An Invitation to Complex Analysis

Ryan Acosta Babb
University of Warwick

Contents

1 Real Analysis in $\mathbb{R}^2$

The complex plane $\mathbb{C}$ can be identified easily with the real plane $\mathbb{R}^2$ via the correspondence:

$$\begin{array}{ccc} \mathbb{C} & \cong & \mathbb{R}^2 \\ x+iy & \longleftrightarrow & (x,y). \end{array}$$

Familiar concepts from multivariable calculus, such as partial derivatives and line integrals can thus be carried over almost effortlessly into the complex setting. We begin by recalling some facts about analysis in $\mathbb{R}^2$ which will then receive a surprising new life in the complex plane.

1.1 The Fundamental Theorem of Calculus for Gradient Fields

Given a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ and a differentiable curve $\gamma : [a,b] \to \mathbb{R}^{\nvDash}$, we define the line integral of F along γ to be

$$\int_\gamma F(\mathbf{x})\,\mathrm{d}\mathbf{x} := \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t.$$

(The $\cdot$ in the right-hand side denotes the vector dot product in $\mathbb{R}^2$). In particular, if γ is a closed curve, meaning $\gamma(a) = \gamma(b)$, then we write

$$\oint_\gamma F(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

to emphasise that the curve is closed.

If $f : \mathbb{R}^2 \to \mathbb{R}$, has differentiable projections $f(\cdot, y)$ and $f(x, \cdot)$, the gradient of f is the vector valued function

$$\nabla f := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

If further $g : \mathbb{R} \to \mathbb{R}^2$, $g(t) = (g_x(t), g_y(t))$, is differentiable, then $f \circ g : \mathbb{R} \to \mathbb{R}$ is differentiable and the chain rule applies

$$(f \circ g)'(t) = \frac{\partial f}{\partial x}(g(t))g_x'(t) + \frac{\partial f}{\partial y}(g(t))g_y'(t) = \nabla f(g(t)) \cdot g'(t)$$

Just as in one-dimensional calculus we can "integrate a derivative" by evaluating it at the end points of an interval, we can "integrate a gradient" along a curve γ by evaluating the function at the end-points of the curve.

Theorem 1.1 (Fundamental Theorem of Calculus for Gradient Fields)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ have partial derivatives on $[a, b]$, and $\gamma : [a, b] \to \mathbb{R}^2$ a differentiable curve. Then*

$$\int_\gamma \nabla f(\mathbf{x})\,\mathrm{d}\mathbf{x} = f(\gamma(b)) - f(\gamma(a)).$$

Proof. The theorem follows from the chain rule and the Fundamental Theorem of Calculus in $\mathbb{R}$:

$$\int_\gamma \nabla f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_a^b \nabla f(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t = \int_a^b (f \circ \gamma)'(t)\,\mathrm{d}t = f(\gamma(b)) - f(\gamma(a)).$$

□

This implies that line integrals of gradients are path-independent. In fact, the converse is also true, which we prove with an additional result which we will "revisit" in §4.2.

Theorem 1.2. *For a continuous vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$, the following are equivalent:*

(i) F is conservative, i.e.

$$\oint_\gamma F(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0.$$

for all closed curves γ.

(ii) All line integrals of F are path independent, i.e. *if γ_1 and γ_2 are curves with the same end-points, then*

$$\int_{\gamma_1} F(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\gamma_2} F(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

(iii) F is a gradient field, i.e. *$F = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}$.*

Proof. The equivalence between (i) and (ii) is straightforward. If all closed paths integrate to zero, then given any two paths γ_1 and γ_2 with the same end point, we can form the path $\Gamma = \gamma_1 - \gamma_2$ understood as "traverse γ_1, then traverse γ_2 in the opposite direction". Clearly Γ is closed, so we have

$$0 = \oint_\Gamma F(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\gamma_1} F(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\gamma_2} F(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Conversely, if the line integrals of F are path independent, then we can calculate any closed path γ by picking a single point $\mathbf{p} \in \gamma$ and considering the constant path $t \mapsto \mathbf{p}$. Hence,

$$\oint_{\gamma} F(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{0}^{0} F(\mathbf{p}) \cdot \mathbf{0}\,\mathrm{d}t = 0.$$

The implication (iii) $\implies$ (ii) follows from Theorem 1.1. For the converse, pick any point $\mathbf{p} \in \mathbb{R}^2$ and consider the path $t \mapsto (1-t)\mathbf{p} + t\mathbf{x}$. We write the line integral, suggestively, as

$$f(\mathbf{x}) = \int_{\mathbf{p}}^{\mathbf{x}} F(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

It can then be shown that $\nabla f = F$. (The proof is similar to that of Theorem 4.7 below.) $\square$

1.2 Continuity in $\mathbb{R}^2$

We now discuss continuity in the plane. Recall that, in $\mathbb{R}$, continuity can be expressed in terms of limits. Thus, a function $f : (a,b) \to \mathbb{R}$ is continuous at $c \in (a,b)$ if, and only if,

$$f(c) = \lim_{x \to c} f(x).$$

This condition is equivalent to the existence of side limits, which must be equal, *i.e.*

$$f(c) = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x).$$

Thus, continuity in one dimension is a very simple matter: it suffices to check limits from the left and from the right.

In the plane however, one can approach a point $\mathbf{p}$ in all sorts of ways: parallel to the x-axis or the y-axis; along arbitrary straight lines; or even more "exotic" curves. For the limit to exist in the plane as $\mathbf{x} \to \mathbf{p}$, *all* such limits along all possible paths must exist and be equal.

Example 1.3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ give by

$$f(x,y) := \begin{cases} \frac{x^2 y}{x^4 + y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Along the x-axis, $\gamma(t) = (t, 0)$, we have

$$f(\gamma(t)) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Similarly, $f(x,y) \to (0,0)$ as $(x,y) \to (0,0)$ along the y-axis $\gamma(t) = (0,t)$. In fact, for any "direction" $(u,v) \in \mathbb{R}^2$, the limit along the straight line $\gamma(t) = (tu, tv)$ is also zero, as

$$f(\gamma(t)) = f(tu,tv) = \frac{(t^2u^2)(tv)}{t^4v^4 + t^2v^2} = \frac{tu^2v}{t^2u^4 + v^2} \to 0.$$

However, if we approach the origin along the parabola $\gamma(t) = (t, t^2)$, we have

$$f(\gamma(t)) = f(t,t^2) = \frac{t^2t^2}{t^4 + t^4} = \frac{t^4}{2t^4} \to \frac{1}{2}.$$

Therefore, the limit of $f(x,y)$ as $(x,y) \to (0,0)$ does *not* exist.

2 Topology in the Plane

In order to study the behaviour of functions on the complex plane, we need to become acquainted with its "geography": in order to take limits we need to known what points can be considered "close" to one another, or if taking limits in a set keeps the limit "inside" the same set, or whether it may escape to the "boundary". We want to known what sets hang together of a piece, and which are "disconnected" into separate pieces, since functions which are isolated to a single "piece" may have different behaviours on different pieces. This mathematical "cartography" is called *topology.* We refer the interested reader to [3].

