Mathematics for Fusion Power part 8

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Interaction of two charges

Reduced dynamics

Reconstructed dynamics

Scattering map

Effect on distribution function

Interaction of charges

- Charged particles exert Coulomb force $\frac{e_1 e_2}{4\pi\varepsilon_0 R^2}$ on each other, where $R = |q_1 - q_2|$ (repulsive for same signs, attractive for opposite signs). Corresponds to potential $V(R) = \frac{e_1 e_2}{4\pi\varepsilon_0 R}$.
- Interaction of charges can lead to cross-field diffusion of particles and of energy, also fusion!
- Generalise to any potential $V(R)$. In particular, can model effect of sea of other charges by Debye-shielded potential $V(R) = \frac{e_1 e_2}{4\pi\varepsilon_0 R} e^{-R/\lambda_D}$, where Debye length $\lambda_D = \sqrt{\frac{\varepsilon_0}{\sum_j n_j e_j^2 \beta_j}}$, with number density n_j and coolness $\beta_j = \frac{1}{k_B T_j}$ of species j. [but response to moving charge?]
- Might want to modify repulsive case to simulate fusion.
- Could allow any $V(r, z)$ where z is separation along field and r perpendicular to field.

Two charges in uniform magnetic field

- Treat non-relativistic motion of two charges in a uniform magnetic field $B = B\hat{z}$.
- 6 DoF Hamiltonian dynamics: $H = \frac{|p_1|^2}{2m_1} + \frac{|p_2|^2}{2m_2} + V(|q_1 - q_2|)$, $\omega = \sum_i dq_{x_i} \wedge dp_{x_i} + dq_{y_i} \wedge dp_{y_i} + dq_{z_i} \wedge dp_{z_i} + k_i dq_{x_i} \wedge dq_{y_i}$, where $k_i = -e_i B$. Note $p_i = m_i \dot{q}_i$. Let $\Omega_i = k_i / m_i$. Write Coulomb case as $V(R) = \frac{k_1 k_2 G}{R}$ with $G = \frac{1}{4\pi\varepsilon_0 B^2}$.
- D Pinheiro, RS MacKay, Interaction of two charges in a uniform magnetic field: I. planar problem, Nonlinearity 19 (2006) 1713-45.
  D Pinheiro, RS MacKay, Interaction of two charges in a uniform magnetic field: II. spatial problem, J Nonlin Sci 18 (2008) 615–666.
- Could add a perpendicular electric field E, but can remove its effect by going to frame with velocity $E \times B/|B|^2$ [contrast incorrect papers on atoms in crossed fields].

Symmetries

- Symmetries under all translations and rotation about z-axis.
- Conserved quantities: $P_z = \sum_i p_{z_i}$, $P_\perp = \sum_i p_{i\perp} + k_i J q_{i\perp}$, $L = \sum_i q_{i\perp}^T J p_{i\perp} - \frac{k_i}{2}|q_{i\perp}|^2$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
- Let gyroradius vector $\rho_i = J\frac{p_{i\perp}}{k_i}$, gyrocentre $Q_i = q_i - \rho_i$.

Hidden symmetry for equal gyrofrequencies

- If $\Omega_1 = \Omega_2$, $\exists$ additional locomotive coupling-rod (LCR) symmetry, conserving $W = |(p_1 + p_2)_\perp|^2$. The symmetry field is $Q_i' = 0$, $(\rho_1 - \rho_2)' = 0$, $(p_1 + p_2)' = -J(p_1 + p_2)$.
- The symmetry action on q_i is $q_i(\theta) = \tilde{Q}_i + R_\theta \bar{\rho}$, where $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\bar{\rho} = J\frac{(p_1+p_2)_\perp}{k_1+k_2} = \frac{k_1\rho_1 + k_2\rho_2}{k_1+k_2}$ and $\tilde{Q}_i = q_i - \bar{\rho}$ ($\tilde{Q}_1 = Q_1 + \frac{k_2(\rho_1-\rho_2)}{k_1+k_2}$, $\tilde{Q}_2 = Q_2 + \frac{k_1(\rho_2-\rho_1)}{k_1+k_2}$).

