{i\neq n,n+1\}} |u_i - v_i| = 0$. As in the proof of 2.2.6 we shall construct states $\underline{u}',\underline{v}'$ which differ from $\underline{u},\underline{v}$ respectively only on a finite segment $[k+1,l-1] \subset \mathbb{Z}$. We shall show that for this finite segment $W_{kl}(\underline{u}') + W_{kl}(\underline{v}') < W_{kl}(\underline{u}) + W_{kl}(\underline{v})$ which contradicts $\underline{u}$ and $\underline{v}$ both being minimal.

Define

$$\Delta_k = h(u_k,v_{k+1}) + h(v_k,u_{k+1}) - h(u_k,u_{k+1}) - h(v_k,v_{k+1})$$
$$= \int_{u_k}^{v_k}\int_{v_{k+1}}^{u_{k+1}} h_{12}(\xi,\xi')\ d\xi\ d\xi'$$

Hence Δ_k has the same sign as $(u_k - v_k)(u_{k+1} - v_{k+1})$. 2.6.3

Also since $h_{12}(\xi,\xi') = h_{12}(\xi+1,\xi'+1)$ and h_{12} is continuous, $h_{12}(\xi,\xi')$ is bounded on $|\xi - \xi'| < 12 + |u_n - v_n|$, say $|h_{12}(\xi,\xi')| \leq C$. Hence

$$|\Delta_k| \leq C\,|(u_k - v_k)|\,|(u_{k+1} - v_{k+1})| \qquad 2.6.4$$

Now consider two cases:

Case 1)
We have a proper crossing i.e. $(u_n - v_n)(u_{n+1} - v_{n+1}) < 0$.
Thus $\Delta_n < 0$. Since we assume that $\liminf_{\{i\neq n,n+1\}} |u_i - v_i| = 0$
$\exists\ m\in\mathbb{Z}$ s.t.

$$|u_m - v_m| < |\Delta_n| / C(6 + |u_n - v_n|), \quad \text{wlog take } m > n+1.$$

But by 2.6.2

$$|u_{m+1} - v_{m+1}| \le 6 + |u_n - v_n|$$

hence by 2.6.3

$$|\Delta_m| < C(6 + |u_n - v_n|)(|\Delta_n| / C(6 + |u_n - v_n|)) = |\Delta_n| .$$

Hence $\Delta_n + \Delta_m < 0$.

Define states $\underline{u}', \underline{v}'$ by (Fig. 2.6.1)

$$u'_i = \begin{cases} u_i & i \le n \text{ or } i \ge m+1 \\ v_i & n+1 \le i \le m \end{cases}$$

$$v'_i = \begin{cases} v_i & i \le n \text{ or } i \ge m+1 \\ u_i & n+1 \le i \le m \end{cases}$$

Then if $k < n$ and $l > m+1$

$$W_{kl}(\underline{u}') + W_{kl}(\underline{v}') - W_{kl}(\underline{u}) - W_{kl}(\underline{v}) = \Delta_n + \Delta_m < 0$$

Hence either $\quad W_{kl}(\underline{u}') < W_{kl}(\underline{u})$

or $\quad W_{kl}(\underline{v}') < W_{kl}(\underline{v})$

This contradicts the minimality of either $\underline{u}$ or $\underline{v}$.

Case 2)

We have a zero in $[n,n+1]$, say $u_n = v_n$. Now simply interchanging $\underline{u}$ and $\underline{v}$ at n does not decrease the total action. However, we can find two nearby states for which the total action is less, as follows:

wlog suppose $u_i < v_i$ for $i < n$. Define $\underline{u}^*, \underline{v}^*$ by

$$u^*_i = \min(u_i, v_i)$$
$$v^*_i = \max(u_i, v_i)$$

Then as in 2.2.6 , case 3) , $\underline{u}^*, \underline{v}^*$ cannot be minimizing segments on $[n-1,n+1]$.

Hence there exist states $\underline{u}^-, \underline{v}^-$ with $u^-_i = u^*_i$, $v^-_i = v^*_i$, $\forall i \ne n$, s.t.

$$W_{n-1,n+1}(\underline{u}^*) > W_{n-1,n+1}(\underline{u}^-)$$
$$W_{n-1,n+1}(\underline{v}^*) > W_{n-1,n+1}(\underline{v}^-)$$

and thus

$$\begin{aligned} \Delta &= W_{n-1,n+1}(\underline{u}^-) + W_{n-1,n+1}(\underline{v}^-) - W_{n-1,n+1}(\underline{u}) - W_{n-1,n+1}(\underline{v}) \\ &= W_{n-1,n+1}(\underline{u}^-) + W_{n-1,n+1}(\underline{v}^-) - W_{n-1,n+1}(\underline{u}^*) - W_{n-1,n+1}(\underline{v}^*) \\ &< 0 \end{aligned}$$

Now proceed as in case 1) using Δ instead of Δ_n:

Find $m \in \mathbf{Z}$ s.t.

$$|u_m - v_m| < |\Delta| / C(6 + |u_n - v_n|) , \quad \text{wlog take } m > n+1.$$

Thus $|\Delta| > |\Delta_m| \Rightarrow \Delta + \Delta_m < 0$

Now $\underline{u}', \underline{v}'$ are defined by

$$u'_i = \begin{cases} u_i & i < n \text{ or } i \ge m+1 \\ u^-_i & n \le i \le m \end{cases}$$

$$v'_i = \begin{cases} v_i & i < n \text{ or } i \geq m+1 \\ v^-_i & n \leq i \leq m \end{cases}$$

Then as before if $k < n$ and $l > m+1$

$$W_{kl}(\underline{u}') + W_{kl}(\underline{v}') - W_{kl}(\underline{u}) - W_{kl}(\underline{v}) = \Delta + \Delta_m < 0$$

which contradicts the minimality of either $\underline{u}$ or $\underline{v}$.

□

§2.7 Minimizing Implies Monotone

2.7.1 Proposition If $\underline{u}$ is a minimizing state then $\underline{u}$ is a monotone state.

Proof Suppose that $\underline{u}$ is not monotone. Then $\exists$ n,r,s $\in \mathbb{Z}$ s.t.

$$\begin{aligned} u_n &> u_{n-r} + s \\ u_{n+1} &\leq u_{n-r+1} + s \end{aligned} \qquad 2.7.2$$

Wlog take $n=0$. Clearly $r \neq 0$, so suppose that $r>0$. The argument for $r<0$ is parallel, with most of the inequalities reversed.

We claim that for some $\delta>0$ (in fact the δ given by Aubry's Fundamental Lemma (2.6.1))

i) $u_{kr} - u_0 \leq ks \qquad \forall k \geq 1$

ii) $u_0 - u_{-kr} > k(s+\delta) \qquad \forall k \geq 1$

Then

i) $\Rightarrow \quad \omega(\underline{u}) \leq s/r$

ii) $\Rightarrow \quad \omega(\underline{u}) \geq (s+\delta)/r > s/r$

giving a contradiction

Proof of i)

Define the states $\underline{u}(k)$ by :

$$u(k)_i = u_{i-kr} + ks$$

These are all minimizing states.

Now by 2.7.2 (Fig. 2.7.1) :

$$u(k)_{kr} = u_0 + ks > u_{-r} + s + ks = u(k+1)_{kr} \qquad \forall k \in \mathbf{Z}$$

and similarly:

$$u(k)_{kr+1} \le u(k+1)_{kr+1}$$

so by Aubry's Fundamental Lemma (2.2.8)

$$u(k)_i \le u(k+1)_i \qquad \forall\, i \ge kr+1 \qquad 2.7.3$$

$$u(k)_i \ge u(k+1)_i \qquad \forall\, i \le kr \qquad 2.7.4$$

Hence $\forall k \ge 1$

$$u(k)_{kr} \ge u(k-1)_{kr} \ge u(k-2)_{kr} \ge \dots \ge u(0)_{kr} \qquad 2.7.5$$

Thus

$$u_{kr} = u(0)_{kr} \le u(k)_{kr} = u_0 + ks \qquad \forall k \ge 1$$

$$u_{kr} - u_0 \le ks \qquad \forall k \ge 1$$

as claimed.

