All Functions are Continuous!

A provocative introduction to constructive analysis

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A dissatisfying proof

Exercise

Find irrational numbers a and b such that a^b is rational.

"Solution".

By cases! Either $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ or $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$.

- $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$. Just take $a = b = \sqrt{2}$.
- $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$. Then take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$:

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2. \qquad \square$$

Sure ... but we didn't find a and b!

The Law of Excluded Middle

- The "culprit" is The Law of Excluded Middle:

$$\varphi \vee \neg\varphi \tag{LEM}$$

- Yields non-constructive proofs!
- What would mathematics without LEM look like?

Five things you wanted to know about constructive mathematics but were too afraid to ask

1. Does constructive mathematics accept contradictions?
2. Does constructive mathematics reject proof by contradiction?
3. Does constructive mathematics reject the Axiom of Choice?
4. Is constructive mathematics compatible with classical mathematics?
5. Is constructive mathematics "poorer" than classical mathematics? Is [your favourite theorem] still valid?

1. Does constructive mathematics accept contradictions?

NO!

- Constructive mathematics does not assume $\varphi \vee \neg\varphi$ in general.
- It does not accept $\varphi \wedge \neg\varphi$ or $\neg(\varphi \vee \neg\varphi)$.
- Compare to agnosticism about Axiom of Choice: we refrain from using it in proofs.

Decidability

- Constructivism does accept LEM in some cases: for decidable propositions.
- $\varphi(x)$ is decidable if there is a "general procedure" ("algorithm") to decide $\varphi(x) \vee \neg\varphi(x)$ for each individual x in finitely may steps!
- Example: $\mathrm{GC}(n) \equiv 2n + 2$ is the sum of two primes. Then:
$$\forall n \in \mathbb{N}(\mathrm{GC}(n) \vee \neg\mathrm{GC}(n)). \qquad (\checkmark)$$
- Just compute all prime numbers up to $2n + 2$ and check each one!
- Compare the full Goldbach Conjecture:
$$\forall n \in \mathbb{N}(\mathrm{GC}(n)) \vee \neg\forall n \in \mathbb{N}(\mathrm{GC}(n)) \qquad (\text{✗})$$

2. Does constructive mathematics reject proof by contradiction?

NO! Compare two types of "proof by contradiction":

Proof of Negation

$$\begin{array}{c}\varphi\\ \vdots\\ \bot\\ \hline \neg\varphi\end{array}$$

OK! E.g. $\sqrt{2} \notin \mathbb{Q}$.

Reductio Ad Absurdum

$$\begin{array}{c}\neg\varphi\\ \vdots\\ \bot\\ \hline \varphi\end{array}$$

Not OK! Assumes LEM!

A closer look:

Reductio Ad Absurdum

$$\dfrac{\dfrac{\begin{matrix}\neg\varphi \\ \vdots \\ \bot\end{matrix}}{\neg\neg\varphi}\ \text{Negation Introduction Rule}}{\varphi}\ \text{Double Negation Rule}$$

"Double Negation Rule" is equivalent to LEM!

$$\varphi \vee \neg\varphi \iff \neg\neg\varphi \to \varphi$$

3. Does constructive mathematics reject the Axiom of Choice?

YES!

Theorem (Diaconescu)

The Axiom of Choice implies LEM.

If we cannot assume LEM, then we must reject Choice!

Proof.

1. Pick a proposition φ and define:
   $U := \{x \in \{0,1\} : (x = 0) \vee \varphi\}, V := \{x \in \{0,1\} : (x = 1) \vee \varphi\}$
2. Choice function f s.t. $f(U) \in U$ and $f(V) \in V$
3. $(f(U) = 0) \vee \varphi$ and $(f(V) = 1) \vee \varphi$
4. $(f(U) \neq f(V)) \vee \varphi$
5. Now, $\varphi \to (U = V)$, so $\varphi \to (f(U) = f(V))$
6. $(f(U) \neq f(V)) \to \neg\varphi$
7. Hence $\varphi \vee \neg\varphi$ by (4) and (6). □

4. Is constructive mathematics compatible with classical mathematics?

YES... and NO... It's complicated!

- YES: Rejecting LEM makes constructive mathematics a generalisation of classical mathematics:

$$\text{Constructivist Mathematics} \subsetneq \text{Classical Mathematics}$$

- Compare: all fields are rings but not all rings are fields!
- NO: Some varieties of constructive mathematics make additional assumptions which are incompatible with classical mathematics.
- More on this later!

