Topics in Harmonic Analysis

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Contents

Preface

These notes are the outcome of a self-study course in Harmonic Analysis taken by the author under the kind supervision of Dr Vedran Sohinger and Prof James C. Robinson, with the approval of Prof José Rodrigo.

The content is based on material found in Muscalu and Schlag (2013) and Stein (1971), expanded with reference to Grafakos (2014), Mattila (2015), Wolff (2003) and Tao (2009). Where appropriate, we have provided further details, discussion and solutions to some exercises of interest.

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Chapter 1

Complex Interpolation

In this chapter we review the methods of complex interpolation for linear operators. The main ideas are contained in the proof of the Riesz-Thorin interpolation theorem, which uses the maximum principle from complex analysis to obtain a uniform bound on a family on integrals, suitably to chosen to result in a bound on the operator norm between L^p spaces. This result will be of use to us when we tackle the "simple proof" of the Stein restriction conjecture. (See Section 4.2.)

1.1 The Riesz-Thorin Theorem

We begin with the lemma from Complex Analysis which will carry the brunt of the proof. It says that, if an analytic function on the strip is bounded on each"side" of the boundary, then it is bounded on each line in the interior. Furthermore, the interpolated bound is log-convex in the abscissa; see Fig. 1.1.

Lemma 1.1 (Hadamard's Three Line Theorem)**.** *Let F be an analytic function on the open strip $S := \{z \in \mathbb{C} : 0 < \Re z < 1\}$, continuous and bounded on its closure, such that*

$$|F(z)| \leqslant B_0 \quad \textit{when} \quad \Re z = 0;$$
$$|F(z)| \leqslant B_1 \quad \textit{when} \quad \Re z = 1,$$

for some positive constants B_0, B_1. Then, for any $\theta \in [0,1]$, we have

$$|F(z)| \leqslant B_0^{1-\theta} B_1^{\theta}.$$

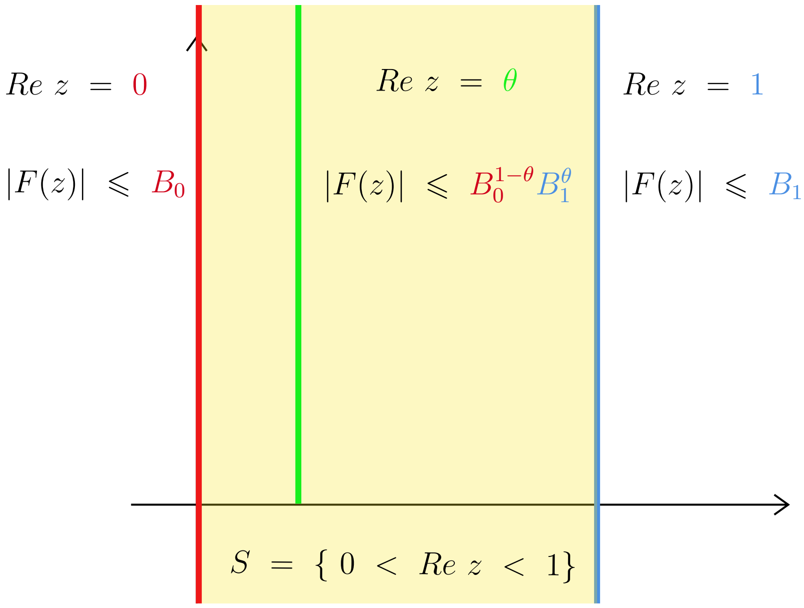

Figure 1.1: Hadamard's Three Line Lemma.

Proof. For $z \in S$, define the functions

$$G(z) := \frac{F(z)}{B_0^{1-z} B_1^z} \quad \text{and} \quad G_n(z) := G(z) e^{(z^2-1)/n}.$$

We will use the maximum principle to bound G by 1 on $\Re z = \theta$ and this will imply the result. To this end, we will study the functions G_n with rapid exponential decay.