Proof of ii)

Since u(0) and u(1) cross in [0,1] , by Aubry's Fundamental Lemma (2.6.1) $\exists \delta > 0$ s.t. (see Fig. 2.7.1)

$$u(1)_i < u(0)_i - \delta \qquad \forall\, i \le 0$$

Thus

$$u_{i-r} + s < u_i - \delta \qquad \forall\, i \le 0$$

$$u_{i-r} < u_i - (s + \delta) \qquad \forall\, i \le 0$$

Hence $\forall k \ge 1$

$$u_{-kr} < u_{-(k-1)r} - (s + \delta) < u_{-(k-2)r} - 2(s + \delta) < \dots < u_0 - k(s+\delta)$$

So

$$u_0 - u_{-kr} > k(s + \delta) \qquad \forall k \ge 1$$

as claimed

□

2.7.6 Corollary Using 1.7.11 we can improve the bound in Hedlund's Lemma 2.5.2 from 3 to 1.

§2.8 Classification of Minimizing Orbits: Irrational Case

Having proved the existence of minimizing states of irrational rotation number ω, we are now ready to classify Min_ω, the set of all such states. It turns out that the set M_ω defined in §2.4 is unique and all the recurrent states of rotation number ω belong to it (recall that we showed that all orbits in M_ω are recurrent). All the other orbits in Min_ω are asymptotic to M_ω. Furthermore recall that in §2.4 we showed that M_ω is a monotone set; this also turns out to be true of Min_ω.

Recall the construction of M_ω from a monotone minimizing state of rotation number ω (2.4.3). Since we now know that all minimizing states of rotation number are monotone we can repeat this construction using any such state $\underline{v}$ of rotation number ω to give $M_\omega(\underline{v})$. Then:

2.8.1 Lemma All minimizing states $\underline{v}$ of rotation number ω give the same M_ω i.e $\forall\, \underline{u},\underline{v} \in Min_\omega$, $M_\omega(\underline{u}) = M_\omega(\underline{v})$.

Proof Let $\underline{u},\underline{v}$ be two minimizing states of rotation number ω. Define $f^\pm$ from $\underline{u}$ as in 2.4.3 and define $g^\pm$ for $\underline{v}$ similarly. We claim that $\exists\, \delta\in\mathbb{R}$ s.t. $f^\pm(\theta) = g^\pm(\theta+\delta)$, $\forall\, \theta\in\mathbb{R}$ and so $M_\omega(\underline{u}) = M_\omega(\underline{v})$. First we show that if $g^+(\beta) > f^+(\alpha)$ then $g^+(\beta+\theta) \geq f^+(\alpha+\theta)$, $\forall\, \theta\in\mathbb{R}$. Suppose not, so for some $\theta'\in\mathbb{R}$, $g^+(\beta+\theta') < f^+(\alpha+\theta')$, hence $g^+(\beta+q\omega) < f^+(\alpha+q\omega)$ for some $q > 0$. Also since ω is irrational and f^+,g^+ are right continuous we can find $m,n\in\mathbb{Z}$, $m > q$ s.t $g^+(\beta+m\omega-n) > f^+(\alpha+m\omega-n)$. So consider states $\underline{u}^+(\alpha)$ and $\underline{v}^+(\beta)$; these are given by

$u^+(\alpha)_i = f^+(i\omega+\alpha)$ and $v^+(\beta)_i = g^+(i\omega+\beta)$. Then $u^+(\alpha)_0 < v^+(\beta)_0$, $u^+(\alpha)_q < v^+(\beta)_q$ and $u^+(\alpha)_m < v^+(\beta)_m$. Hence $\underline{u}^+(\alpha)$ and $\underline{v}^+(\beta)$ cross twice in $[0,m]$ contradicting Aubry's Fundamental Lemma (2.2.6). So now define $\delta^+ = \inf\{\, \delta\in\mathbb{R} :\ g^+(\delta+\theta) \geq f^+(\theta),\ \forall\theta\in\mathbb{R}\,\}$. By right continuity $g^+(\delta^++\theta) \geq f^+(\theta)$, $\forall\theta\in\mathbb{R}$. But if $g^+(\delta^++\theta') > f^+(\theta')$ for some $\theta\in\mathbb{R}$, then by right continuity $g^+(\delta^++\theta') > f^+(\theta'+\varepsilon)$ for a sufficiently small $\varepsilon>0$ and hence by above $g^+(\delta^+-\varepsilon+\theta) \geq f^+(\theta)$, $\forall\theta\in\mathbb{R}$, ※. Thus $g^+(\delta^++\theta) = f^+(\theta)$, $\forall\theta\in\mathbb{R}$. Similarly we can find δ^- s.t. $g^-(\delta^-+\theta) = f^-(\theta)$, $\forall\theta\in\mathbb{R}$. But $f^+ = f^-$ and $g^+ = g^-$ everywhere except on a countable set (2.4.9) , hence $\delta^+ = \delta^-$ as required. □

2.8.2 Corollary If $\underline{v}$ is a recurrent minimizing state of rotation number ω then it belongs to M_ω.

Proof Use 2.4.11 .

2.8.3 Corollary If $\underline{v}$ is a non-recurrent minimizing state of rotation number ω, there exists an $\alpha\in\mathbb{R}$ s.t.

$$f^-(n\omega+\alpha) < v_n < f^+(n\omega+\alpha) \qquad \forall\, n\in\mathbb{Z}$$

Proof Construct $\underline{v}^\pm(\beta)$ as usual. By 2.4.11, $v^-(\beta)_n \leq v(\beta)_n \leq v^+(\beta)_n$, $\forall n\in\mathbb{Z}$. By 2.8.1 , $v^\pm(\beta)_n = g^\pm(n\omega+\beta) = f^\pm(n\omega+\beta+\delta)$, for some $\delta\in\mathbb{R}$, so take $\alpha = \beta + \delta$. □

2.8.4 Corollary If $M^\#_\omega$ is a circle then every minimizing state of rotation number ω is recurrent and lies in $M^\#_\omega$. If $M^\#_\omega$ is a Cantor set then every minimizing state $\underline{v}$ of rotation number ω is either in $M^\#_\omega$ or lies in some gap of $M^\#_\omega$ and is asymptotic to the orbits of the endpoints defining that gap (recall 1.4.12).

Proof If $\underline{v}$ is recurrent then $\underline{v} \in M_\omega$, otherwise $u^-(\alpha)_n < v_n < u^+(\alpha)_n$ for some $\alpha \in \mathbf{R}$. But $f^+(\theta+1) = f^+(\theta)+1$ implies that

$$0 < \sum_{n\in\mathbf{Z}} (u^+(\alpha)_n - u^-(\alpha)_n) = \sum_{n\in\mathbf{Z}} (f^+(n\omega+\alpha) - f^-(n\omega+\alpha)) \le 1$$

and hence $|u^+(\alpha)_n - u^-(\alpha)_n| \to 0$ as $n \to \pm\infty$. Thus $\underline{v}$ is asymptotic to both $\underline{u}^\pm(\alpha) \in M_\omega$.

□

2.8.5 Remark In §5.X we will see that if there exists a rotational invariant circle Γ of rotation number ω then all the orbits in Γ are minimizing.

2.8.6 Proposition Min_ω = {minimizing states of rotation number ω} is a monotone set.

Proof The only case we have to worry about is that of two orbits lying in the same gap of M_ω i.e. $u^-(\alpha)_n < v_n, v'_n < u^+(\alpha)_n$, $\forall n \in \mathbf{Z}$. But by Aubry's Fundamental Lemma (2.6.1) $\underline{v}, \underline{v}'$ cannot cross since they are asymptotic at $\pm\infty$. Hence wlog $u^-(\alpha)_n < v_n < v'_n < u^+(\alpha)_n$, $\forall n \in \mathbf{Z}$, as required. □

§ 2.9 Classification of Minimizing Orbits: Rational Case

In this section we give a description of the set $Min_{p/q}$ of all minimizing states with a rational rotation number p/q. From 1.7.14 we know that any such state must either be periodic or advancing or retreating, thus the recurrent orbits are all periodic. Here we shall show that the rest are heteroclinic to a pair of adjacent periodic orbits. Furthermore for each such neighbouring pair of periodic orbits there are at least two minimizing heteroclinic orbits, one advancing and one retreating. The proofs in this section are almost entirely direct translations of the corresponding results in [Morse, 1924].