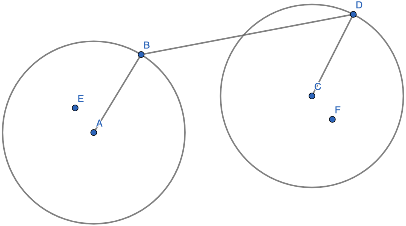


Figure: Charges (equal) at B,D; GCs at E,F; LCR rotation about A,C

Reduction

- By P_z: Let $M = m_1 + m_2$, $m = \frac{m_1 m_2}{M}$, $q_z = q_{z_1} - q_{z_2}$, $p_z = (m_2 p_{z_1} - m_1 p_{z_2})/M$, $Q_z = (m_1 q_{z_1} + m_2 q_{z_2})/M$. Write Q_i for $Q_{i\perp}$, e.g. $R = \sqrt{|Q_1 + \rho_1 - Q_2 - \rho_2|^2 + q_z^2}$. Then

$$H = \frac{k_1 \Omega_1}{2} |\rho_1|^2 + \frac{k_2 \Omega_2}{2} |\rho_2|^2 + V(R) + \frac{p_z^2}{2m} + \frac{P_z^2}{2M},$$

$$\omega = \sum_i k_i (dQ_{x_i} \wedge dQ_{y_i} - d\rho_{x_i} \wedge d\rho_{y_i}) + dq_z \wedge dp_z + dQ_z \wedge dP_z.$$

  wlog $P_z = 0$.
- In gyro-variables, remaining integrals of motion are $P_\perp = J \sum_i k_i Q_i$, $L = \sum_i \frac{k_i}{2} (|\rho_i|^2 - |Q_i|^2)$, $W = |\sum_i k_i \rho_i|^2$.

Case $k_1 + k_2 \neq 0$

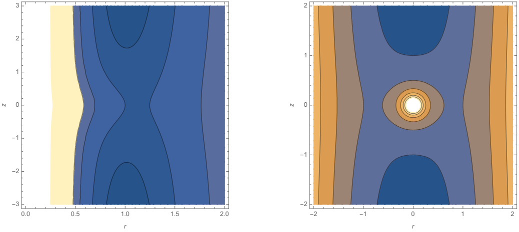
- e.g. e^-- e^-, H^+- H^+ $(H = p, D, T)$, H^+- He^{2+}, He^{2+}- e^-
- Write $K = k_1 + k_2$, $k = \frac{k_1 k_2}{K}$ and $\bar{Q} = \frac{1}{K}\sum_i k_i Q_i$, $\bar{\rho} = \frac{1}{K}\sum_i k_i \rho_i$, $S = Q_1 - Q_2$, $s = \rho_1 - \rho_2$. So $\rho_1 = \bar{\rho} + \frac{k_2}{K}s$, $\rho_2 = \bar{\rho} - \frac{k_1}{K}s$ and similar for Q_j.
- Then $L = \frac{K}{2}(|\bar{\rho}|^2 - |\bar{Q}|^2) + \frac{k}{2}(|s|^2 - |S|^2)$, $W = K^2|\bar{\rho}|^2$, $H = \frac{P_z^2}{2M} + \frac{p_z^2}{2m} + \frac{\bar{\Omega}K}{2}|\bar{\rho}|^2 + k\delta\Omega\,\bar{\rho}^T s + \frac{k^2}{2m}|s|^2 + V(R)$, $\omega = dQ_z \wedge dP_z + dq_z \wedge dp_z + K(d\bar{Q}_x \wedge d\bar{Q}_y - d\bar{\rho}_x \wedge d\bar{\rho}_y) + k(dS_x \wedge dS_y - ds_x \wedge ds_y)$, where $\bar{\Omega} = \frac{1}{K}\sum_i k_i \Omega_i$, $\delta\Omega = \Omega_1 - \Omega_2$, $R = \sqrt{|S+s|^2 + q_z^2}$.
- So $\bar{Q}$ conserved (equivalent to $P_\perp = JK\bar{Q}$), thus no nett perpendicular displacement of charge. wlog $\bar{Q} = 0$.
- To reduce by L (simultaneous rotation of $\bar{Q}, \bar{\rho}, s, S$), let $q = (q_1 - q_2)_\perp$, suppose $r = |q| \neq 0$ and use coordinates X, Y along and $\perp$ to q, i.e. let ϕ be direction of q in (x, y) and write $(s_x, s_y) = (s_X \cos\phi - s_Y \sin\phi, s_X \sin\phi + s_Y \cos\phi)$, etc.