The geometric intuitions sketched above will be familiar from analysis in $\mathbb{R}$. In one dimension. we use the absolute value function to define a "distance" between points on the line:

$$d(x,y) := |x - y|.$$

This takes care of "closeness" in terms of open intervals, $(a-\varepsilon, a+\varepsilon)$, which localise the points that are within ε of a. A further useful property of open intervals is that they allow for "wiggle" room: given any point $x \in (a,b)$, we can always find a small enough $\delta > 0$ such that $(x-\delta, x+\delta) \subset (a,b)$. This property is very useful when studying limits and continuous functions, since we don't need to worry about "falling out" of the set where the function is defined.

Notice that closed intervals, $[a,b]$, fail to allow for wiggle room at the end points, where moving to the left of a or the right of b will lead us out of the interval we want to be in. In this sense, the points a and b form the "boundary" of the interval, delimiting the "inside" from the "outside" of the set.

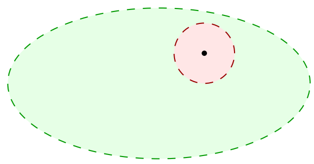

Figure 1: An open set in $\mathbb{C}$. Every point has a small "wiggle ball" around it. The dashed line indicates that the boundary is not included.

It should be recalled that the presence of the end points makes closed intervals suitable for taking limits: if $(x_n)_n$ is a convergent sequence of points in $[a, b]$, then $\lim x_n$ will also be a point of $[a, b]$. Open intervals, however, fail to have this property precisely because limits may "escape to the end points", as is the case of $(\frac{1}{n+1})_n \subset (0, 1)$ converging to $0 \notin (0, 1)$.

The goal of this section is to formalise and generalise these familiar concepts so that we can translate them to the complex plane. Subsequent sections will rely on many of these notions, especially when it comes to complex integration.

2.1 Open and Closed Sets

Just as in $\mathbb{R}$, the complex modulus function allows us to define a distance between points:

$$d(z, w) := |z - w|.$$

However, in two dimensions the equation $|z - w| = r$ does not define an interval around z of length $2r$ (r on either side); instead, we have a disk or ball. The concept of a ball is fundamental, so we introduce the following definitions:

Definition 2.1. The *open ball* of radius $r > 0$ centred at $a \in \mathbb{C}$ is defined to be the set

$$B(a, r) := \{z \in \mathbb{C} : |z - a| < r\}.$$

The *closed ball* of radius $r > 0$ centred at $a \in \mathbb{C}$ is

$$\overline{B}(a, r) := \{z \in \mathbb{C} : |z - a| \leqslant r\}.$$

Balls provide a natural way to consider "wiggle room". (See Figure 1.)

Definition 2.2. Let $G \subset \mathbb{C}$. A point $a \in G$ is called *interior* to G if there exists an $\varepsilon > 0$ such that $B(a, \varepsilon) \subset G$. The set G is *open* if every point of G is an interior point.

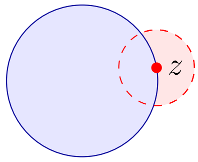

Figure 2: Any ball around the point z on the boundary intersects both the "interior" and the "exterior" of the ball (shaded red).

Notice that open balls are indeed open sets. Closed balls, however, are *not open*: picking a point on the boundary circle $|z - a| = r$, any open ball around z will intersect the complement of the ball. (See Figure 2.)

Balls also give a topological characterisation of limits: a sequence $(z_n)_n$ converges to z if for each $\varepsilon > 0$ there is an N such that $z_n \in B(z, \varepsilon)$ for all $n \geqslant N$. In other words, any ball around z contains infinitely many terms of the sequence.

Definition 2.3. Let $F \subset \mathbb{C}$. A point $z \in \mathbb{C}$ is a *limit point* of F if every ball around z contains points of F. The set F is *closed* if it contains all of its limit points.

One again, note that closed balls are closed sets according to this definition, but open balls are *not closed*: any ball around a point on the boundary circle intersects the inside of the circle, making these points limit points. (See again Figure 2.)

We have used the term "boundary" informally, but it has a simple formal definition which clearly captures the geometric meaning we are used to:

Definition 2.4. The *boundary* of a set $U \subset \mathbb{C}$ is the collection of points $z \in \mathbb{C}$ such that all open balls about z meet U and its complement. We denote this set by ∂U.

The reader may be tempted to think that: "a set is open if and only if it is not closed", but this is certainly not true given the definitions above. Indeed, even in $\mathbb{R}$, an open may be neither open no closed, as in the case of $(a, b]$. We invite the reader to construct a similar example in in $\mathbb{C}$. Perhaps more strikingly, the set $\mathbb{C}$ is *both open and closed* according to the definitions.[1]

The "true" relationship between open and closed sets is the following:

Proposition 2.5. *A set $F \subset \mathbb{C}$ is closed if, and only if, its complement $\mathbb{C} \setminus F$ is open.*

[1]For the pedantically inclined, the empty set is also both open and closed.

Proof. Let F be a closed. We want to show that any point $z \in \mathbb{C} \setminus F$ is interior to $\mathbb{C} \setminus F$. Suppose it was not, *i.e.* there is no $\varepsilon > 0$ such that $B(z, \varepsilon) \subset \mathbb{C} \setminus F$. Then all balls around z meet F and, therefore, z is a limit point of F. But then $z \in F$, as F is closed, contradicting $z \in \mathbb{C} \setminus F$.

Conversely, suppose $\mathbb{C} \setminus F$ is open. We want to show that every limit point of F lies in F. So suppose z is a limit point, *i.e.* $B(z, \varepsilon) \cap F \neq \varnothing$ for all $\varepsilon > 0$. Since $\mathbb{C} \setminus F$ is open and there is not ball around z lying entirely in $\mathbb{C} \setminus F$, it follows that $z \in F$. □

2.2 Continuity in $\mathbb{C}$

Functions $f : \mathbb{C} \to \mathbb{C}$ can also be continuous, and the definition is identical to the real definition, the "only" change being the use of the complex modulus instead of the absolute value: a function $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ is *continuous* at $a \in U$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|z - a| < \delta \implies |f(z) - f(w)| < \varepsilon.$$

We chose the domain U to be open to avoid worrying about whether some z with $|z - a| < \delta$ may not be in U, where f is defined. The condition above is easily rephrased using open balls, and we adopt this formulation as our definition.

Definition 2.6. A function $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ is *continuous* at $a \in U$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(B(a, \delta)) \subset B(f(a), \varepsilon)$.

That is, all points in the ball $B(a, \delta)$ around a are mapped into the ball $B(f(a), \varepsilon)$ around $f(a)$. Thus, continuous functions "keep points together".

The reader is reminded that continuity in the plane is more subtle than on the line. We refer back the the discussion in §1.2, and in particular Example 1.3.

For many purposes, the following, more topological, characterisation of continuity is useful.

Proposition 2.7. *A function $f : U \to \mathbb{C}$ is continuous if, and only if, the preimage of every open set under f is open in U.*

The reader may wonder "why the *pre-image*?" A naive guess might be that continuous functions map open sets to open sets, but this is certainly not the case. All constant maps take open sets to a single point, which is clearly not open. The reason behind the formulation with pre-images should become clearer during the proof.

Proof. We want to show that for every open set $G \subset \mathbb{C}$ the set $f^{-1}(G)$ is open in U.

Suppose f is continuous and let $G \subset \mathbb{C}$ be open, and pick any point $x \in f^{-1}(G)$. We want to show that there is a ball $B(x, \delta) \subset f^{-1}(G)$. Indeed, as G is open and $f(x) \in G$, there is a ball $B(f(x), \varepsilon) \subset G$ for some $\varepsilon > 0$. By continuity of f, there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. In other words, $f(y) \in B(f(x), \epsilon)$ for all $y \in B(x, \delta)$, so $B(x, \delta) \subset f^{-1}(G)$. This shows that $f^{-1}(G)$ is open.