5. Is constructive mathematics "poorer" than classical mathematics?

It may seem that Constructive Mathematics "loses" something by avoiding LEM: many classical theorems become unprovable. Casualties include

1. Tychonov's Theorem.
2. Zorn's Lemma.
3. Intermediate Value Theorem.
4. Bolzano-Weierstraß Theorem.
5. Trichotomy for $(\mathbb{R}, <)$, *i.e* $\forall x \in \mathbb{R}(x < 0 \vee x = 0 \vee x > 0)$.

But we also gain some new theorems! More on this later. . . .

Five things you wanted to know about constructive mathematics but were too afraid to ask

1. Does constructive mathematics accept contradictions? NO
2. Does constructive mathematics reject proof by contradiction? NO
3. Does constructive mathematics reject the Axiom of Choice? YES
4. Is constructive mathematics compatible with classical mathematics? Erm...
5. Is constructive mathematics “poorer” than classical mathematics? Is [your favourite theorem] still valid? Erm...

3/5 – not bad!

Building blocks of analysis

- We accept arithmetic on $\mathbb{N}$ and $\mathbb{Z}$ “as usual”!
- Atomic statements about integers, *i.e.* $a = b$ and $a < b$, are decidable. In particular:

$$\forall x, y \in \mathbb{Z}(x < y \vee x = y \vee x > y).$$

- Arithmetic on $\mathbb{Q}$ can be reduced to arithmetic on $\mathbb{Z}$! In particular:

$$\forall x, y \in \mathbb{Q}(x < y \vee x = y \vee x > y).$$

N.B. Hereafter, quantifiers of the form $\forall n$ and $\exists n$ range over $\mathbb{N}$.

Real number generators

We introduce real numbers as "Cauchy sequences" over $\mathbb{Q}$.

Definition

A real number generator (RNG) is a Cauchy sequence of rationals, *i.e* an $(a_n) \subset \mathbb{Q}$ such that

$$\forall k \exists n \forall p (|a_{n+p} - a_n| < 1/k).$$

Definition

Two RNG's (a_n) and (b_n) coincide if

$$\forall k \exists n \forall p (|a_{n+p} - b_{n+p}| < 1/k).$$

We write: $(a_n) \approx (b_n)$.

We identify real numbers with RNGs modulo coincidence:

$$a = b \iff (a_n) \approx (b_n)$$

A curious theorem

Classically:

Theorem

If $a \neq b$ is impossible, then $a = b$.

Proof.

SERIOUSLY?! □

Constructively things are little more subtle. Recall:

$$a \not\approx b \iff a \approx b \to \bot$$

Theorem

If $(a_n) \not\approx (b_n)$ is impossible, then $(a_n) \approx (b_n)$.

Proof (Heyting, 1966).

1. Fix k and let n be such that, for all p:
$$|a_{n+p} - a_n| < 1/4k \quad \text{and} \quad |b_{n+p} - b_n| < 1/4k.$$
2. Suppose $|a_n - b_n| \geqslant 1/k$. Then
$$|a_{n+p} - b_{n+p}| > 1/2k \qquad \text{for all } p \in \mathbb{N}.$$
3. Hence, $(a_n) \not\approx (b_n)$ – but this is impossible!
4. Thus, $|a_n - b_n| < 1/k$. [$a_n, b_n \in \mathbb{Q}$!]
5. It follows that
$$|a_{n+p} - b_{n+p}| < 2/k \qquad \text{for all } p \in \mathbb{N}.$$
6. Therefore, $(a_n) \approx (b_n)$. □

Is this necessary?

YES! Constructively: $\varphi \not\Leftrightarrow \neg\neg\varphi$!

Example (Fundamental Theorem of "Algebra")

Let $\varphi \equiv \forall p \in \mathbb{C}[x] \exists z \in \mathbb{C}(p(z) = 0)$.

- A proof of φ is an "algorithm" that, given a polynomial $p \in \mathbb{C}[x]$, yields (i) a $z \in \mathbb{C}$ and (ii) a proof that $p(z) = 0$.
- A proof of $\neg\neg\varphi$ is an "algorithm" that given a proof of $\neg\varphi$ produces a contradiction.

In general, a construction of a contradiction does not tell us how to construct a root $z \in \mathbb{C}$!