2.9.1 Definition

$Min_{p/q}$ = {minimizing states of rotation number p/q}

$M_{p/q}$ = {minimizing recurrent states of rotation number p/q}

= {minimizing periodic states of rotation number p/q}

$M^+_{p/q}$ = {minimizing advancing states of rotation number p/q}

$M^-_{p/q}$ = {minimizing retreating states of rotation number p/q}

2.9.2 Definition Let $\underline{v}^-, \underline{v}^+ \in M_{p/q}$ be two states s.t. $v^-_n < v^+_n$, $\forall n \in \mathbf{Z}$. We say $\underline{v}^-, \underline{v}^+$ are *adjacent* if there is no other state $\underline{v} \in M_{p/q}$ s.t. $v^-_n < v_n < v^+_n$, $\forall n \in \mathbf{Z}$. Note that by Aubry's Fundamental Lemma 2.2.6, if $v^-_k < v_k < v^+_k$, for some $k \in \mathbf{Z}$ then $v^-_n < v_n < v^+_n$, $\forall n \in \mathbf{Z}$.

2.9.3 Proposition Every minimizing state $\underline{u}$ of rotation number p/q is either periodic or asymptotic to distinct periodic states as $n \to \pm\infty$.

Proof Suppose $\underline{u}$ is not periodic, then by 1.7.14 either $\underline{u}$ is advancing i.e.

$u_{n+q} - p - u_n > 0$, $\forall n \in \mathbf{Z}$, or retreating: $u_{n+q} - p - u_n < 0$, $\forall n \in \mathbf{Z}$. Define states $\underline{u}(k)$ by $u(k)_n = u_{n+kq} - kp$. Suppose that $\underline{u}$ is advancing, then $\forall\, k,n \in \mathbf{Z}$, $u(k)_n > u(k-1)_n$. On the other hand by 1.7.11 $|u(k)_n - u_n| \leq 2$, hence $u(k)_n$ converges as $k \to \infty$ and $k \to -\infty$, say

$$u(k)_n \nearrow v^+_n \qquad \text{as } k \to \infty$$
$$u(k)_n \searrow v^-_n \qquad \text{as } k \to -\infty$$

By 1.7.15 and 2.4.2 $\underline{v}^\pm$ are minimizing states with rotation number p/q. But $v^\pm_{n+q} = \lim_{k\to\pm\infty} u(k)_{n+q} = \lim_{k\to\pm\infty} u(k+1)_n + p = v^\pm_n + p$, and thus $\underline{v}^\pm$ are periodic. Given $\varepsilon > 0$, take $N \in \mathbf{N}$ s.t. $|u(k)_i - v^+_i| < \varepsilon$, $\forall\, k \geq N$, $0 \leq i \leq q-1$. Then if $j \geq Nq$, write $j = kq + i$ with $k \geq N$, $0 \leq i \leq q-1$, so $|u_j - v^+_j| = |u_{i+kq} - v^+_{i+kq}| = |u(k)_i + kp - (v^+_i + kp)| < \varepsilon$. Thus $\underline{u}$ is asymptotic to $\underline{v}^+$ as $n \to \infty$, and similarly to $\underline{v}^-$ as $n \to -\infty$. Finally $\underline{v}^-$ and $\underline{v}^+$ are distinct since by construction $v^-_n < v_n < v^+_n$, $\forall\, n \in \mathbf{Z}$. If $\underline{u}$ is retreating the proof is entirely analogous.

□

Thus $\underline{u}$ is trapped between the two states $\underline{v}^-$ and $\underline{v}^+$. Next we show that in this situation there is no other state in $M_{p/q}$ between $\underline{v}^-$ and $\underline{v}^+$, in other words that $\underline{u}$ is asymptotic to two adjacent periodic states.

2.9.4 Proposition ([Morse, 1924], [Aubry and Le Daeron, 1983]) Let $\underline{v}^-, \underline{v}^+ \in M_{p/q}$, $\underline{u} \in Min_{p/q}$ be states s.t. $v^-_n < u_n < v^+_n$, $\forall\, n \in \mathbf{Z}$, and either $\underline{u}$ is asymptotic to $\underline{v}^+$ as $n \to \infty$ and to $\underline{v}^-$ as $n \to -\infty$, or vice versa. Then $\underline{v}^-$ and $\underline{v}^+$ are adjacent.

Proof Suppose not, so there is some $\underline{v} \in M_{p/q}$, s.t. $v^-_n < v_n < v^+_n$, $\forall\, n \in \mathbf{Z}$. We will show that this contradicts $\underline{u}$ being minimizing. The technique we use is essentially that which we used to prove the strong form of Aubry's Fundamental Lemma (2.6.1). The reader should refer back to §2.6 for detailed calculations, which we omit here. So, wlog suppose that $\underline{u}$ is advancing. By Aubry's Fundamental Lemma (2.2.6) there exists a unique $N \in \mathbf{Z}$ s.t.

$$u_n < v_n \qquad \text{for } n \leq N$$
$$u_{N+1} \geq v_{N+1}$$
$$u_n > v_n \qquad \text{for } n > N+1$$

First suppose that $u_{N+1} > v_{N+1}$, i.e. we have a proper crossing . Define $\Delta_N = h(u_N,v_{N+1}) + h(v_N,u_{N+1}) - h(u_N,u_{N+1}) - h(v_N,v_{N+1})$. Recall from 2.6.3 that Δ_N has the same sign as $(u_N - v_N)(u_{N+1} - v_{N+1})$. Thus $\Delta_N < 0$. Since $\underline{u}$ is asymptotic to $\underline{v}^+$ as $n \to \infty$ we can find an $M > N+q$, s.t (e.g. recall 2.4.2) ,

$$|W_{M,M+q}(\underline{u}) - W_{M,M+q}(\underline{v}^+)| \leq |\Delta_N|/4$$
$$|h(u_{M+q},u_{M+q+1}) - h(v^+_{M+q},v^+_{M+q+1})| \leq |\Delta_N|/4 \qquad 2.9.5$$
$$|h(u_M+p,u_{M+q+1}) - h(v^+_M+p,v^+_{M+q+1})| \leq |\Delta_N|/4$$
$$(\text{use } v^+_M+p = v^+_{M+q})$$

Now define a state $\underline{u}'$ by (see Fig. 2.9.1):

$$u'_i = \begin{cases} u_i & i \le N \\ v_i & N < i \le N+q \\ u_{i-q} + p & N+q < i \le M+q \\ u_i & M+q < i \end{cases}$$

Now

$$\begin{aligned} \Delta W &= W_{N,M+q}(\underline{u}') - W_{N,M+q}(\underline{u}) \\ &= h(u_N,v_{N+1}) + W_{N+1,N+q}(\underline{v}) + h(v_{N+q},u'_{N+q+1}) \\ &\quad + W_{N+q+1,M+q}(\underline{u}') + h(u'_{M+q},u_{M+q+1}) - h(u_N,u_{N+1}) \\ &\quad - W_{N+1,M}(\underline{u}) - W_{M,M+q}(\underline{u}) - h(u_{M+q},u_{M+q+1}) \end{aligned}$$

$$\begin{aligned} \Delta W &= [h(u_N,v_{N+1}) + h(v_N+p,u_{N+1}+p) - h(u_N,u_{N+1}) - h(v_N,v_{N+1})] \\ &\quad + [W_{N,N+q}(\underline{v}) - W_{M,M+q}(\underline{u})] + [W_{N+q+1,M+q}(\underline{u}') - W_{N+1,M}(\underline{u})] \\ &\quad + [h(u'_{M+q},u_{M+q+1}) - h(u_{M+q},u_{M+q+1})] \end{aligned}$$