continued

- Then $ds_x \wedge ds_y = ds_X \wedge ds_Y + \frac{1}{2} d|s|^2 \wedge d\phi$. Thus,
  $K(d\bar{Q}_x \wedge d\bar{Q}_y - d\bar{\rho}_x \wedge d\bar{\rho}_y) + k(dS_x \wedge dS_y - ds_x \wedge ds_y) =$
  $K(d\bar{Q}_X \wedge d\bar{Q}_Y - d\bar{\rho}_X \wedge d\bar{\rho}_Y) + k(dS_X \wedge dS_Y - ds_X \wedge ds_Y) - dL \wedge d\phi$.
- But $q = S + s$ so eliminate S by $S_X = r - s_X, S_Y = -s_Y$:
  $|S|^2 = (s_X - r)^2 + s_Y^2$ & $dS_X \wedge dS_Y - ds_X \wedge ds_Y = -dr \wedge ds_Y$.
  Write $p_r = -ks_Y$ and use
  $r^2 = |S + s|^2 = |S|^2 + 2S^T s + |s|^2 = |S|^2 - |s|^2 + 2rs_X$ to write
  $s_X = \frac{r}{2} + \frac{\tilde{L}}{kr}$ with $\tilde{L} = L + \frac{K}{2}(|\bar{Q}|^2 - |\bar{\rho}|^2)$.
- Thus, obtain $\omega =$
  $dQ_z \wedge dP_z + dq_z \wedge dp_z + K(d\bar{Q}_X \wedge d\bar{Q}_Y - d\bar{\rho}_X \wedge d\bar{\rho}_Y) + dr \wedge dp_r - dL \wedge d\phi$,
  $H = \frac{P_z^2}{2M} + \frac{p_z^2}{2m} + \frac{\bar{\Omega} K}{2}|\bar{\rho}|^2 + \delta\Omega\, k\bar{\rho}^T s + \frac{1}{2m}(\frac{kr}{2} + \frac{\tilde{L}}{r})^2 + \frac{p_r^2}{2m} + V(R)$,
  with $k\bar{\rho}^T s = \bar{\rho}_X(\frac{kr}{2} + \frac{\tilde{L}}{r}) - \bar{\rho}_Y p_r$, $R = \sqrt{r^2 + q_z^2}$ and $\tilde{L}$ as above.
- Family of 3DoF systems on $(\bar{\rho}, r, p_r, q_z, p_z)$ param. by $P_z, \bar{Q}, L$.
- Assumption $r > 0$ is automatic for $A = \frac{2L}{K} + |\bar{Q}|^2 < 0$ because $r = 0$ implies $|S| = |s|$ implies $A = |\bar{\rho}|^2 \geq 0$.
- But for other sign of A, should make another coordinate patch to cover $r = 0$, e.g. take coordinates along and $\perp$ to $\bar{\rho}$, which would produce a form closer to that for $K = 0$ to come.