For the converse, pick $x \in U$ and $\varepsilon > 0$. Then $B(f(x), \varepsilon)$ is an open set, so

$$f^{-1}(B(f(x), \varepsilon)) = \{y \in U : |f(x) - f(y)| < \varepsilon\}$$

is open in U. Hence, there is a $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$, since $x \in f^{-1}(B(f(x), \varepsilon))$. Thus,

$$|x - y| < \delta \implies y \in B(x, \delta) \implies f(y) \in B(f(x), \varepsilon) \implies |f(x) - f(y)| < \varepsilon,$$

which establishes continuity of f. □

As an example of the power of the topological characterisation, we prove the following:

Proposition 2.8. *The composition of two continuous functions is a continuous function.*

Proof. Let $f, g : \mathbb{C} \to \mathbb{C}$ be continuous functions, so $g \circ f : \mathbb{C} \to \mathbb{C}$. Pick any open set $G \subset \mathbb{C}$. Since g is continuous, $g^{-1}(G)$ is open. Hence, using that f is continuous, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is open. □

2.3 Connectedness and Compactness

Continuous functions in $\mathbb{R}$ satisfy two important properties:

Intermediate Value Theorem If $f : (a, b) \to \mathbb{R}$ is continuous, then it attains all values between $f(a)$ and $f(b)$.

Extreme Value Theorem If $f : [a, b] \to \mathbb{R}$ is continuous, then it is bounded and attains its bounds.

The crucial assumption behind the IVT is that the domain of f consists of a single "piece".

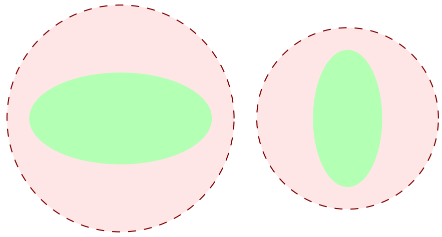

Figure 3: A disconnected subset of the plane (green), covered by disjoint open sets (red).

Example 2.9. Let f be defined on $(-2,-1) \cup (1,2)$ as follows:

$$f(x) := \begin{cases} 0, & \text{if } x \in (-2,-1); \\ 1, & \text{if } x \in (1,2). \end{cases}$$

This function is continuous according to the $\varepsilon - \delta$ definition, but it does not satisfy the intermediate value property, since f never achieves $1/2$.

The property of a set being "a single piece" or "not breaking up" is called connectedness.

Definition 2.10. A set $R \subset \mathbb{C}$ is *connected* if there are no non-empty, disjoint open sets $U, V \subset \mathbb{C}$ such that $R \subset U \cup V$.

In other words, a set is *disconnected* if we can cover it by disjoint open sets which "isolate its pieces". (See Figure 3.)

It should come as no surprise that continuous functions preserve connectedness, as they "keep points together".

Proposition 2.11. *The continuous image of a connected set is connected.*

Proof. Let $f : U \to \mathbb{C}$ be a continuous map and suppose $f(U)$ is not connected. Hence, there exist open sets $A, B \subset \mathbb{C}$ such that $A \cap B = \varnothing$ and $f(U) = A \cup B$. But then $U = f^{-1}(A) \cup f^{-1}(B)$ with $f^{-1}(A)$ and $f^{-1}(B)$ disjoint and open (by continuity of f). Therefore, U is not connected. $\square$

It can be shown that the only connected subsets of $\mathbb{R}$ are intervals, *i.e.* sets $I \subset \mathbb{R}$ such that whenever $x, y \in I$ and $x \leqslant z \leqslant y$, then $z \in I$. Hence, the special case of the above proposition in $\mathbb{R}$ reads: continuous functions map intervals to intervals, which is precisely the content of the IVT.

There is another "intuitive" notion of what it means for a set to be connected: any two points can be "joined together" by a path without leaving the set.

Definition 2.12. A *path* is a continuous map $\gamma : [0, 1] \to \mathbb{C}$. A set $U \subset \mathbb{C}$ is *path connected* if any two points $a, b \in U$ can be joined by a path in U. That is, there exists a path $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(0) = a, \gamma(1) = b$ and $\gamma([0, 1]) \subset U$.

A special case of path connected sets are convex sets. These are the sets which contain all straight lines between their points. The "official definition" is

Definition 2.13. A set $C \subset \mathbb{C}$ is *convex* if for all $a, b \in C$ and any $t \in (0, 1)$, we have $(1 - t)a + tb \in C$.

The paths in this case are the straight lines $t \mapsto (1 - t)a + tb$. Clearly

Proposition 2.14. *Balls (open or closed) are convex, and therefore path connected.*

Since we will often consider functions defined in open connected sets, we introduce the following

Definition 2.15. An open and connected subset of $\mathbb{C}$ is called a *domain*.

It can be shown that an open subset of $\mathbb{C}$ (or indeed $\mathbb{R}^n$) is connected if, and only if, it is also path connected, although we shall not prove this here. Suffice it to show that both definitions are convenient for different purposes, and that there is no risk of confusion by using them interchangeably for domains.

We now turn to the EVT. The reader will recall from real analysis that the conditions that the interval be bounded *and* closed are essential to the theorem. A continuous function on an unbounded interval may of course be unbounded (a trivial example is $f(x) = x$ on $(0, \infty)$).

Example 2.16. The function $f : (0, 1) \to \mathbb{R}$ given by $f(x) := \frac{1}{x}$ is continuous on the bounded interval $(0, 1)$, but is clearly unbounded as $x \to 0^+$.

The generalisation to $\mathbb{C}$ involves closed and bounded subsets of the plane. But in topology it is useful to work with the even more general notion of compactness.

Definition 2.17. A set $K \subset \mathbb{C}$ is called *compact* if whenever $\{G_j\}_{j \in J}$ are open sets such that $K \subset \cup_j G_j$, there exist finitely many $G_{j_1}, \ldots G_{j_n}$ such that $K \subset G_{j_1} \cup \cdots \cup G_{j_n}$. We call a collection $\{G_j\}_{j \in J}$ as above an *open cover* of K.

It is fairly easy to see that compact sets are bounded: cover K by the balls $B(x,1)$, $x \in K$. Then K is contained in a union of finitely many balls of radius 1, and therefore is bounded. We invite the reader to show that compact sets are also closed. (Proposition 2.5 might be helpful.)

In $\mathbb{C}$ (and more generally, $\mathbb{R}^n$), the compact sets are precisely the closed and bounded sets, a fact known as the Heine-Borel Theorem.

Theorem 2.18 (Heine-Borel)**.** *A subset of $\mathbb{C}$ is compact if, and only if, it is closed and bounded.*

Will we not prove this theorem here.[2] But we do remark that this Theorem *only* holds in $\mathbb{R}^n$, and not in more general metric or topological spaces. In particular, we have that closed and bounded intervals $[a,b]$ are compact in $\mathbb{R}$, and closed balls $\overline{B}(z,r)$ are compact in $\mathbb{C}$.