In a sense, these are different facts – constructive mathematics is sensitive to this difference in a way that classical mathematics is not!

Logical fine-print

Consider the three statements:

1. $(a_n) \approx (b_n) \iff \forall k \exists n \forall p (|a_{n+p} - b_{n+p}| < 1/k)$
2. $(a_n) \not\approx (b_n) \iff \forall k \exists n \forall p (|a_{n+p} - b_{n+p}| < 1/k) \to \bot$
3. $\neg((a_n) \not\approx (b_n)) \iff \neg \forall k \exists n \forall p (|a_{n+p} - b_{n+p}| < 1/k) \to \bot$

- Classically, (1) and (3) are equivalent.
- Constructively, there is no *a priori* reason why (3) should yield the positive statement (1): the latter specifies the distance between two RNGs.

This is because we lose the (classical) equivalence:

$$\neg \forall x \varphi(x) \iff \exists x \neg \varphi(x)$$

due to the stronger constructive reading of $\exists$.

Positive and negative notions

Compare:

- "Negative inequality": $\neg\forall k\exists n\forall p(|a_{n+p} - b_{n+p}| < 1/k)$
- "Positive inequality": $\exists k\exists n\forall p(|a_{n+p} - b_{n+p}| > 1/k)$

The positive statement tells us exactly how "separated" the RNGs (a_n) and (b_n) are, and is thus a stronger statement.

Definition

Two RNGs (a_n) and (b_n) lie apart if

$$\exists k\exists n\forall p(|a_{n+p} - b_{n+p}| > 1/k).$$

We write: $(a_n) \# (b_n)$, and also $a \# b$.

We already proved:

$$\neg((a_n) \# (b_n)) \iff (a_n) \approx (b_n).$$

Non-decidability of equality

Define the following RNG:

$$a_n := \begin{cases} 1/n & \text{if } \forall k \leqslant n \mathrm{GC}(k); \\ 1/m & \text{if } \exists m \leqslant n (\neg \mathrm{GC}(m) \wedge \forall k < m \mathrm{GC}(k)). \end{cases}$$

This is a well-defined RNG. It is clear that

- $a = 0 \iff$ the Goldbach Conjecture is proved.
- $a \# 0 \iff$ the Goldbach Conjecture is refuted.

The only way to (constructively) decide if $a = 0$ or $a \# 0$ is to have a proof or counterexample for the Goldbach Conjecture.

Non-decidability of equality for $\mathbb{R}$

Constructively: $\mathbb{R} \neq \{x : x = 0 \vee x \# 0\}$.

Proof that (a_n) is an RNG.

1. Fix $k \in \mathbb{N}$ and take $n = 2k > k$.
2. For each $l \leqslant n$, decide whether GC(l) or $\neg$GC(l) holds.
3. If $\forall l \leqslant n \mathrm{GC}(l)$, then $a_n = 1/n$ and $a_{n+p} \leqslant 1/n$, for all p, so
$$|a_{n+p} - a_n| \leqslant 1/n + 1/n < 1/k.$$
4. Otherwise, if m is the least number less than n such that $\neg$GC(m) holds, then $a_n = a_{n+p} = 1/m$ for all p, so
$$|a_{n+p} - a_n| = 0 < 1/k.$$
5. Therefore, $\forall p(|a_{n+p} - a_n| < 1/k)$. □

The proof only requires "algorithms" to find the right choices of n given k – no need to solve Goldbach!

Order relation for RNGs

Definition

Let a and b be reals. We say that $a < b$ if

$$\exists k \exists n \forall p (b_{n+p} - a_{n+p} > 1/k).$$

We write $a \leqslant b$ for $\neg(b < a)$. We also let

$$[a, b] := \{x : a \leqslant x \leqslant b\}.$$

Warning

$$a \leqslant b \not\Leftrightarrow (b < a \vee a = b)$$

It can be shown that:

$$|x + y| \leqslant |x| + |y|$$

.

Non-decidability of trichotomy

Construct (a_n) and (b_n) simultaneously as follows:

1. If $\forall m \leqslant n \mathrm{GC}(m)$, then $a_n = b_n = 1/n$.
2. If $m < n$ is the least counterexample s.t. $\neg\mathrm{GC}(m)$ and m is odd, then $a_n = 1/m$ and $b_n = 1/n$.
3. If m as above is even, then $a_n = 1/n$ and $b_n = 1/m$.