Case $\delta\Omega = 0$

- e.g. two of the same species, also D^+- $^4He^{2+}$.
- If $\delta\Omega = 0$ then system is rotation-symmetric in $\bar{\rho}$, so $|\bar{\rho}|^2 = W/K^2$ is conserved and obtain reduction to 2DoF on (r, p_r, q_z, p_z), parametrised by $P_z, \bar{Q}, \tilde{L}, |\bar{\rho}|^2$, with
  $$H = \frac{P_z^2}{2M} + \frac{p_z^2}{2m} + \frac{\bar{\Omega}K}{2}|\bar{\rho}|^2 + \frac{1}{2m}\left(\frac{kr}{2} + \frac{\tilde{L}}{r}\right)^2 + \frac{p_r^2}{2m} + V(R),$$
  and $\omega = dr \wedge dp_r + dq_z \wedge dp_z$. Potential for $\tilde{L} \neq 0$ below left.
- $\tilde{L} = \frac{k}{2}(|s|^2 - |S|^2)$, so $r = 0$ $(S = -s)$ implies $\tilde{L} = 0$; treat this as a special case, with $r \in \mathbb{R}$ replacing $r > 0$. Repulsive potential bounds motion away from $(0, 0)$ (right).



Case $K = 0$

- e$^-$- H^+ or e$^-$- e$^+$ (positron).
- Write $k = k_1 = -k_2$, $q = (q_1 - q_2)_\perp$, $p = -\frac{k}{2}J(\rho_1 + \rho_2)$, $C = -\frac{1}{2}J(q_1 + q_2)_\perp$, $\Pi = k(Q_1 - Q_2)_\perp$, $\alpha = \frac{m_2 - m_1}{2m_1 m_2}$.
- Then

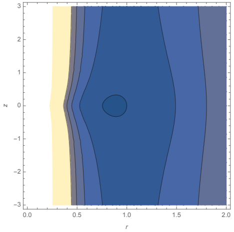
$$
\begin{aligned}
H =& \frac{1}{2m}|p|^2 + \frac{k^2}{8m}|q|^2 - \frac{k}{4m}\Pi^T q + \frac{1}{8m}|\Pi|^2 + V(R) \\
&+ \alpha(kq - \Pi)^T Jp + \frac{p_z^2}{2m} + \frac{P_z^2}{2M},
\end{aligned}
$$

  with $R = \sqrt{|q|^2 + q_z^2}$, and
  $\omega = \sum_{j=1}^{3} dq_j \wedge dp_j + \sum_{j=1}^{2} dC_j \wedge d\Pi_j + dQ_z \wedge dP_z$.
- So Π and P_z are conserved and (H, ω) reduces to 3DoF in (q, p, q_z, p_z).
- $L = q^T Jp + C^T J\Pi$ also conserved but unless $\Pi = 0$, does not constrain reduced variables nor commute with Π.

Subcase $\Pi = 0$

- $\Pi = 0$ means GCs on same fieldline
- By writing $q = re_r$, $p = p_r e_r + \frac{p_\theta}{r} e_\theta$, p_θ is conserved and system reduces to 2DoF in (r, p_r, q_z, p_z):

$$H = \frac{1}{2m}\left(p_r^2 + p_z^2 + \frac{p_\theta^2}{r^2}\right) + \frac{k^2}{8m} r^2 + V(R) + \frac{P_z^2}{2M} + \alpha k p_\theta,$$

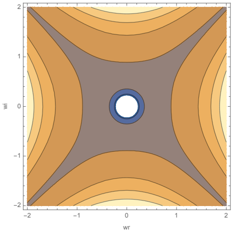


$R = \sqrt{r^2 + q_z^2}$, $\omega = dr \wedge dp_r + dq_z \wedge dp_z$.

- If $p_\theta = 0$ (i.e. $\rho_1 + \rho_2 \perp q_1 - q_2$) then dynamics is singular at $(r, q_z) = (0, 0)$ (V is attractive). wlog $q_y = 0$ and $H = \frac{1}{2m}(p_x^2 + p_z^2) + \frac{k^2}{8m} q_x^2 + V(R)$, with $R = \sqrt{q_x^2 + q_z^2}$, $\omega = dq_x \wedge dp_x + dq_z \wedge dp_z$. Can Levi-Civita regularise.