Using Proposition 2.7, it is easy to show that

Proposition 2.19. *If $f : U \to \mathbb{C}$ is continuous and $K \subset U$ is compact, then $f(K)$ is compact*

Proof. Pick any cover $\{G_j\}_{j\in J}$ of $f(K)$. Since the G_j are open and f is continuous, Proposition 2.7 implies that the $f^{-1}(G_j)$ are all open. Furthermore,

$$f(K) \subset \bigcup_{j\in J} G_j \iff K \subset \bigcup_{j\in J} f^{-1}(G_j),$$

so the compactness of K means that we can pick finitely many $j_1, \ldots, j_n$ such that

$$K \subset \bigcup_{k=1}^{n} f^{-1}(G_{j_k}).$$

Therefore, applying the same reasoning as above,

$$f(K) \subset \bigcup_{k=1}^{n} G_{j_k}.$$

□

From the Heine Borel Theorem, we can deduce that the continuous image of a closed bounded set in the plane is again closed and bounded. This fact will be useful in §5.1.

Example 2.20. The function defined by $f(z) = \frac{1}{z}$ is continuous on the bounded set $\overline{B}(0,1) \setminus \{0\}$, but it is certainly not bounded! This "failure" is due to the fact that the punctured disk is not compact, since it is not closed: the origin $0 \in \mathbb{C}$ is clearly a limit point of $\overline{B}(0,1) \setminus \{0\}$. (Compare with Example 2.16.)

[2] A proof may be found in any text on general topology such as [3] , or in [1].

We note that the same argument works for functions $f : K \to \mathbb{R}$. In particular, $f(K)$ is a closed and bounded subset of $\mathbb{R}$, and hence contained in a bounded interval. We can thus give a simple proof of the Extreme Value Theorem:

Theorem 2.21 (Extreme Value Theorem)**.** *Let $f : K \to \mathbb{C}$ be a continuous function on a compact set $K \subset \mathbb{C}$. Then $|f|$ is bounded and attains its bounds.*

Proof. By Proposition 2.8, $|f| : K \to \mathbb{R}$ is continuous as f and $|\cdot|$ are continuous. By Proposition 2.19, $|f|(K)$ is compact, hence closed and bounded. By Proposition 2.11, $|f|(K)$ is connected. Therefore, $|f|(K)$ is a closed bounded interval $[a, b] \subset \mathbb{R}$. In particular, $a = \min |f|$ and $b = \max |f|$ are achieved at some points in K, for otherwise $a, b \notin |f|(K)$. □

3 Differentiability in $\mathbb{C}$

In §2.2, we saw how the definition of continuity carried over seamlessly to $\mathbb{C}$. We will see how, formally, the definition of a derivative of a complex function is identical. Yet looks are deceiving, for a complex derivative is a far stranger object than a real one, as we will have ample opportunity to explore in the remainder of these notes.

3.1 Differentiability

Limits in the complex plane are defined using the complex modulus. The usual shorthand may be used to convey the definition

$$\lim_{z\to a} f(z) = L \iff \forall \varepsilon > 0 \exists \delta > 0 \forall z \in U(|z - a| < \delta \implies |f(z) - f(a)| < \varepsilon),$$

where $f : U \to \mathbb{C}$ is a complex function defined on a domain U. (See Definition 2.15.) We also remind the reader that limits in the complex plane are required to exists and be equal in *all* directions of approach, as in Example 1.3.

With these preliminaries out of the way, it is routine to define a complex derivative.

Definition 3.1. A complex function $f : U \to \mathbb{C}$ on a domain U is *differentiable* at $a \in U$ if the following (complex) limit exists and is finite:

$$\lim_{z\to a} \frac{f(z) - f(a)}{z - a}$$

In that case we denote the limit by $f'(a)$, the *derivative* of f at a. If $f : U \to \mathbb{C}$ is differentiable at every point of U (open), we shall call f a *holomorphic* function on U. In particular, a holomorphic function on $\mathbb{C}$ is called *entire*.

It is routine to check the complex differentiability implies continuity, and that the usual algebraic properties of derivatives carry over to complex derivatives. (The proofs, being identical to the real case, are omitted.)

Theorem 3.2. *Let $f, g : \mathbb{C}$ be differentiable functions and $\alpha, \beta \in \mathbb{C}$. Then,*

(i) f and g are continuous;

(ii) $\alpha f + \beta g$, fg and (where $g(z) \neq 0$) f/g are differentiable with derivatives

$$(\alpha f + \beta g)' = \alpha f' + \beta g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2};$$

(iii) Chain Rule: $f \circ g$ is differentiable with $(f \circ g)' = (f' \circ g)g'$.

Example 3.3. The function $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = z^2$ is differentiable at every $z \in \mathbb{C}$ with $f'(z) = 2z$:

$$\frac{(z+h)^2 - z^2}{h} = \frac{2zh + h^2}{h} = 2z + h \to 2z \text{ as } h \to 0.$$

Example 3.4. The function $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = \bar{z}$ is nowhere differentiable on $\mathbb{C}$. Consider $h \in \mathbb{R}$. Then, along the real axis, we have

$$\frac{\overline{(z+h)} - \bar{z}}{h} = \frac{h}{h} \equiv 1.$$

But along the imaginary axis, with ih, we have

$$\frac{\overline{(z+ih)} - \bar{z}}{ih} = \frac{-ih}{ih} \equiv -1.$$

Since the "partial limits" do not agree, the limit does not exists.

This simple example gives us some indication of how complex differentiability differs from real differentiability. In real analysis, it is very complicated to construct a function which is everywhere continuous but nowhere differentiable. In the complex case, we have a very simple example of this phenomenon in complex conjugation.

Some authors call holomorphic functions "analytic". We reserve this term for a more specific use:

Definition 3.5. A holomorphic $f : U \to \mathbb{C}$ is *analytic* if it has complex derivatives of all orders and its Taylor series converges to f on U.

Note that, in the real setting, having derivatives of all orders is *not* enough for analyticity.

Example 3.6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) := \begin{cases} e^{-1/x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is infinitely differentiable, but its Taylor series does not converge to $f(0) = 0$ at $z = 0$. So f is not analytic.

So far we only known that a holomorphic function is continuous, but know nothing about its derivative, which may not even be continuous. We will examine the properties of holomorphic functions and their derivatives in greater detail with the help of complex integrals in §4.

3.2 The Cauchy-Riemann Equations

Before progressing to complex integration, we derive an important characterisation of holomorphic functions.

Suppose $f : U \to \mathbb{C}$ is holomorphic on U and pick $a \in U$. Recall that we can write $f(x + iy) = u(x, y) + iv(x, y)$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

We now turn to the derivative of f at a, which it is convenient to write in the form

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

We known that the limit must be the same in all directions. If we take $h \to 0$ along the x-axis:

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{u(\Re a + h, \Im a) - u(\Re a, \Im a)}{h} + i \lim_{h \to 0} \frac{v(\Re a + h, \Im a) - v(\Re a, \Im a)}{h} \\ &= \frac{\partial u}{\partial x}(\Re a, \Im a) + i \frac{\partial v}{\partial x}(\Re a, \Im a). \end{aligned}$$

A similar calculation taking $ih \to 0$ ($h \in \mathbb{R}$) along the y-axis shows

$$f'(a) = \frac{\partial v}{\partial y}(\Re a, \Im a) - i \frac{\partial u}{\partial y}(\Re a, \Im a).$$

Equating real and imaginary terms in both calculations yields the *Cauchy-Riemann Equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We have shown that holomorphic functions satisfy the Cauchy-Riemann Equations. The converse, however, is not true.