By assumption, F is bounded on $\overline{S}$ and

$$\left|B_0^{1-z} B_1^z\right| \geqslant \min(1, B_0) \min(1, B_1) > 0$$

for $z \in \overline{S}$, so G is uniformly bounded on $\overline{S}$. Let M be a bound for G. Now,

$$|G_n(x+iy)| \leqslant M e^{-y^2/n} e^{(x^2-1)/n} \leqslant M e^{-y^2/n},$$

so $G_n(x+iy) \to 0$ uniformly in $0 \leqslant x \leqslant 1$ as $|y| \to \infty$.

Choose $y(n) > 0$ large enough that

$$|G_n(x+iy)| \leqslant 1 \quad \text{for all} \quad x \in [0,1], |y| \geqslant y(n).$$

By the boundedness of F on the boundary lines and the maximum principle, G_n is uniformly bounded by 1 on the domain $[0,1] + i[-y(n), y(n)] \subset \mathbb{C}$. Hence, $|G_n(z)| \leqslant 1$ uniformly on $\overline{S}$. Taking $n \to \infty$, we further conclude that $|G(z)| \leqslant 1$ uniformly on $\overline{S}$.

Hence, for $z = \theta + it$, we obtain

$$|F(z)| \leqslant \left|B_0^{1-\theta-it} B_1^{\theta+it}\right| = B_0^{1-\theta} B_1^{\theta}. \qquad \square$$

Theorem 1.2 (Riesz-Thorin)**.** *Let T be a linear operator defined on simple functions in $\mathbb{R}^n$ mapping into measurable functions. Let $p_0, p_1, q_0, q_1 \in [1, +\infty]$. Suppose that*

$$\|Tf\|_{L^{q_0}(\mathbb{R}^n)} \leqslant M_0 \|f\|_{L^{p_0}(\mathbb{R}^n)};$$
$$\|Tf\|_{L^{q_1}(\mathbb{R}^n)} \leqslant M_1 \|f\|_{L^{p_1}(\mathbb{R}^n)},$$

for some positive constants M_0, M_1 and all simple functions f. For $\theta \in (0,1)$, define

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{1.1}$$

Then, T extends to a bounded operator $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ such that

$$\|Tf\|_{L^q(\mathbb{R}^n)} \leqslant M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We will make use of the duality estimate for the norm:

$$\|Tf\|_{L^q(\mathbb{R}^n)} = \sup_{\substack{g \in L^p: \\ \|g\|_p \leqslant 1}} \left| \int_{\mathbb{R}^n} (Tf) g \right|.$$

In particular, we may take both f and g to be finitely simple functions:

$$f = \sum_{k=1}^{m} a_k e^{i\alpha_k} \chi_{A_k} \quad \text{and} \quad g = \sum_{j=1}^{n} b_j e^{i\beta_j} \chi_{B_j},$$

where $a_k, b_j > 0$, α_k, β_j are real and A_k and B_j, respectively, form pairwise disjoint collections of subsets of $\mathbb{R}^n$ with finite (Lebesgue) measure.

As in Hadamard's Theorem, we extend our θ-variable to $z \in S$. Define the polynomials

$$P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1} z \quad \text{and} \quad Q(z) = \frac{q}{q_0}(1-z) + \frac{q}{q_1} z,$$

so clearly $P(\theta) = Q(\theta) = 1$. We now extend our functions f and g to $z \in \overline{S}$-indexed families

$$f_z = \sum_{k=1}^{m} a_k^{P(z)} e^{i\alpha_k} \chi_{A_k} \quad \text{and} \quad g_z = \sum_{j=1}^{n} b_j^{Q(z)} e^{i\beta_j} \chi_{B_j}.$$

From the above observation on P and Q, we see $f = f_\theta$ and $g = g_\theta$. Finally, here is our analytic function

$$F(z) = \int_{\mathbb{R}^n} (Tf_z) g_z. \tag{1.2}$$

Hence, we have

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leqslant \sup_g |F(\theta)|. \tag{1.3}$$