So using $h(x+1,x'+1) = h(x,x')$:

$$\begin{aligned} \Delta W &= \Delta_N + [W_{N,N+q}(\underline{v}) - W_{M,M+q}(\underline{u})] + [W_{N+1,M}(\underline{u}) - W_{N+1,M}(\underline{u})] + \\ &\quad + [h(u_M+p,u_{M+q+1}) - h(u_{M+q},u_{M+q+1})] \end{aligned}$$

But $W_{N,N+q}(\underline{v}) = W_{M,M+q}(\underline{v}^+)$, since both $\underline{v}$ and $\underline{v}^+$ are global minima of $W_{p/q}$, so from 2.9.5 we have:

$$\Delta W < \Delta_N + |\Delta_N|/4 + |\Delta_N|/4 + |\Delta_N|/4 < 0 \qquad *$$

Now if $u_{N+1} = v_{N+1}$, then $\Delta_N = 0$. However if we define $\underline{u}'$ as above, then $\underline{u}'$ is not stationary, and hence not minimal. Then as in the proof of 2.6.1 we can decrease the action of $\underline{u}'$ by modifying u'_N and u'_{N+q} .

Then as above if we choose M sufficiently large we obtain a contradiction to $\underline{u}$ being a minimizing segment on [N,M+q+1].

□

Finally we show that if $\underline{v}^+$ and $\underline{v}^-$ are adjacent periodic states in $M_{p/q}$ then there exist states in $Min_{p/q}$ heteroclinic from $\underline{v}^+$ to $\underline{v}^-$ and from $\underline{v}^-$ to $\underline{v}^+$. This incidentally shows that unless $M_{p/q}$ forms a whole circle then there are non-recurrent minimizing states of rotation number p/q.

2.9.6 Proposition ([Morse, 1924], [Aubry and Le Daeron, 1983])

Let $\underline{v}^+,\underline{v}^- \in M_{p/q}$ be adjacent states. Then $\exists$ $\underline{u}^+,\underline{u}^- \in Min_{p/q}$ s.t. $v^-_n < u^+_n, u^-_n < v^+_n$, $\forall n \in \mathbb{Z}$, and

$$\begin{aligned} u^+_n &\to v^+_n \quad \text{and} \quad u^+_{-n} \to v^-_{-n} \quad \text{as } n \to \infty \\ u^-_n &\to v^-_n \quad \text{and} \quad u^-_{-n} \to v^+_{-n} \quad \text{as } n \to \infty \end{aligned}$$

Proof (following [Morse,1924]) We will show how to construct $\underline{u}^+$; the method for $\underline{u}^-$ is entirely analogous. For $n \in \mathbb{N}$, let $\underline{u}(n)$ be a minimizing segment from v^-_{-n} to v^+_n (Fig. 2.9.2). This exists by 2.2.1 . Note that by 2.2.6 , $v^-_i < u(n)_i < v^+_i$, $\forall -n < i < n$. For any $n \in \mathbb{N}$ we can find $i_n \in \mathbb{Z}$ s.t $-n \le i_n < n$ and

$$\begin{aligned} v^+_{i_n} - u(n)_{i_n} &\ge u(n)_{i_n} - v^-_{i_n} \\ v^+_{i_n+1} - u(n)_{i_n+1} &\le u(n)_{i_n+1} - v^-_{i_n+1} \end{aligned}$$

There must exist a $j \in \mathbb{Z}$ with $0 \le j \le q-1$ s.t. $i_n \equiv j \pmod q$ for infinitely many i_n , so wlog suppose $i_n \equiv j \pmod q$, $\forall n \in \mathbb{N}$. Define $k_n = (i_n - j)/q$ and minimizing segments $\underline{u}'(n)$ on $[-n+j-i_n, n+j-i_n]$ by $u'(n)_i = u(n)_{i+k_nq} - k_np$. Then the sequence $(u'(n)_j, u'(n)_{j+1})$ is bounded as $n \to \infty$, so let (u^+_j, u^+_{j+1}) be a limit point, and generate the state $\underline{u}^+$ by stationarity. Now, either $\liminf(-n+j-i_n) = -\infty$ or $\limsup(n+j-i_n) = \infty$. Wlog suppose the latter and let $N = \liminf(-n+j-i_n)$. We will show below that $N = -\infty$. Assuming so, then $\underline{u}^+$ is the limit of minimizing segments and hence is a minimizing state (see 2.4.2). It is trapped between $\underline{v}^+$ and $\underline{v}^-$, thus its rotation number must be p/q. Since $v^+_j - u^+_j \geq u^+_j - v^-_j$ and $v^+_{j+1} - u^+_{j+1} \leq u^+_{j+1} - v^-_{j+1}$, $\underline{u}^+$ cannot be either $\underline{v}^+$ or $\underline{v}^-$, and hence since $\underline{v}^+$ and $\underline{v}^-$ are adjacent, cannot be periodic. Thus by 2.9.3 it is asymptotic to two distinct periodic states. By 2.9.4 these must be precisely $\underline{v}^+$ and $\underline{v}^-$. Finally note that

$$u(n)_{-n+q} - p \geq v^-_{-n+q} - p = v^-_{-n} = u(n)_{-n}$$
$$u(n)_n - p \geq v^+_n - p = v^+_{n-q} = u(n)_{n-q}$$

Thus by 2.2.6, $\underline{u}(n)$ satisfies $u(n)_{i+q} - p \geq u(n)_i$, $\forall -n \leq i \leq n-q$. Hence $u^+_{i+q} - p \geq u^+_i$, $\forall i \in \mathbb{Z}$, thus since $\underline{u}^+$ is not periodic, it must be advancing and thus $u^+_n \to v^+_n$ as $n \to \infty$ and $u^+_n \to v^-_n$ as $n \to -\infty$, as claimed.

To show that $N = -\infty$ suppose that $N > -\infty$. Then by the definition of N, there are infinitely many $n \in \mathbb{N}$ s.t. $u'(n)_N = v^-_N$. We can take our convergent subsequence from these and thus wlog assume that $u^+_N = v^-_N$. Note that $\underline{u}^+$ is minimizing on $[N,\infty)$ i.e. $\forall N \leq m < n-1$, the segment $\underline{u}^+_{mn}$ is minimizing. Next we show that $\exists\ \underline{v} \in M_{p/q}$ s.t. s.t. $u^+_n \to v_n$ as $n \to \infty$. The proof is essentially the same as that of 2.9.3. By above $u(n)_{i+q} - p \geq u(n)_i$, $\forall -n \leq i \leq n-q$, hence $u^+_{i+q} - p \geq u^+_i$, $\forall i \in [N,\infty)$. As in 2.9.3 define $u(k)_i = u_{i+kq} - kp$. For fixed $i \in \mathbb{Z}$, this is a bounded monotone sequence in k for $k > (M-i)/q$ and so $u(k)_i$ converges as $k \to \infty$, say $u(k)_i \nearrow v_i$. As before, the limit

sequence $\underline{v}$ is in $M_{p/q}$ and $\underline{u}^+$ is asymptotic to it as $n \to \infty$. Furthermore $v^-_i \leq u^+_i \leq v_i$, $\forall i \in [N,\infty)$, and as we remarked above $\underline{u}^+ \neq \underline{v}^-$. Thus $\underline{v}^- \neq \underline{v}$. To summarize, $\underline{u}^+$ is minimizing on $[N,\infty)$, asymptotic to $\underline{v}$ as $n \to \infty$ and crosses $\underline{v}^-$ at N; with $\underline{v}^- \neq \underline{v}$ in $M_{p/q}$ (Fig. 2.9.3). This contradicts the following lemma which is just a slightly stronger form of 2.9.4:

2.9.7 Lemma Let $\underline{u}$ be a stationary state and suppose $\exists\ N \in \mathbb{Z}$, s.t. $\underline{u}$ is minimizing on $[N,\infty)$, i.e. s.t. $\forall N \leq m < n-1$, $\underline{u}_{mn}$ is a minimizing segment on $[m,n]$. Suppose that $\underline{u}$ is asymptotic to some $\underline{v} \in M_{p/q}$ as $n \to \infty$. Then for any $\underline{v}' \in M_{p/q}$ with $\underline{v} \neq \underline{v}'$, $\underline{u}$ cannot cross $\underline{v}'$ in $[N,\infty)$.