Example 3.7. Let $f(z) := e^{-1/z^4}$ if $z \neq 0$ and $f(0) := 0$. It is easy to show that f satisfies the Cauchy-Riemann Equations at 0. But f is not continuous at 0, hence not differentiable.

The precise theorem is as follows.

Theorem 3.8. *Let $f : U \to \mathbb{C}$ be a function given by $f(x + iy) \equiv u(x, y) + iv(x, y)$ on a domain $U \subset \mathbb{C}$. Then, f is complex differentiable at $a \in U$ if, and only if, the function $(x, y) \mapsto (u(x, y), v(x, y))$ is real differentiable at a and satisfies the Cauchy-Riemann Equations.*

We omit the proof, as we have not developed real-differentiability in $\mathbb{R}^2$ in sufficient detail; it can be found in [1].

4 Complex Integration

The complex theory of integration plays a central rôle in complex analysis. Many of the more subtle and interesting results from the discipline are most easily proved using integration. In this section we explore two key results: Cauchy's Integral Theorem (§4.2) and one of its richest consequences, Cauchy's Integral Formula (§4.3). Both results will shed light on the richness of complex differentiation, setting it well apart from real differentiation (§4.4). We will also see how they can be applied to more specific problems in §5.

4.1 Line and Arc-Length Integrals

Complex line integrals can be defined in much the same way as real line integrals. (C.f. §1.)

Let $\gamma : [a, b] \to \mathbb{C}$ be a differentiable path in the plane and $f : U \to \mathbb{C}$ a complex function such that $\gamma([a, b]) \subset U$. The *line integral* of f along γ is defined to be

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

We will often integrate along the boundary of a bounded domain Ω in the plane, or around a circle $|z-a|=r$, in which cases the more explicit notation

$$\oint_{\partial\Omega} f(z)\,\mathrm{d}z \quad \text{or} \quad \oint\limits_{|z-a|=r} f(z)\,\mathrm{d}z$$

will be used. It is understood that all curves are traversed anticlockwise, and it is easy to show, by a change of variable, that a change of orientation implies a change of sign. Thus, if $-\gamma$ denotes the curve γ traversed in the opposite orientation, we have

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z.$$

Circles $|z-a|=r$ are most easily parametrised by $t \to a + re^{it}$ with $0 \leqslant t \leqslant 2\pi$.

Example 4.1. Let $f(z) := z$ on $\mathbb{C}$. Then

$$\oint\limits_{|r-a|=r} f(z)\,\mathrm{d}z = \int_0^{2\pi} (a+re^{it})(ire^{it})\,\mathrm{d}t = 0.$$

This result should come as no surprise, since z is the complex derivative of $\frac{z^2}{2}$. (Compare with Theorem 1.1.)

Example 4.2. Let $f(z) = \frac{1}{z-a}$. Then

$$\oint\limits_{|z-a|=r} \frac{1}{z-a}\,\mathrm{d}z = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}\,\mathrm{d}t = 2\pi i.$$

Another useful integral is the *arc-length* integral:

$$\int_{\gamma} f(z)\,|\mathrm{d}z| := \int_a^b f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t.$$

This integral should be familiar from multivaribale calculus. In particular, we have

$$\int_{\gamma} |\mathrm{d}z| = \mathrm{length}(\gamma).$$

We will often use this integral in conjunction with the following lemma.

Lemma 4.3 (Estimation Lemma)**.** *Let $f : U \to \mathbb{C}$ be a complex function and γ a differentiable path. Then*

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leqslant \int_\gamma |f(z)|\,|\mathrm{d}z|.$$

Proof. We first establish the estimate

$$\left|\int_a^b \varphi(t)\,\mathrm{d}t\right| \leqslant \int_a^b |\varphi(t)|\,\mathrm{d}t. \tag{$*$}$$

Note that, by linearity of the integral, we have

$$\int_a^b e^{-i\theta}\varphi(t)\,\mathrm{d}t = e^{-i\theta}\int_a^b \varphi(t)\,\mathrm{d}t.$$

for all $\theta \in [0, 2\pi)$. Hence

$$\Re\left[e^{-i\theta}\int_a^b \varphi(t)\,\mathrm{d}t\right] = \int_a^b \Re[e^{-i\theta}\varphi(t)]\,\mathrm{d}t \leqslant \int_a^b |\varphi(t)|\,\mathrm{d}t.$$

Setting $\theta = \arg\left[\int_a^b \varphi(t)\,\mathrm{d}t\right]$,

$$\Re\left[e^{-i\theta}\int_a^b \varphi(t)\,\mathrm{d}t\right] = \left|\int_a^b \varphi(t)\,\mathrm{d}t\right|$$

which proves ($*$). The lemma follows by taking $\varphi(t) = f(\gamma(t))\gamma'(t)$. □

4.2 Cauchy's Integral Theorem

This section introduces a fundamental result of complex analysis, with far-reaching consequences:

Theorem 4.4 (Cauchy's Integral Theorem)**.** *Let $f : U \to \mathbb{C}$ be holomorphic on a domain $U \subset \mathbb{C}$. Then, for any closed curve γ in U that can be "shrunk to a point in U", we have:*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

The assumption that γ may be "shrunk to a point" can be given a rigorous meaning. Geometrically, it is quite clear (see Figure 4), and the main application of this theorem in §4.3 will also be obvious. It is a crucial condition, as the following example shows.

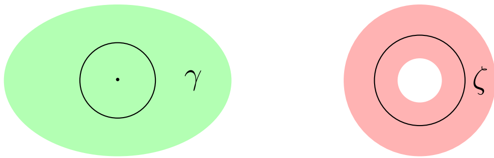

Figure 4: The curve γ (right) can be shrunk to a point in the green set. The curve ζ (right) cannot be shrunk to a point in the red set as it wraps around the "hole".

Example 4.5. The function $f(z) = \frac{1}{z}$ is holomorphic on $B(0,1) \setminus \{0\}$, but

$$\oint_{|z|=1} \frac{1}{z}\,\mathrm{d}z = 2\pi i.$$

Cauchy's Theorem does not apply to the boundary circle $|z| = 1$ since it cannot be "shrunk to a point" in $B(0,1) \setminus \{0\}$ as it encircles the "puncture" at the origin. (Compare this to the curve ζ in Figure 4.)

Cauchy's Integral Theorem is analogous to Theorem 1.2. Just as in $\mathbb{R}^2$, if f happens to be the derivative of a function, then the result easily follows from the Fundamental Theorem of Calculus (or Theorem 1.1). But we are only assuming that f has a derivative, without additional assumptions such as that f has an antiderivative.

We will prove Cauchy's Theorem for the special case of a ball. The more general proof is a little more complex, and we will omit it; it can be found in Chapter 4 of [1].[3] Note that it is this more general version which we shall need to prove Theorem 4.8 below.

We begin with an even simpler case: a triangle. This proof is due to Goursat.

Theorem 4.6 (Goursat's Theorem)**.** *Let $f : U \to \mathbb{C}$ be holomorphic on a domain U. Then, for every triangle $\Delta \subset U$ (with $\partial\Delta \subset U$ too), we have*

$$\oint_{\partial\Delta} f(z)\,\mathrm{d}z = 0.$$

Proof. We Divide the triangle δ into four smaller triangles $\Delta'_1, \ldots \Delta'_4$ as per Figure 5. Then

$$\oint_{\partial\Delta} f(z)\,\mathrm{d}z = \sum_{j=1}^{4} \oint_{\partial\Delta'_j} f(z)\,\mathrm{d}z,$$

[3]If the reader is familiar with Green's Theorem, we invite them to prove the general version of Cauchy's Integral Theorem using the Cauchy Riemann Equations from §3.2.