By linearity, we can see that

$$F(z) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_k^{P(z)} b_j^{Q(z)} e^{i\alpha_k} e^{i\beta_j} \int_{\mathbb{R}^n} (T\chi_{A_k})\chi_{B_j}.$$

Hölder's inequality implies

$$\left| \int_{\mathbb{R}^n} (T\chi_{A_k})\chi_{B_j} \right| \leqslant \left\| \chi_{A_j} \right\|_{L^{q_0}(\mathbb{R}^n)} \left\| \chi_{B_j} \right\|_{L^{q_0'}(\mathbb{R}^n)} < \infty \quad \text{where} \quad \frac{1}{q_0} + \frac{1}{q_0'} = 1.$$

And since the $a_k, b_j > 0$, F is analytic in $z \in S$, bounded and continuous on $\overline{S}$. It thus remains to check the bounds on the two boundary lines.

For $\Re z = 0$, write $z = it$ and note that $\left| a_k^{P(it)} \right| = a_k^{\frac{p}{p_0}}$. Hence, using the disjointness of the sets A_k, we have

$$\|f_{it}\|_{L^{p_0}(\mathbb{R}^n)}^{p_0} = \sum_{k=1}^{m} a^p \, |A_j| = \|f\|_{L^p(\mathbb{R}^n)}^{p}.$$

Similarly, $\|g_{it}\|_{L^{q_0'}(\mathbb{R}^n)}^{q_0'} = \|g\|_{L^{q'}(\mathbb{R}^n)}^{q'}$. Thus, by the boundedness assumptions on T, we have

$$\begin{aligned} |F(it)| &\leqslant \|Tf_{it}\|_{L^{q_0}(\mathbb{R}^n)} \|g_{it}\|_{L^{q_0'}(\mathbb{R}^n)} \\ &\leqslant M_0 \|f_{it}\|_{L^{p_0}(\mathbb{R}^n)} \|g_{it}\|_{L^{q_0'}(\mathbb{R}^n)} \\ &= M_0 \|f\|_{L^p(\mathbb{R}^n)}^{\frac{p}{p_0}} \|g\|_{L^{q'}(\mathbb{R}^n)}^{\frac{q'}{q_0'}}. \end{aligned}$$

For $\Re z = 1$, we take $z = 1 + it$ and similarly derive

$$|F(1+it)| \leqslant M_1 \|f\|_{L^p(\mathbb{R}^n)}^{\frac{p}{p_1}} \|g\|_{L^{q'}(\mathbb{R}^n)}^{\frac{q'}{q_1'}}.$$

Thus, by Hadamard's theorem, we have

$$\begin{aligned} |F(z)| &\leqslant \left(M_0 \|f\|_{L^p(\mathbb{R}^n)}^{\frac{p}{p_0}} \|g\|_{L^{q'}(\mathbb{R}^n)}^{\frac{q'}{q_0'}} \right)^{1-\theta} \left(M_1 \|f\|_{L^p(\mathbb{R}^n)}^{\frac{p}{p_1}} \|g\|_{L^{q'}(\mathbb{R}^n)}^{\frac{q'}{q_1'}} \right)^{\theta} \\ &= M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{q'}(\mathbb{R}^n)}, \end{aligned}$$

using Eq. (1.1) to simplify the exponents. The theorem now follows from Eq. (1.3). □

This theorem is sometimes referred to as the "Riesz Convexity Theorem" as it shows that the set of pairs $(1/p, 1/q) \in [0,1]$ for which $T : L^p \to L^q$ is bounded forms a convex set. See Fig. 1.2.