Proof Observe that to get the contradiction in the proof of 2.9.4 we do not actually require $\underline{u}$ to be a minimizing state, merely for it to be minimizing on $[N,\infty)$ (we also need a suitable modification to 2.6.1).

2.9.8 Summary

$Min_{p/q} = M_{p/q} \cup M^+_{p/q} \cup M^-_{p/q}$.

$M_{p/q}$ is a closed invariant non-empty monotone set.

For each gap in $M_{p/q}$ there are points in $M^+_{p/q}$ and in $M^-_{p/q}$ in that gap.

$M_{p/q} \cup M^+_{p/q}$ is a monotone set, and so is $M_{p/q} \cup M^-_{p/q}$.

$Min_{p/q}$ is not monotone unless $Min_{p/q} = M_{p/q}$, or equivalently unless $M_{p/q}$ is a circle.

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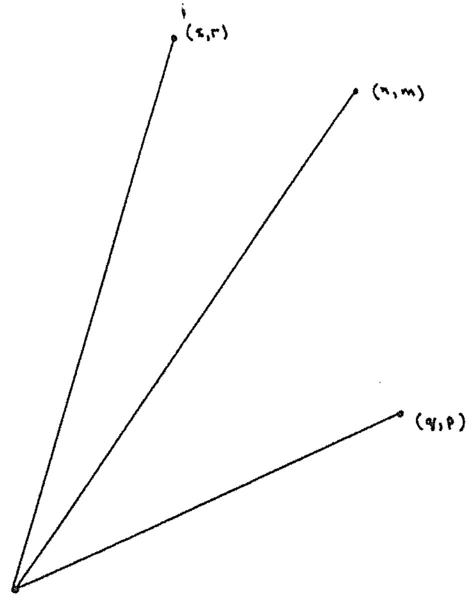