Then:

1. $a = b = 0 \iff$ Goldbach's Conjecture is true.
2. $a = 0 < b \iff \min\{m : \neg\mathrm{GC}(m)\}$ is even.
3. $a > 0 = b \iff \min\{m : \neg\mathrm{GC}(m)\}$ is odd.

Setting $c := a - b$, it is clear that we cannot decide

$$(c < 0) \vee (c = 0) \vee (c > 0).$$

The Intermediate Value Theorem

Let c be as before, so we cannot decide $(c \leqslant 0) \vee (c \geqslant 0)$, and "define" the function $g : [0,3] \to \mathbb{R}$ as follows:

$$g(x) \text{ ``} := \text{''} \begin{cases} x-1, & \text{if } 0 \leqslant x \leqslant 1; \\ 0, & \text{if } 1 \leqslant x \leqslant 2; \\ x-2, & \text{if } 2 \leqslant x \leqslant 3. \end{cases}$$

WARNING

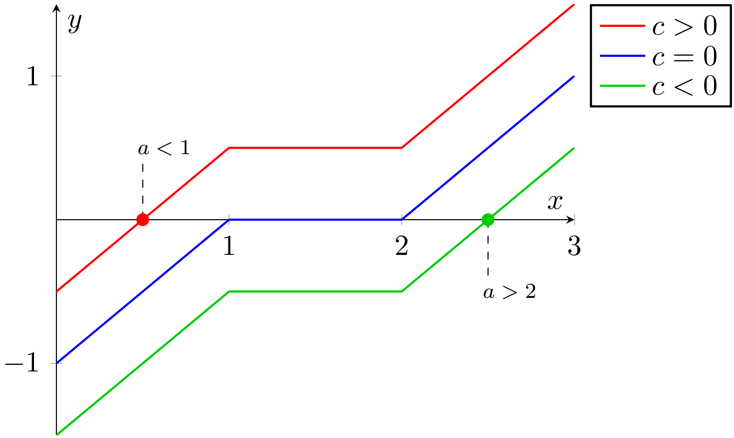
N.B. This is not strictly a valid constructive definition, since we cannot always decide, for $x \in [0,3]$, whether

$$(0 \leqslant x \leqslant 1) \vee (1 \leqslant x \leqslant 2) \vee (2 \leqslant x \leqslant 3).$$

But we can fix it by using Cauchy sequences directly.

This function is continuous on $[0,3]$, and so is $f(x) := g(x) + c$.

The function $f(x)$



IVT is contradictory

- Clearly $f(0) = c - 1 < 0$ and $f(3) = c + 1 > 0$. So IVT implies $f(a) = 0$ for some $0 < a < 3$.
- Since (a_n) is Cauchy, let $n \in \mathbb{N}$ be such that:

$$\forall p(|a_{n+p} - a_n| < 1/4).$$

- Note that $(a_n \leqslant 3/2) \vee (a_n > 3/2)$ [as $a_n \in \mathbb{Q}$!].
  - (A) $a_n \leqslant 3/2 \implies a_{n+p} < 1/4 + a_n \leqslant 7/4 < 2$ for all $p \in \mathbb{N}$. So $a < 2$.
  - (B) $a_n > 3/2 \implies a_{n+p} > a_n - 1/4 > 5/4 > 1$ for all $p \in \mathbb{N}$. So $a > 1$.
- On the other hand:
  - (A) $a < 2 \implies \neg(c < 0)$, *i.e.* $c \geqslant 0$.
  - (B) $a > 1 \implies \neg(c > 0)$, *i.e.* $c \leqslant 0$.
- Therefore, $(c \leqslant 0) \vee (c \geqslant 0)$ – contradiction!

Constructive IVT(s)

Theorem

Suppose $f : [a, b] \to \mathbb{R}$ *is continuous and* $f(a) \leqslant m \leqslant f(b)$*. Then,*

$$\forall \varepsilon > 0 \exists c \in [a, b](|f(c) - m| < \varepsilon).$$

Theorem

Suppose $f : [a, b] \to \mathbb{R}$ *is continuous,* $f(a) < 0 < f(b)$ *and for each* $x, y \in [a, b]$*, with* $x < y$*, there is a* $z \in [x, y]$ *such that* $f(z) \# 0$*. Then there is* $c \in [a, b]$ *such that* $f(c) = 0$*.*

Both are classically equivalent to the (classical) IVT, but not so constructively!