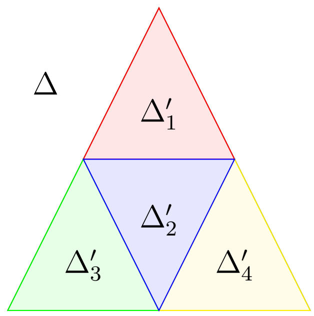

Figure 5: Triangle decomposition used in the proof of Goursat's Theorem.

as the integrals over the edges *inside* of Δ cancel out. By the triangle inequality.

$$\left|\oint_{\partial\Delta} f(z)\,\mathrm{d}z\right| \leqslant \sum_{j=1}^{4}\left|\oint_{\partial\Delta'_j} f(z)\,\mathrm{d}z\right|.$$

Let Δ_1 be the Δ'_j with largest integral in the sum above, so that

$$\left|\oint_{\partial\Delta} f(z)\,\mathrm{d}z\right| \leqslant 4\left|\oint_{\partial\Delta_1} f(z)\,\mathrm{d}z\right|.$$

We proceed by induction, obtaining a sequence $(\Delta_n)_n$ of triangles satisfying

(i) $\Delta \supset \Delta_1 \supset \cdots \supset \Delta_n \cdots$;

(ii) $\operatorname{diag}(\Delta_n) \leqslant 2^{-n}\operatorname{diag}(\Delta)$;

(iii) $\operatorname{length}(\partial\Delta_n) \leqslant 2^{-n}\operatorname{length}(\partial\Delta)$;

for which we have the inequalities

$$\left|\oint_{\partial\Delta} f(z)\,\mathrm{d}z\right| \leqslant 4^n\left|\oint_{\partial\Delta_n} f(z)\,\mathrm{d}z\right|. \tag{$*$}$$

Picking arbitrary points $z_n \in \Delta_n$, the sequence $(z_n)_n$ is Cauchy, hence convergent to some $z^* \in \Delta \cup \partial\Delta$. In particular, $z^* \in U$, so f is differentiable at z^*. Hence, for any $\varepsilon > 0$ there is a $\delta > 0$ small enough that $B(z^*, \delta) \subset \Delta$ and

$$\left|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\right| < \varepsilon$$

for all $0 < |z - z^*| < \delta$. We can re-write this as

$$|f(z) - f(z^*) - (z - z^*)f'(z^*)| < \varepsilon|z - z^*|.$$

By Theorem 1.1, we have that

$$\oint_{\partial\Delta_n} \mathrm{d}z = 0 \quad \text{and} \quad \oint_{\partial\Delta_n} z\,\mathrm{d}z = 0$$

since $z \mapsto 1$ and $z \mapsto z$ have complex antiderivatives everywhere. Therefore,

$$\oint_{\partial\Delta_n} f(z)\,\mathrm{d}z = \oint_{\partial\Delta_n} [f(z) - f(z^*) - (z - z^*)f'(z^*)]\,\mathrm{d}z,$$

and the Estimation Lemma implies

$$\begin{aligned}\left|\oint_{\partial\Delta_n} f(z)\,\mathrm{d}z\right| &\leqslant \varepsilon \oint_{\partial\Delta_n} |z - z^*|\,|\mathrm{d}z| \\ &\leqslant \varepsilon\,\mathrm{diag}(\Delta_n)\,\mathrm{length}(\partial\Delta_n) \\ &\leqslant \varepsilon(2^{-n}\,\mathrm{diag}(\Delta))(2^{-n}\,\mathrm{length}(\partial\Delta)) \\ &= \varepsilon 4^{-n}\,\mathrm{diag}(\Delta)\,\mathrm{length}(\partial\Delta).\end{aligned}$$

Thus, $(*)$ implies

$$\left|\oint_{\partial\Delta} f(z)\,\mathrm{d}z\right| \leqslant \varepsilon\,\mathrm{diag}(\Delta)\,\mathrm{length}(\partial\Delta).$$

Since $\varepsilon > 0$ is arbitrary and $\mathrm{diag}(\Delta)\,\mathrm{length}(\partial\Delta) > 0$ is a constant independent of ε, it follows that $\oint_{\partial\Delta} f(z)\,\mathrm{d}z = 0$. □

Goursat's Theorem can be used to prove a more general statement for closed curves in a disk. The proof fill's in the gap left in Theorem 1.2, where we used an integral along a path from a fixed point $\mathbf{p}$ to $\mathbf{x}$ to define an antiderivative of F.

Theorem 4.7 (Cacuchy's Integral Theorem for Balls)**.** *Let $f : U \to \mathbb{C}$ be holomorphic on a domain $U \subset \mathbb{C}$ and $B(a, r) \subset U$. Then, for any closed curve γ in $B(a, r)$, we have*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. We show that f has an antiderivative in $B(a, r)$. Then, the result follows from Theorem 1.1.

Since balls are convex, let λ_z denote the straight line from a to $z \in B(a, r)$. Define the function $F : B(a, r) \to \mathbb{C}$ as follows:

$$F(z) := \int_{\lambda_z} f(\xi)\,\mathrm{d}\xi.$$

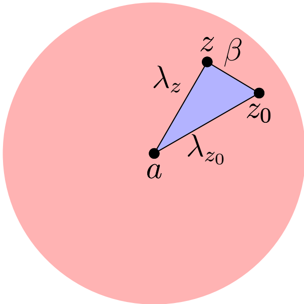

Figure 6: The triangle $\langle a, z_0, z\rangle$, oriented anticlockwise

We claim that $F' = f$.

To see this, fix $z_0 \in B(a, r)$. Since open balls are open, there is a $\rho > 0$ such that $B(z_0, \rho) \subset B(a, r)$. Let β denote the straight line from z_0 to z. Then, the three lines λ_{z_0}, λ_z and β form a triangle in $B(a, r)$, since the ball is convex. (Figure 6.) Applying Goursat's Theorem, we have

$$\int_{\lambda_{z_0}} f(\xi)\,\mathrm{d}\xi + \int_\beta f(\xi)\,\mathrm{d}\xi - \int_{\lambda_z} f(\xi)\,\mathrm{d}\xi = 0.$$

Hence,

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0}\left(\int_{\lambda_z} f(\xi)\,\mathrm{d}\xi - \int_{\lambda_{z_0}} f(\xi)\,\mathrm{d}\xi\right) = \frac{1}{z - z_0}\int_\beta f(\xi)\,\mathrm{d}\xi.$$

Writing down $\beta(t) = (1 - t)z_0 + tz$ explicitly, we have

$$\frac{1}{z - z_0}\int_\beta f(\xi)\,\mathrm{d}\xi = \frac{1}{z - z_0}\int_0^1 f((1-t)z_0+tz)(z-z_0)\,\mathrm{d}t = \int_0^1 f((1-t)z_0+tz)\,\mathrm{d}t.$$

As $z \to z_0$, we can use the continuity of to f exchange the limit and the integral to conclude that the right hand side is $f(z_0)$. □

4.3 Cauchy's Integral Formula

We are now ready to prove one of the most remarkable and beautiful results of complex analysis. Its power we become evident in the consequences we will derive from it.