Fig. 1·4·1

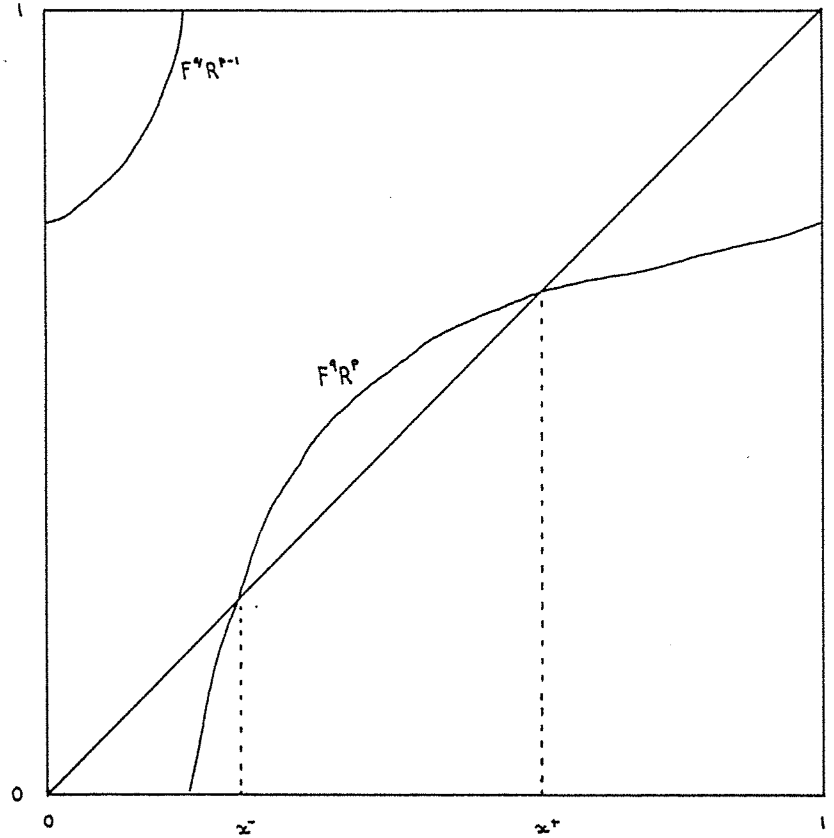


Fig. 1·4·2

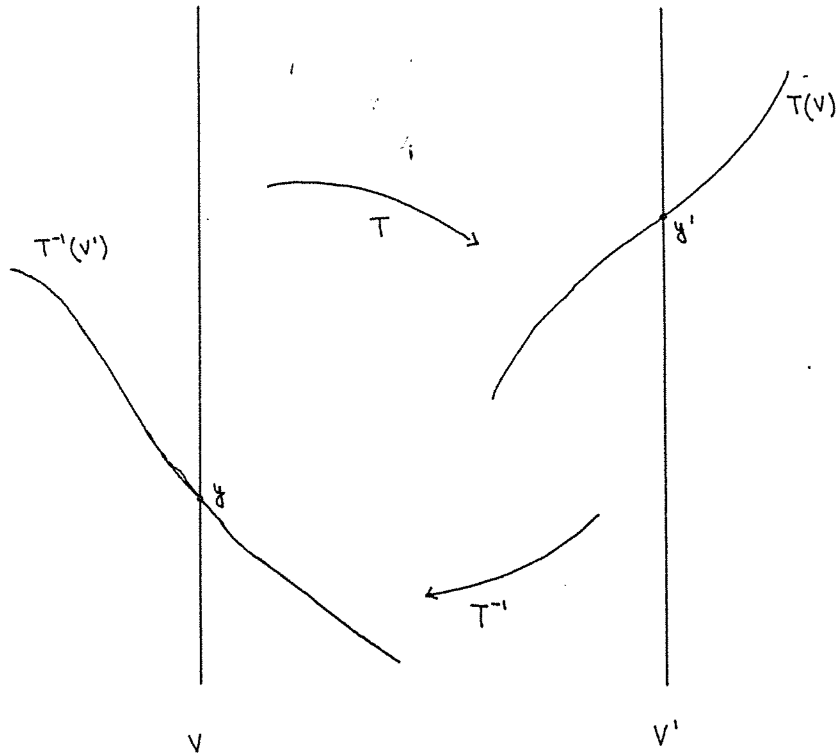


Fig 1·5·1

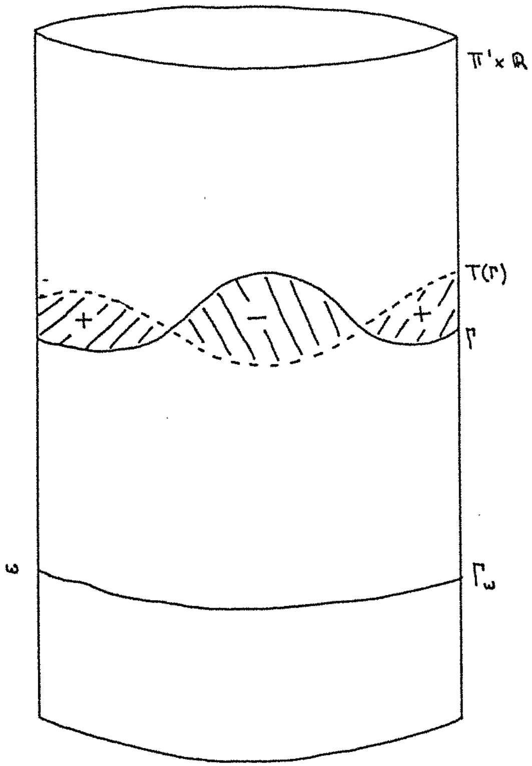


Fig 1·6·1

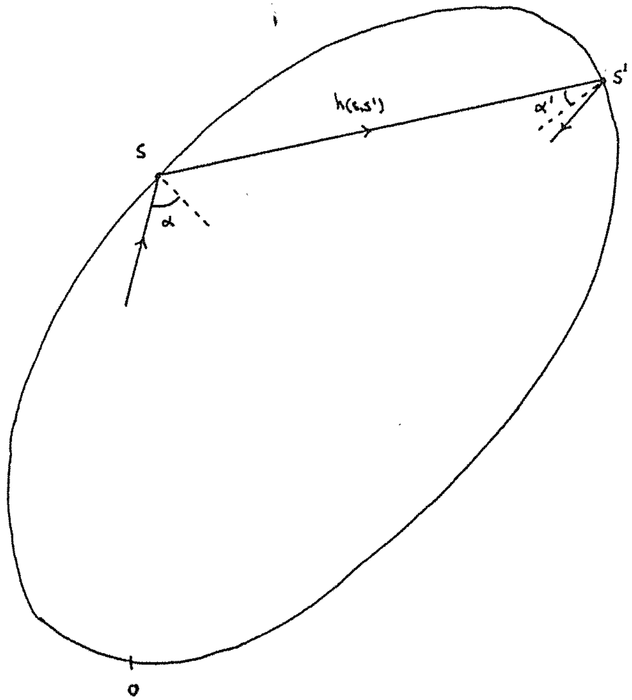


Fig. 1·6·2

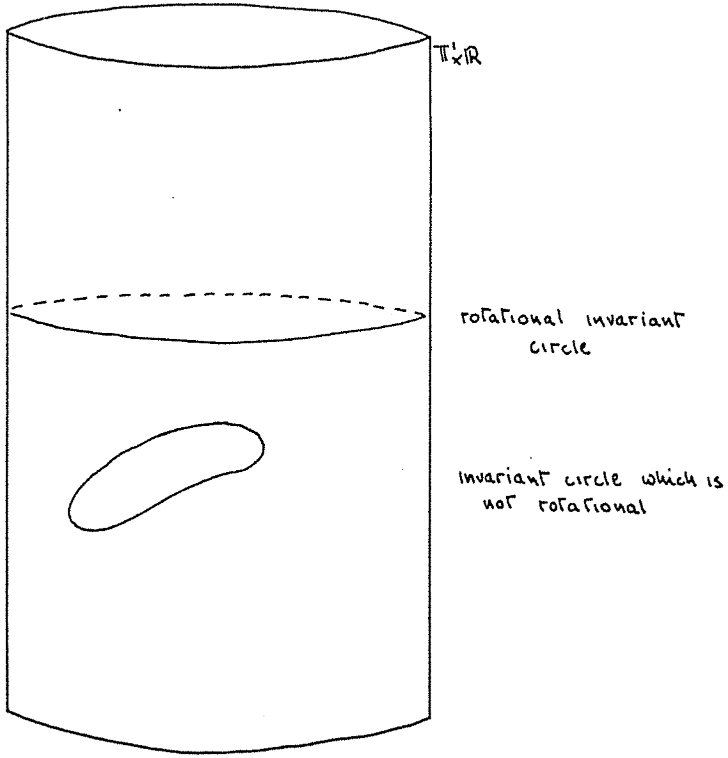
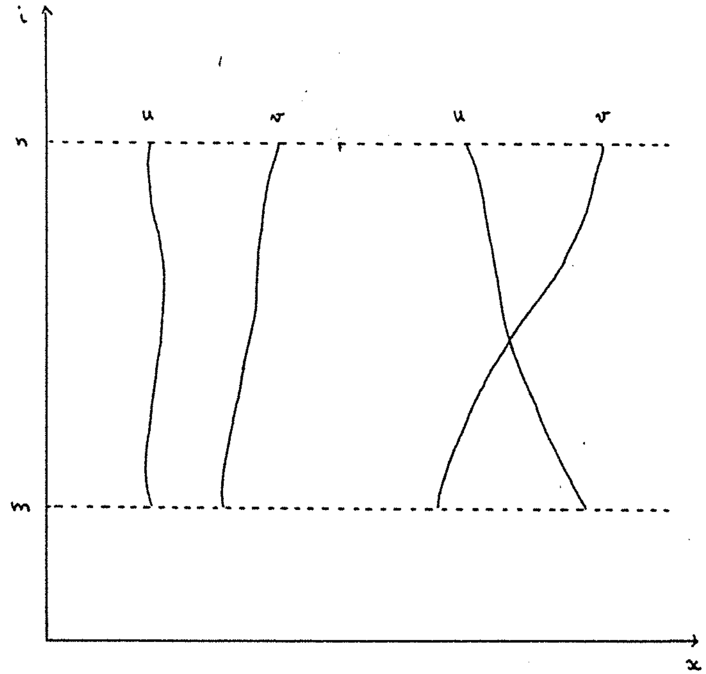
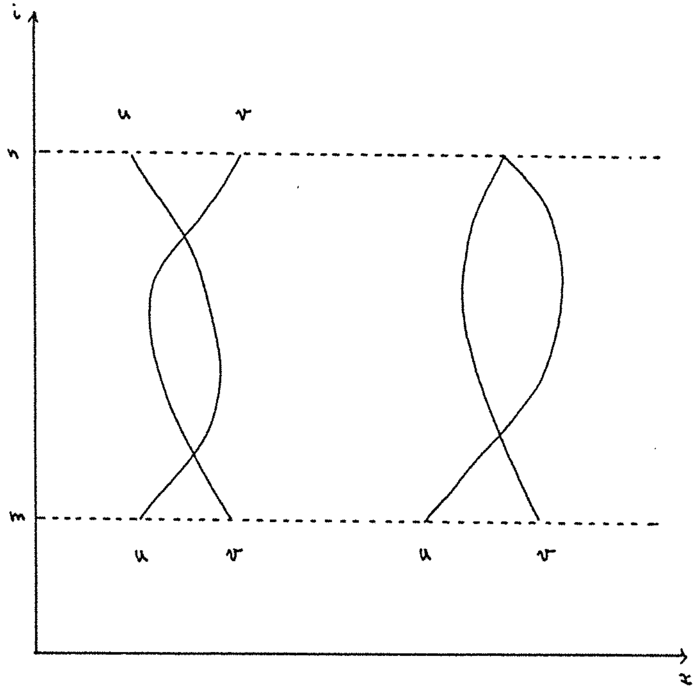