Levi-Civita regularisation

- Write $q = q_x + iq_z = w^2$, $w \in \mathbb{C}$, $p = p_x + ip_z \in \mathbb{C}$, and set $\pi = 2w^* p \in \mathbb{C}$. Then $p^* dq = \pi^* dw$. Note $k_1 k_2 < 0$.
- $H = \frac{|p|^2}{2m} + \frac{k^2}{8m} q_x^2 + \frac{k_1 k_2 G}{|q|} = \frac{|\pi|^2}{8m|w|^2} + \frac{k^2}{8m}(w_r^2 - w_i^2)^2 + \frac{k_1 k_2 G}{|w|^2}$.
- Scale to a new time s by $\frac{ds}{dt} = 4m|w|^2$.
- Then the original dynamics on $H = E$ transforms to that of $\tilde{H} = \frac{1}{2}|\pi|^2 + \frac{1}{2}k^2(w_r^2 - w_i^2)^2|w|^2 - 4mE|w|^2 + 4mk_1k_2G$ on $\tilde{H} = 0$ and continues smoothly through $w = 0$.



- Accessible region in w-plane for various E:
- Copes with Debye shielding too.

Planar dynamics

- All cases have invariant subspace $q_z = p_z = 0$. Dynamics reduces by 1 DoF.
- Case $\delta\Omega = 0$: $H = \frac{k^2}{8m} r^2 + \frac{\tilde{L}^2}{2mr^2} + \frac{p_r^2}{2m} + V(r) + cst$. For $\tilde{L} \neq 0$ or $V(r) \to +\infty$ as $r \to 0$, get periodic oscillations about equilibrium r_0 minimising H. Its energy $E_0 = \frac{P_z^2}{2M} + \frac{K\bar{\Omega}}{2}|\bar{\rho}|^2 + |\bar{\Omega}\tilde{L}| + V\left((\frac{2|\tilde{L}|}{|k|})^{\frac{1}{2}} \right)$ to first order in V. Coulomb case $V(r) = \frac{kKG}{r}$ gives elliptic functions. $V = 0$ has $r = \sqrt{A + B \sin \Omega t}$ with $A = \frac{4E}{k\Omega}$, $B = \frac{2}{k}\sqrt{\frac{4E^2}{\Omega^2} - \tilde{L}^2}$ (r = distance between two particles with same gyrofrequency).
- Case $K = 0, \Pi = 0$: same with p_θ replacing $\tilde{L}$ and reinterpretation of k. For $p_\theta \neq 0$ and (attractive) Coulomb potential there is precisely one equilibrium and get oscillations about it. But when $p_\theta = 0$, r reaches 0 in finite time (can restrict the regularisation to q real to continue through).

General planar cases: $K \neq 0$

- INCOMPLETE NOTES IN PROGRESS FROM HERE ON
- Easiest studied in $\bar{Q}, S, \rho_1, \rho_2$ coordinates.
  $H = \sum_j \frac{k_j \Omega_j}{2} |\rho_j|^2 + V(|S + \rho_1 - \rho_2|)$,
  $L = \sum_j k_j |\rho_j|^2 - K|\bar{Q}|^2 - k|S|^2$ and $\bar{Q}$ conserved.
- Use coordinate system so that $S + \rho_1 - \rho_2$ is on positive x-axis.
- Then for repulsive V, the dynamics oscillates around a unique equilibrium (minimum). Call its energy E_{00}.
- Can study conditions to reach fusion.