Laws and sequences

Question: What is an infinite sequence?
Answer: A rule assigning $n \mapsto a_n$(?)

$\implies$ Only countably many rules. (Countable language!)

$\implies$ Only countably many reals!

$\implies$ $\mathbb{R}$ has (Lebesgue) measure zero!!

We need to expand our notion of "sequence"!

Choice sequences

A choice sequence is a sequence which can be indefinitely extended, whether or not a rule is specified that governs "all future choices".

N.B. We are only guaranteed a finite initial segment of any given choice sequence! (Since we cannot assume a rule which determines "all future choices").

A topology on sequences

Consider the set of sequences of $\mathbb{N}$, denoted $\mathbb{N}^{\mathbb{N}}$. Two sequences are "close" if they agree on some initial segment, denoted by $\bar{\alpha}(n) = (\alpha_1, \ldots, \alpha_n)$. This topology is generated by the basis sets:

$$V_{\bar{\alpha}(n)} := \{\beta \in \mathbb{N}^{\mathbb{N}} : \bar{\beta}(n) = \bar{\alpha}(n)\}.$$

Continuity on sequences

A function $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is continuous if and only if

$$\forall \alpha \exists n \forall \beta \in \bar{\alpha}(n)(\Phi(\alpha) = \Phi(\beta)).$$

Here and after, α, β range over $\mathbb{N}^{\mathbb{N}}$ and $\beta \in \bar{\alpha}(n)$ abbreviates $\bar{\beta}(n) = \bar{\alpha}(n)$.

The Weak Continuity Axiom

The only functions which are meaningfully defined for all choice sequences are the continuous operations on $\mathbb{N}^{\mathbb{N}}$, since they are completely specified on some (finite) initial segment of the sequence.

Weak Continuity

Let $\varphi(\alpha, n)$ be a formula depending on sequences and integers.

$$\forall\alpha\forall n\varphi(\alpha, n) \to \forall\alpha\exists n, m\forall\beta \in \bar{\alpha}(m)\varphi(\beta, n) \qquad \text{(WC-}\mathbb{N}\text{)}$$

Translation

If φ can proved for all sequences α, then it must be determined by an initial segment of any given sequence, *i.e.* there is a "cut-off" point m such that only the values $\alpha_1, \ldots \alpha_m$ matter to decide φ.

Refutation of LEM

WC-$\mathbb{N}$ is incompatible with classical logic.

WC-$\mathbb{N}$ refutes LEM.

1. Assume LEM in the form: $\forall\alpha(\forall n(\alpha_n = 0) \vee \neg\forall n(\alpha_n = 0))$.
2. By WC-$\mathbb{N}$ [Exercise!]:
$$\forall\alpha\exists m\big(\forall\beta \in \bar{\alpha}(m)\forall n(\beta_n = 0) \vee \forall\beta \in \bar{\alpha}(m)\neg\forall n(\beta_n = 0)\big).$$
3. Let $\alpha = (0, 0, \ldots)$ and pick m such that
$$\forall\beta \in \bar{\alpha}(m)\forall n(\beta_n = 0) \vee \forall\beta \in \bar{\alpha}(m)\neg\forall n(\beta_n = 0).$$
4. Left disjunct is false: consider $\beta = (0, \overset{(m \text{ zeros})}{\ldots}, 0, 1, \ldots)$.
5. Right disjunct is false: consider $\beta = \alpha = (0, 0, \ldots)$.
6. This contradicts (1)! □

(Non-canonical) Canonical RNGs

- We can represent each $x \in [0,1]$ by a "decimal sequence" $(\alpha_n 10^{-n})$ where $\alpha \in \mathbb{N}^{\mathbb{N}}$.
- Example: for $x = 0.12$, $\alpha = (1, 12, 120, 1200, \ldots)$.
- Note that α satisfies:

$$\text{(i)}\ \forall n(\alpha_n \leqslant 10^n) \quad \text{and} \quad \text{(ii)}\ \forall n(|10\alpha_n - \alpha_{n+1}| \leqslant 9).$$

Definition (Non-standard terminology!)

I call a sequence α satisfying (i) and (ii) above a canonical number generator (CNG). We write $\alpha \in \mathscr{G}$ and $x^\alpha = \lim \alpha_n 10^{-n}$.