Fig. 1·6·3



Possible Configurations

Fig 2·2·1



Impossible configurations

Fig 2.2.2

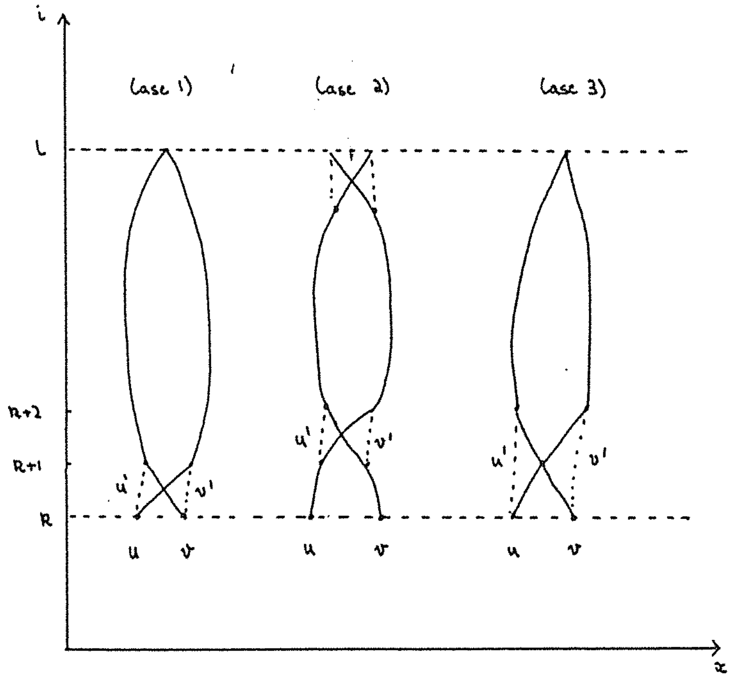


Fig 2.2.3

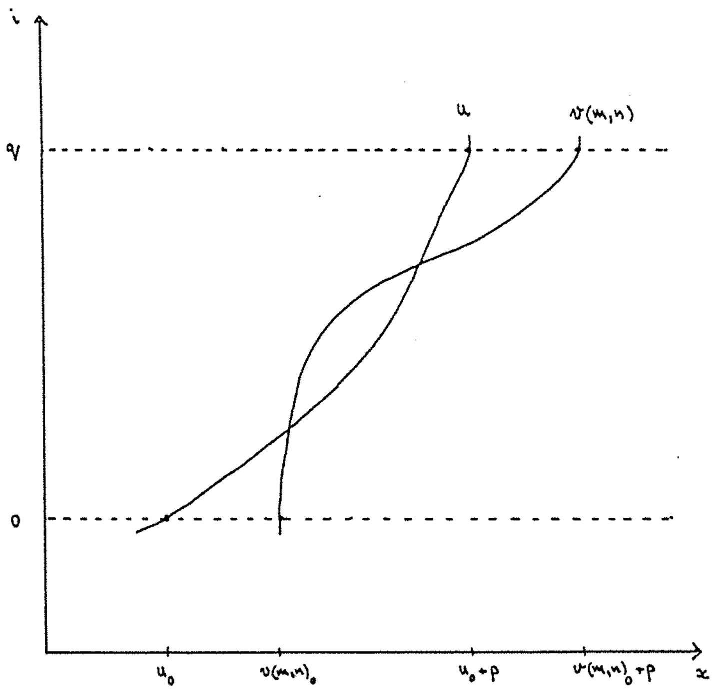


Fig 2.3.1

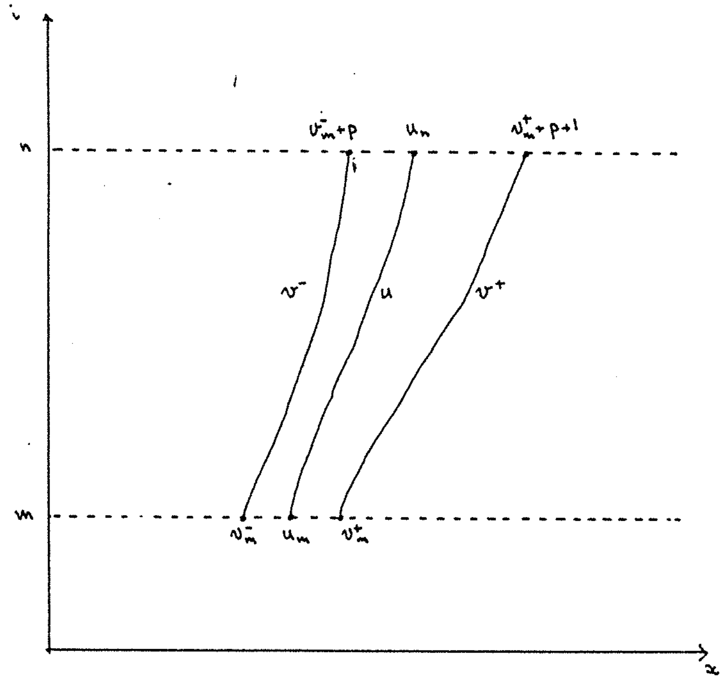


Fig 2.5.1