Opposite signs

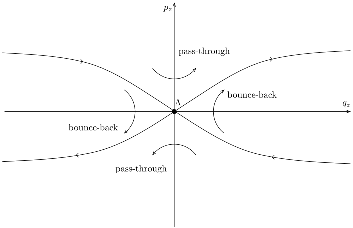
- For opposite sign charges, have to consider $r \in \mathbb{R}$ and after regularisation get oscillations about $\bar{\rho}_X = 0$, $r = 0$.
- Although extending after collision is fictitious, many regularised trajectories continue for small $\tilde{L}$ to true trajectories with repeated near-collisions, making a chaotic subshift. We proved this for all high enough energies if $\Omega_1 + \Omega_2 \neq 0$. [EXPLAIN?]

$K = 0$

- $H = \frac{|p|^2}{2m} + \frac{|q|^2}{8m} + \alpha k q^T J p - \Pi^T(\frac{k}{4m} q + \alpha J p) + V(|q|) + cst.$
- Absorb the linear terms in p into the quadratic. Left with positive-definite in $q - q_1$, for some $q_1(\Pi)$, plus negative V. Get oscillations about 0 (after regularisation) and possible saddle to a potential well about some q_0 (e.g. two widely separated gyromotions).
- Get chaotic subshift again for all high enough energies if $\Omega_1 + \Omega_2 \neq 0$ (but excludes e$^-$- e$^+$) and $\Pi \neq 0$.
- Consequences for e$^-$- e$^+$ annihilation ($\alpha = 0$).

Spatial dynamics

- Reduced z-dynamics is $\dot{q}_z = \frac{p_z}{m}, \dot{p}_z = -q_z \frac{V'(R)}{R}$, coupled to the other DoF via $R = \sqrt{q_z^2 + r^2}$.
- Same signs of charge (Λ is 4D planar subspace):

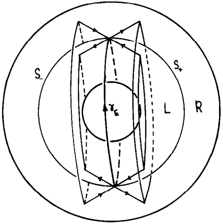


- For $E < E_{00}$ all trajectories bounce back. For $E > E_{00}$, subset Λ_E with energy E is an $\mathbb{S}^3$, it is 'morally' normally hyperbolic and its 4D forward contracting submanifolds separate the 5D energy level into points whose trajectories pass through and those that bounce back.
- For fusion, want to be on $W^+(\Lambda)$ with asymptotic trajectory on Λ approaching $r = 0$.
- For $\delta\Omega = 0$: extra integral $|\bar{\rho}|^2$, 2D lower, and critical energy E_0.

Flux over a saddle

- For 3DoF Hamiltonian system (H, ω), vector field X is defined by $i_X\omega = dH$ and energy-level volume μ by $\mu \wedge dH = \frac{1}{6}\omega^{\wedge 3}$. Then flux of μ, $i_X\mu = \frac{1}{2}\omega^{\wedge 2}$.
- Let $S_E = \{x \in H^{-1}(E) : q_z = 0\}$. It is an $\mathbb{S}^4$ that separates $H^{-1}(E)$ into $q_z < 0$ and $q_z > 0$ ("dividing surface").
- Decompose $S_E = S_E^{\pm} \cup \Lambda_E$ according to the orientation of $i_X\mu$: $\pm = \operatorname{sgn} p_z$ and Λ_E forms the "equator" $p_z = 0$.

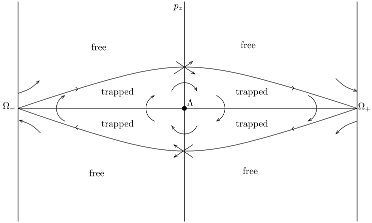
Illustration for 2DoF:



- Flux of μ from $q_z < 0$ to $q_z > 0$ is $\int_{S_E^+} \frac{1}{2}\omega \wedge \omega = \int_{\Lambda_E} \omega \wedge \alpha$, where α is a primitive of ω. Same for other direction.
- RS MacKay, Flux over a saddle, Phys Lett A 145 (1990) 425-7