We can determine an open neighbourhood of a CNG by an initial segment. If $\alpha, \beta \in \mathscr{G}$, then

$$x^\beta \in B(x^\alpha, 10^{-n-1}) \implies \bar{\alpha}(n) = \bar{\beta}(n). \qquad (*)$$

The Interior Covering Lemma

Theorem (Interior Covering Lemma (ICL))

If $[0,1] \subset \bigcup_n X_n$, *then* $[0,1] \subset \bigcup_n (X_n)^\circ$.

Recall: $X^\circ = \{x \in [0,1] : \exists \delta > 0(B(x,\delta) \subset X)\}$.

A counterexample?

$[0,1] \subset [0,1/2] \cup [1/2,1]$, but $[0,1] \not\subset [0,1/2) \cup (1/2,1]$? No!

Note that $[0,1] \subset [0,1/2] \cup [1/2,1]$ means

$$\forall x \in [0,1](x \leqslant 1/2 \vee x \geqslant 1/2)$$

But we cannot decide this for an arbitrary real in $[0,1]$!

Can you construct a weak counterexample?

Proof of ICL

Recall that for $\alpha, \beta \in \mathscr{G}$:

$$x^\beta \in B(x^\alpha, 10^{-m-1}) \implies \bar{\alpha}(m) = \bar{\beta}(m). \qquad (*)$$

Proof.

1. Each $x \in [0,1]$ has a CNG α, *i.e* $x = x^\alpha$.
2. If we can decide whether $x^\alpha \in X_n$, then WC-$\mathbb{N}$ implies that we can show this from a finite segment $\bar{\alpha}(m)$.
3. Hence, all other CNGs β with this initial segment will also satisfy $x^\beta \in X_n$.
4. By $(*)$, this means that the entire ball $B(x, 10^{-m-1}) \subset X_n$.
5. Therefore, $x \in (X_n)^\circ$. □

The Continuity Theorem

Theorem (Brouwer, 1923)

Any function $f : [0,1] \to \mathbb{R}$ *is continuous.*

Proof (in Troelstra and van Dalen, 1988).

1. Fix $\varepsilon \in \mathbb{Q}_{>0}$ and enumerate the rationals: $\{r_n\} = \mathbb{Q}$.
2. Consider the sets $X_n := \{x \in [0,1] : |f(x) - r_n| < \varepsilon/2\}$.
3. Clearly $\{X_n\}$ is a cover of $[0,1]$.
4. By the ICL, $\{(X_n)^\circ\}$ is an open cover of $[0,1]$.
5. Let $x \in [0,1]$, so $x \in (X_n)^\circ$ for some n.
6. Hence, there is a $\delta > 0$ such that: $|x - y| < \delta \implies y \in X_n$.
7. But if $x, y \in X_n$, then
$$|f(x) - f(y)| \leqslant |f(x) - r_n| + |f(y) - r_n| < \varepsilon.$$ □

Conclusion

Reasons to study Constructive Mathematics

- A more grounded foundation for mathematics emphasising proofs. (Brouwer)
- A more hands on approach to mathematics, giving concrete, computational content to theorems. (Bishop)
- A useful metamathematical tool to study existing proofs and the assumptions that they rely on.
- A source of new and interesting mathematics to explore!

THANK YOU

Bibliography

- George, A. and Velleman, D. J. (2002) *Philosophies of Mathematics*. Oxford: Blackwell.

  Contains detailed philosophical discussion and historical background, as well as mathematical proofs.

- Heyting, A. (1966) *Intuitionism: An Introduction*. Amsterdam: North-Holland.

  Classic populariser of the subject, contains plenty of examples contrasting classical and intuitionistic theorems. It is a little dated!

- Troelstra, A. S. and van Dalen, D. (1988) *Constructivism in Mathematics: An Introduction*. Volume 1. Amsterdam: Elsevier.

  Superb and thorough treatment of the subject, including a survey of different varieties of constructivism. Beware it was written by logicians!

Introductory Material on Constructivism

There is good introductory material in the Stanford Encyclopedia of Philosophy:

- Bridges, D. and Palmgren, E.: "Constructive Mathematics".
- Iemhoff, R.: "Intuitionism in the Philosophy of Mathematics".
- van Atten, M.: "The Development of Intuitionistic Logic".
- van Atten, M.: "Luitzen Egbertus Jan Brouwer".

For the sceptics among you, Andrej Bauer's fanstastic talk "Five Stages to Accepting Constructivism" is a must see!