Theorem 4.8 (Cauchy's Integral Formula)**.** *Let $f : U \to \mathbb{C}$ be holomorphic on a bounded domain $U \subset \mathbb{C}$ and $\overline{B}(a, r) \subset U$. Then*

$$f(a) = \frac{1}{2\pi i}\oint_{\partial B(a,r)} \frac{f(z)}{z - a}\,\mathrm{d}z.$$

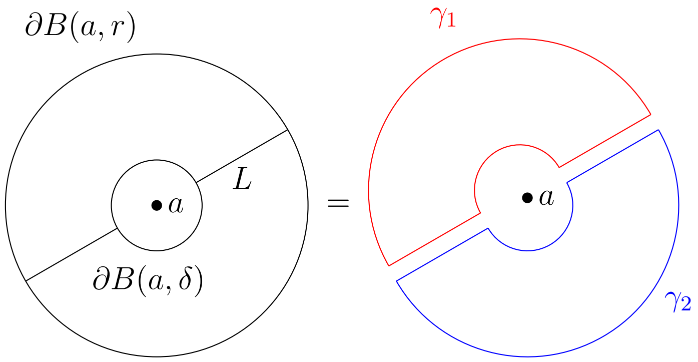

Figure 7: The contour used in the proof of Theorem 4.8.

Proof. Consider the function $F : B(a,r) \setminus \{a\} \to \mathbb{C}$ defined by

$$F(z) = \frac{f(z) - f(a)}{z - a}.$$

Note that F is holomorphic. To apply Cauchy's integral theorem to F, we will "cut off" a smaller ball $B(a,\delta) \subset B(a,r)$ at a and "integrate around it", as shown in Figure 7.

Consider the paths γ_1 and γ_2 shown in Figure 7. Then

$$\oint\limits_{|z-a|=r} F(z)\,\mathrm{d}z - \oint\limits_{|z-a|=\delta} F(z)\,\mathrm{d}z = \oint_{\gamma_1} F(z)\,\mathrm{d}z + \oint_{\gamma_2} F(z)\,\mathrm{d}z$$

where we integrate anticlockwise around $|z-a| = r$ and clockwise around $|z-a| = \delta$. Hence, the integrals along the line segments L cancel out as they are traversed once in each direction. Since the function F is holomorphic inside the domains bounded by γ_1 and γ_2, and these curves can be shrunk to a point in $B(a,r) \setminus \{a\}$, it follows from Cauchy's Integral Theorem that $\oint_{\gamma_j} F(z)\,\mathrm{d}z = 0$. Therefore,

$$\oint\limits_{|z-a|=r} F(z)\,\mathrm{d}z = \oint\limits_{|z-a|=\delta} F(z)\,\mathrm{d}z.$$

We now use the continuity of f to estimate the integral on the right hand side. Fix $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$$|z-a| < \delta \implies |f(z) - f(a)| < \varepsilon/2\pi.$$

(We require this δ to be small enough so that $B(a,\delta) \subset B(a,r)$, where f is holomorphic!) Then, using the Estimation Lemma,

$$\left|\oint_{\partial B(a,\delta)} F(z)\,\mathrm{d}z\right| \leqslant \oint_{|z-a|=\delta} \frac{|f(z)-f(a)|}{|z-a|}\,|\mathrm{d}z| \leqslant \oint_{|z-a|=\delta} \frac{\varepsilon/2\pi}{\delta}\,|\mathrm{d}z| = \frac{\varepsilon}{2\pi\delta}(2\pi\delta) = \varepsilon.$$

We have therefore proved that

$$\int_{|z-a|=r} F(z)\,\mathrm{d}z = \lim_{\delta\to 0^+} \int_{|z-a|=\delta} F(z)\,\mathrm{d}z = 0.$$

Expanding the integral of $F(z)$ we see that

$$\begin{aligned}\oint_{\partial B(a,r)} \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z &= \oint_{\partial B(a,r)} \frac{f(z)}{z-a}\,\mathrm{d}z - f(a)\int_{\partial B(a,r)} \frac{1}{z-a}\,\mathrm{d}z \\ &= \oint_{\partial B(a,r)} \frac{f(z)}{z-a}\,\mathrm{d}z - 2\pi i f(a),\end{aligned}$$

and the result follows from noting that the left hand side is zero. □

In fact, the integral formula can be applied to general curves encircling a, not just circles. We will use this more general version in §5.2.

This theorem is a remarkable departure from real analysis: we can determine every value of a complex-differentiable function on a domain by considering the integral *only along its boundary*. But it has a much more powerful consequence.

4.4 Complex Analytic Functions

Let us rename our variables and write

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi.$$

Since $z \neq \xi$ along $\partial\Omega$, we can take the derivative of the integrand with respect to z for each fixed $z \in \Omega$. Hence, differentiating under the integral sign:

$$f'(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\xi)}{(\xi - z)^2}\,\mathrm{d}\xi.$$

This shows that if f is holomorphic on Ω, then f' is also continuous, as the integrand on the right hand side is continuous. In fact, it is also *differentiable*! Iterating differentiation by induction yields

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial\Omega} \frac{f(\xi)}{(\xi - z)^{n+1}}\,\mathrm{d}\xi.$$

for each $n \in \mathbb{N}$. Hence, *holomorphic functions are infinitely differentiable.*

This is most certainly *not true* of real differentiable functions: having one derivative, even in an open and connected set, in no way guarantees even the continuity of the derivative, let alone being smooth! But we can show more.

Fix a point $z_0 \in \Omega$ and write

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0}\frac{\xi - z_0}{\xi - z} = \frac{1}{\xi - z_0}\frac{1}{\frac{\xi - z}{\xi - z_0}} = \frac{1}{\xi - z_0}\frac{1}{1 - \frac{z-\xi}{\xi - z_0}}.$$

For $|z - z_0| < |\xi - z_0|$, the second fraction on he right hand side can be expanded as a geometric series:

$$\frac{1}{1 - \frac{z-\xi}{z_0-\xi}} = \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^k.$$

Plugging this development back into Cauchy's Integral Formula yields

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\xi)}{\xi - z}\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\xi)}{\xi - z_0}\sum_{k=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^k \mathrm{d}z \\ &= \sum_{k=0}^{\infty}\left[\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\xi)}{(\xi - z_0)^{k+1}}\,\mathrm{d}z\right](z - z_0)^k \\ &= \sum_{k=0}^{\infty}\frac{f^{(k)}}{k!}(z - z_0)^k, \end{aligned}$$

where the exchange of the integral and the series can be justified by the uniform convergence in ξ of the series. Therefore, we have "proved"[4]

Theorem 4.9. *A holomorphic function $f : U \to \mathbb{C}$ on a domain $U \subset \mathbb{C}$ is analytic on U.*

Having proved this result, we will hereafter refer to holomorphic functions as "analytic", following common practise in the literature.

5 Applications

5.1 The Fundamental Theorem of Algebra

A remarkable consequence of Theorem 4.9 is

[4]We still have to justify the differentiation under the integral sign! This is done, for example, in Lemma 3 at [1, p. 121].