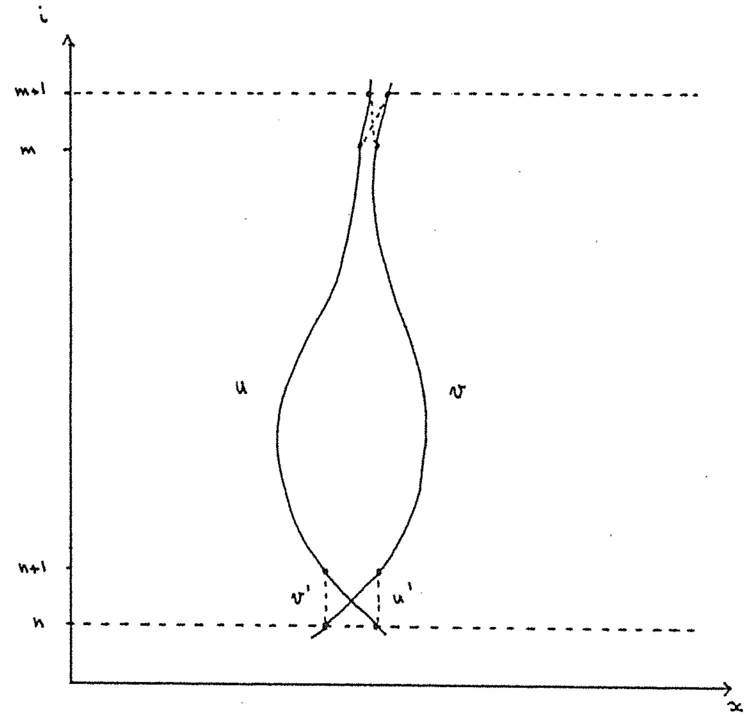


Fig 2·6·1

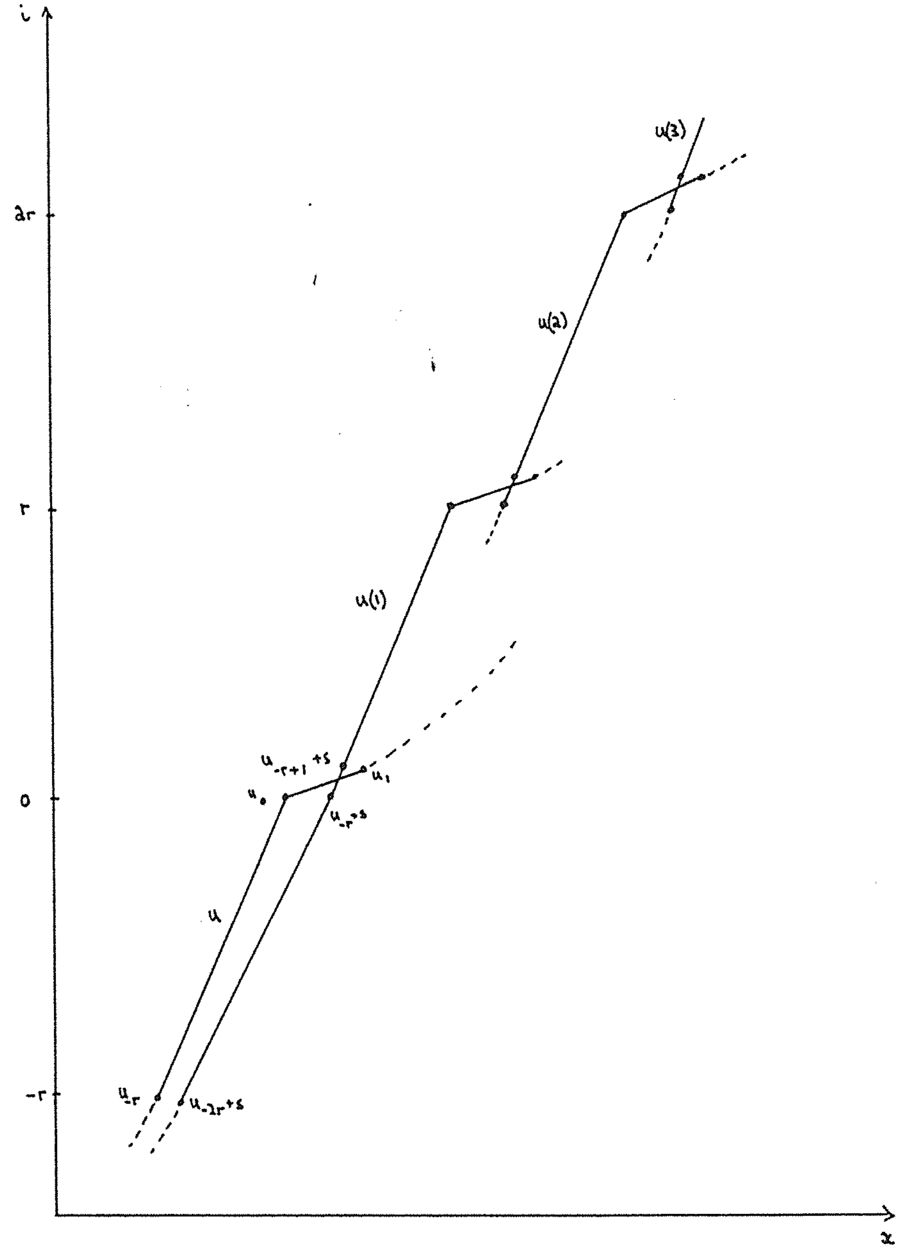


Fig 2·7·1

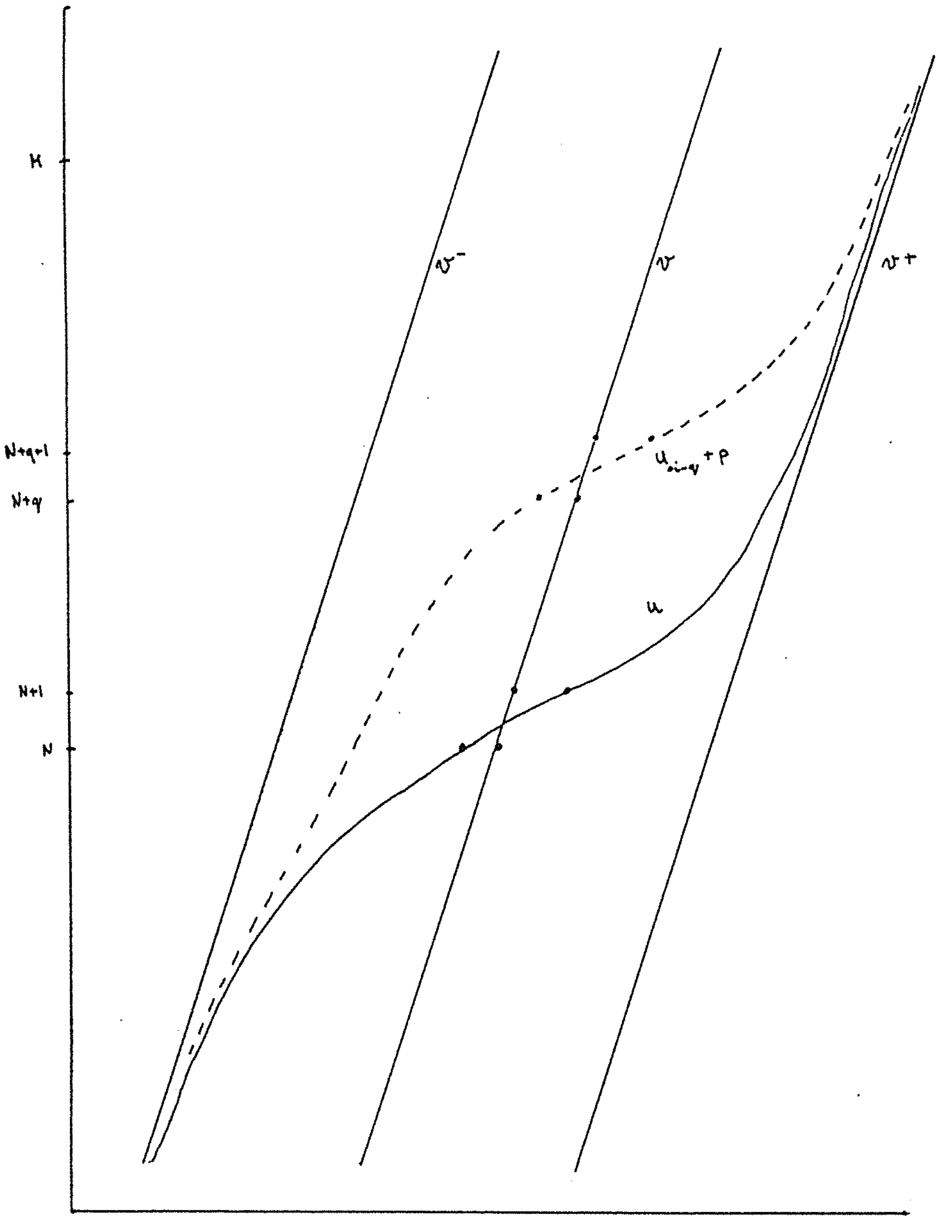


Fig. 2.9.1

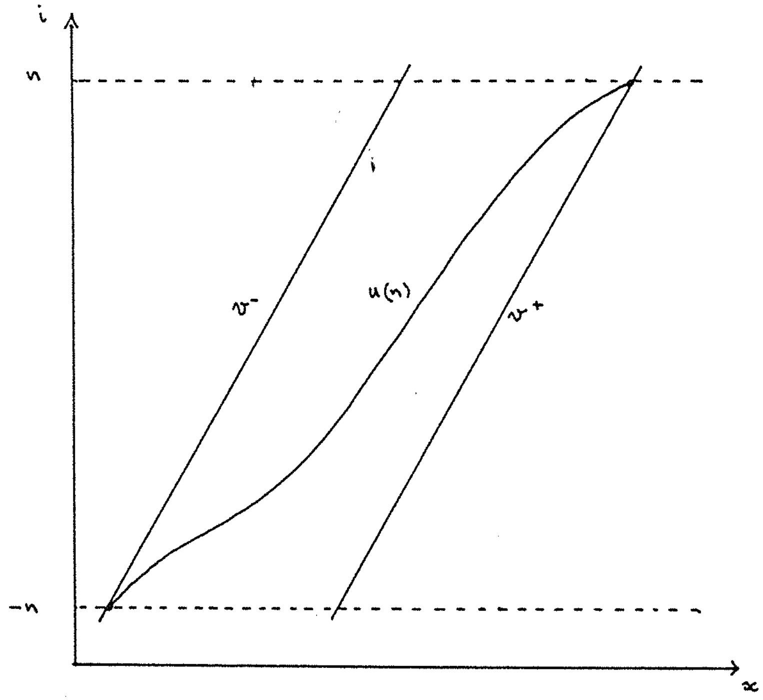


Fig 2.9.2

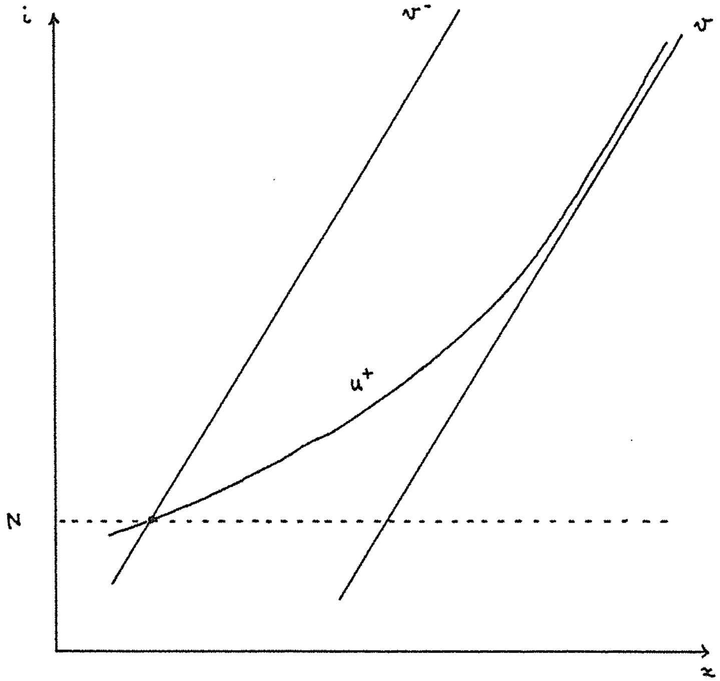


Fig 2.9.3