Opposite signs

- Let $E_\infty = \frac{P_z^2}{2M}$. For $E < E_\infty$, $H^{-1}(E)$ is bounded.
- Regularise $\Omega_\pm = \{q_z = \pm\infty, p_z = 0\}$ by setting $q_z = \pm\sigma^{-2}$ locally and using new time s with $ds/dt = \sigma^3$. Then $\frac{d\sigma}{ds} = -\frac{p}{2m}$, $\frac{dp}{ds} = \frac{k_1 k_2 G \sigma}{(1+r^2\sigma^4)^{3/2}}$, making $\Omega_\pm$ 'morally' normally



hyperbolic.
- Their 5D contracting submanifolds separate points which bounce off $\Omega_\pm$ from those that go through infinity (in s).
- Let $\Omega_\pm(E)$ be intersections with $H^{-1}(E)$ for $E > E_\infty$.
- Their 4D contracting submanifolds intersect $q_z = 0$ in non-coincident 3-spheres; get fluxes between "bound" and "free". Can bind only once and unbind only once.
- 2D lower for the case $K = 0, \Pi = 0$ with extra integral p_θ.

Reconstructed motion

- $p_{z_1} = \frac{m_1}{M} P_z + p_z, p_{z_2} = \frac{m_2}{M} P_z - p_z$, and get q_{z_j} by integrating $\dot{q}_{z_j} = p_{z_j}/m_j$.
- Reconstructing the rest from reduced dynamics is more messy.
- But computation of full dynamics is easy:

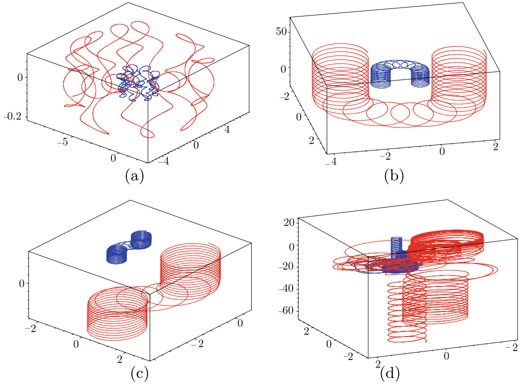


Figure: (a),(d) opposite signs; (b),(d) equal Ω

Scattering map

- Trajectories starting from $|q_z| = \infty$ at $t = -\infty$ go to $|q_z| = \infty$ at $t = +\infty$ except for (same sign): those asymptotic to Λ, (opposite sign): those that become bound forever (does this have positive measure or not?).
- Define scattering map from state at $t = -\infty$ to $t = +\infty$. Asymptotic gyroradii $|\rho_i|$, guiding centre fieldlines Q_i, parallel momenta p_{z_i} well-defined. For gyrophases θ_i, can define $\phi_i = \theta_i - \Omega_i t$, which has asymptotic limits (possibly $\pm\infty$).
- Only problem is q_{z_i}: would like to claim $q_{z_i} - \frac{p_{z_i}}{m_i} t$ has asymptotic limits but need screening for this, e.g. 1D case $H = \frac{1}{2}p_z^2 + \frac{G}{z} = E > 0$ on $z > 0$ gives $z - pt = \frac{G}{E}\left(1 - \operatorname{sgn} G \sqrt{1 - \frac{G}{Ez}} \tanh^{-1}(1 - \frac{G}{Ez})^{\operatorname{sgn}(G)/2}\right) \sim \frac{G}{E}(1 - \operatorname{sgn}(G)\frac{1}{2}\log\frac{4Ez}{|G|})$. [CHECK $G < 0$] But not important.
- The conserved quantities constrain the scattering map.
- Change in p_z leads to change in parallel KE: $\Delta E_{\parallel 1} = \Delta p_z(\frac{p_z^+ + p_z^-}{2m_1} + \frac{P_z}{M})$ and similar.