Theorem 5.1 (Liouville's Theorem)**.** *An entire bounded function is constant.*

Proof. Recall that an entire function is a function which is analytic (holomorphic) on all of $\mathbb{C}$. Suppose that there is an $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$. Using the derivative forms of Cauchy's Integral Formula we have

$$|f^{(n)}(z)| \leqslant \frac{n!}{2\pi} \int\limits_{|\xi - z| = r} \frac{|f(\xi)|}{|\xi - z|^{n+1}} \, |\mathrm{d}\xi| \leqslant \frac{n! M}{r^n}.$$

For $n \geqslant 1$, the right hand side can be made arbitrarily small as $r \to \infty$, and f remains analytic since it is entire. Hence, all coefficients of the Taylor series of f vanish except for the the constant term.[5] □

Once again, this result violates all of our expectations from real analysis, where even bounded analytic functions can be injective (consider tanh).

Example 5.2. The real restriction of sin is bounded by 1, as is well known; yet this function is analytic. Indeed, we *defined it* to be a power series when we extended it to the complex plane. Since sin is entire and clearly not constant, Liouville's theorem implies that it must be unbounded. As a complex function, we have

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Choosing $z_n = -in$, we have $e^{iz_n} = e^n \to \infty$ as $n \to \infty$ while $e^{-iz_n} = e^{-n} \leqslant 1$, so $\sin(z_n)$ is unbounded.

Liouville's Theorem also yields a simple and elegant proof of the Fundamental Theorem of Algebra.

Theorem 5.3 (Fundamental Theorem of Algebra)**.** *Any non-constant polynomial with complex coefficients has at least one complex root.*

Proof Sketch. Let $p(z) = a_n z^n + \cdots + a_0$ be a polynomial of degree $n \geqslant 1$ (so $a_n \neq 0$) with complex coefficients $a_k \in \mathbb{C}$. Suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then, the function

$$g(z) = \frac{1}{p(z)}$$

is entire. Since $|p(z)| \to \infty$ as $|z| \to \infty$, there exists an $R > 0$ such that $|p(z)| > |a_0|$ for all $|z| > R$, and therefore $|g(z)| \leqslant 1/|a_0|$ whenever $|z| >$

[5] Alternatively, notice that for $n = 1$ we have that $|f'(z)|$ vanishes for all z, so $f' \equiv 0$. Hence, as in the real case, f is constant.

R. On the other hand, $|z| \leqslant R$ defines the closed ball $\overline{B}(0, R)$. Since g is continuous, Proposition 2.19 implies that g is also bounded on $\overline{B}(0, R)$. Hence, g is bounded on the entire plane. By Liouville's theorem, g must be constant. Hence p must be constant. But this contradicts the assumption that p is a polynomial of degree $n \geqslant 1$. Therefore, p must have at least one root in the complex plane. □

The more usual form of the Fundamental Theorem of Algebra, which grantees the factorisation of p into n linear factors $\prod_{k=1}^{n}(z - z_k)$ where $z_1, \ldots, z_n$ are the (possibly repeated) roots of p, follows by the remainder theorem and induction on n.

5.2 Contour Integration

In this final section we will apply the results from §4 to compute a "hard" real integral.

We wish to calculate

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 1} \, \mathrm{d}x.$$

Using Euler's identity, we can re-write the integral as

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 1} \, \mathrm{d}x = \Re \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2 + 1} \, \mathrm{d}z.$$

For convenience, we will set

$$f(z) := \frac{e^{iz}}{z^2 + 1} = \frac{e^{iz}}{(z - i)(z + i)}, \quad g(z) := \frac{e^{iz}}{z + i}.$$

Note that f is analytic on $\mathbb{C} \setminus \{\pm i\}$ and g is analytic on $\mathbb{C} \setminus \{-i\}$.

We now consider the contour Γ as described in Figure 8. The point $-i$ lies outside of the region bounded by Γ and, for $R > 1$, i is inside this region. We can therefore apply Cauchy's Integral Formula to the analytic function g inside Γ:

$$\int_{\Gamma} f(z) \, \mathrm{d}z = \int_{\Gamma} \frac{g(z)}{z - i} \, \mathrm{d}z = 2\pi i g(i) = \frac{2\pi i e^{-1}}{2i} = \frac{\pi}{e}.$$

We can now recover our initial integral by calculating the line integrals

$$\int_{C_R} f(z) \, \mathrm{d}z \quad \text{and} \quad \int_{-R}^{R} f(z) \, \mathrm{d}z,$$

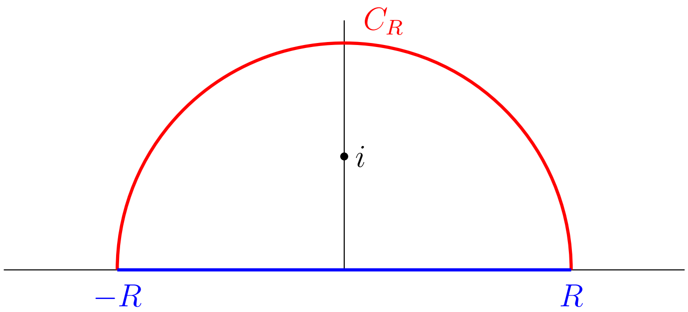

Figure 8: The (anticlockwise) contour Γ consists of two pieces: the straight line $[-R, R]$ and the upper semicircle C_R.

since

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+1}\,\mathrm{d}z = \lim_{R\to\infty}\int_{-R}^{R} f(z)\,\mathrm{d}z = \lim_{R\to\infty}\left[\int_{\Gamma} f(z)\,\mathrm{d}z - \int_{C_R} f(z)\,\mathrm{d}z\right].$$

Ideally, we want to show that the integral along C_R vanishes as $R \to \infty$, so that we are left with the finite integral over Γ, which we already computed to be $\frac{\pi}{e}$. Indeed, this is so, by an application of the Estimation Lemma.

First note that

$$|e^{iz}| = |e^{ix}||e^{-y}| = |e^{-y}| \leqslant 1$$

since $0 \leqslant \Im z \leqslant R$ on the upper semicircle C_R. Furthermore, using the reverse triangle inequality[6], we see that

$$|z^2+1| \geqslant ||z|^2 - 1| = R^2 - 1$$

as $|z| = R > 1$ along C_R. Putting these estimates together, we have

$$\left|\int_{C_R} f(z)\,\mathrm{d}z\right| \leqslant \int_{C_R} \frac{|e^{iz}|}{|z^2+1|}\,|\mathrm{d}z| \leqslant \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+1}\,\mathrm{d}z = \lim_{R\to\infty}\left[\int_{\Gamma} f(z)\,\mathrm{d}z - \int_{C_R} f(z)\,\mathrm{d}z\right] = \frac{\pi}{e}.$$

Since this integral is real, our initial integral evaluates to

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,\mathrm{d}x = \frac{\pi}{e}.$$

[6] *I.e.* $|x-y| \geqslant ||x| - |y||$. This is an easy consequence of the regular triangle inequality.

6 Bibliography

[1] Lars V. Ahlfors: *Complex Analysis* (Third Edition). McGraw-Hill. (1979)

[2] T. M. Apostol: *Mathematical Analysis* (Second Edition). Addison-Wesley Publishing Co. (1974)

[3] J. R. Munkres: *Topology*. Prentice-Hall. (2000)

[4] David Preiss: Lecture notes for the course "Metric Spaces" at the University of Warwick. (2015)

[5] T. J. Sullivan: Lecture notes for the course "Differentiation" at the University of Warwick.

[6] Hendrik Weber: Lecture notes for the course "Complex Analysis" at the University of Warwick. (2018)