Scattering map for same sign charges

- Except for trajectories in or asymptotic to Λ, scattering map is well-defined and smooth. Subdivides into pass-through and bounce-back.
- Case $\delta\Omega = 0$: P_z, $\bar{Q}$, $|\bar{\rho}|^2$, $\tilde{L}$, H conserved. Tidiest to write in terms of p_z, S, s. $\frac{k\Omega}{2}|s|^2 + \frac{p_z^2}{2m} = cst$, $|S|^2 - |s|^2 = cst$. So there is an allowed interval I in p_z and for each $p_z \in I$ there is a unique $|S|, |s|$. One or other goes to 0 at the ends of I, depending on the sign of $\tilde{L}$. Except for $\tilde{L} = 0$ the allowed space is an $\mathbb{S}^3$. But given s there is also an $\mathbb{S}^1$ for (ρ_1, ρ_2). So the scattering map is on $\mathbb{S}^3 \times \mathbb{S}^1$. Change in S changes the fieldlines for the GCs, change in $|s|$ changes the gyroradii. Both could lead to cross-field transfer of perpendicular KE.

continued

- Formulate using coordinates for the individual particles but in centre of mass frame. Write $\rho_i = \sqrt{2\nu_i} e_{\theta_i}$ (ν_i is scaled μ_i).
- $H = \frac{p_z^2}{2m} + \sum_i k_i \Omega_i \nu_i + V(R)$.

Weak interaction for $\delta\Omega = 0$

- If helices are far apart there is another adiabatic invariant, for (r, p_r). Write $V = V(r, q_z)$. Then for fixed q_z, $H_r = \frac{k^2}{8m}r^2 + \frac{\tilde{L}^2}{2mr^2} + \frac{p_r^2}{2m} + V(r, q_z)$ performs oscillations at frequency roughly $\bar{\Omega}$. So if effect of changing q_z is small in a gyroperiod then action $I = \int p_r dr$ is an adiabatic invariant.
- When V negligible, $H_r = \frac{1}{2}\bar{\Omega}k|s|^2$ & $I = \frac{1}{2}k|s|^2$. So scattering map preserves $|s|$, hence p_z^2 and $|S|$ too. Thus no change in magnetic moments. Ignoring the gyrophase, only changes are a possible sign-change of p_z and rotation of S about $\bar{Q}$.
- Can compute the rotation of S. But it will produce no transport of charge, nor mass nor KE in a cross-field gradient. Only perpendicular to the gradient.
- Bouncing could produce parallel transport of parallel KE, but is limited to a small fraction of encounters.

$\delta\Omega \neq 0$

- If assume ratio between gyrofrequencies is not a low-order rational, the helices are widely separated and p_z not too large, then can two-phase average, deduce conservation of the magnetic moments again and compute the angle of rotation of S. But produces transport perpendicular to gradient.

Opposite sign charges

- Scattering map defined for initial conditions that do not get bound forever, but "chaotic" because the pair of GCs may become bound temporarily. But perhaps irrelevant in plasma because other particles interact too.
- Bound region is usually claimed to be negligible ("fully ionised") but I'm not so sure.
- Have to treat neutral case ($k_1 + k_2 = 0$) separately. In averaging approximation, get a shift of guiding centres perpendicular to their separation. But this produces flow perpendicular to a density gradient.

Non-widely separated helices

- So I think we have to analyse the non-widely separated case to find significant effects of encounters between particles.
- Usual mantra (e.g. Helander & Sigmar, Collisional transport in magnetized plasmas, 2002) is that transport is dominated by accumulation of small deflections from approaches b much larger than $b_{\min} = \frac{e_1 e_2}{4\pi\varepsilon_0 k_B T}$ (for 90^o deflection) but less than λ_D. This applies to magnetised plasma for gyroradii larger than λ_D. But ITER is planned to have gyroradii smaller than λ_D. So needs a more refined analysis. Approximate treatments by Psimopoulos&Li, Dubin&O'Neil. I think the result will be roughly that the usual $\log \Lambda$ with Debye cutoff is replaced by gyroradius cutoff.

Distribution function

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