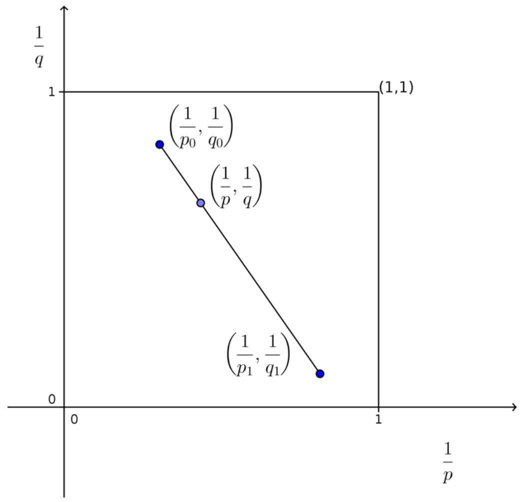

Figure 1.2: The Riesz-Thorin "Convexity Square". (Source: Oswald.luc, CC0, via Wikimedia Commons.)

The theorem is slightly less general than the "real" Marcinkievicz Interpolation Theorem. (The theorem is "real" in the sense that it does not rely on results form Complex Analysis, such as the maximum modulus principle.) While the latter is more general (*e.g.* it applies to sub-linear operators too), it yields rather awkward constants in the final estimate. We will need the cleaner exponents from Riesz-Thoring in Section 4.2.

1.2 Stein's Theorem for Analytic Families of Operators

A further advantage of the Complex Method is that the proof of Theorem 1.2 generalises almost verbatim to families $\{T_z\}_{z\in\mathbb{C}}$ of operators which depend analytically on z. This powerful result, due to Stein (1956), will provide an interesting alternative proof of the restriction theorem. See Section 4.3.1.

Suppose $\{T_z\}_{z\in\overline{S}}$ and there is a $\tau_0 \in (0,\pi)$ such that

$$\log\left|\int_{\mathbb{R}^n} T_z(f)g\right| \lesssim_{f,g} e^{\tau_0|y|} \tag{1.4}$$

for all simple functions f, g. Here $z = x + iy$.

Theorem 1.3. *Let $\{T_z\}_{z\in\overline{S}}$ be an analytic family of linear operators satisfying the growth condition Eq. (1.4). Let $M_0, M_1 : \mathbb{R} \to (0,\infty)$ such that there is a $\tau_1 \in (0,\pi)$ and*

$$\sup_{y\in\mathbb{R}} e^{-\tau_1|y|}\log(M_j(y)) < \infty \quad \textit{for} \quad j = 0,1. \tag{1.5}$$

With $\theta \in (0,1)$*, define* p *and* q *as in Eq.* (1.1)*. If*

$$\|T_{0+iy}f\|_{L^{q_0}(\mathbb{R}^n)} \leqslant M_0(y)\,\|f\|_{L^{p_0}(\mathbb{R}^n)}\,,$$
$$\|T_{1+iy}f\|_{L^{q_1}(\mathbb{R}^n)} \leqslant M_1(y)\,\|f\|_{L^{p_1}(\mathbb{R}^n)}\,,$$

for all simple functions f *and all* $y \in \mathbb{R}$*, then there is a function* $M : (0,1) \to (0,\infty)$ *such that*

$$\|T_\theta f\|_{L^q(\mathbb{R}^n)} \leqslant M(\theta)\,\|f\|_{L^p(\mathbb{R})} \quad \text{for all} \quad f \in L^p(\mathbb{R}^n).$$

Fefferman (perhaps somewhat optimistically) notes that

> Remarkably, the proof of the theorem comes from that of the Riesz Convexity Theorem by adding a single letter of the alphabet. (2014, p. 3)

Indeed, the change amounts to adding z to the operator T in Eq. (1.2):

$$F(z) = \int_{\mathbb{R}^n} (T_z f_z) g_z.$$

However, Hadamard's lemma must be replaced with a different, technically complicated result to yield the bound M, whose explicit expression is rather daunting. Thankfully, we will not need such "precision" in our application of Theorem 1.3.

See the original paper by Stein (1956) or the more modern treatment in (Grafakos, 2014, Sect. 1.3.3.) for the precise statements and full proofs.

Chapter 2

Oscillatory Integrals

The Fourier transform is a special case of so-called *oscillatory integrals*:

$$I(\lambda) = \int_{\mathbb{R}} e^{i\lambda\varphi(x)}\psi(x)\,\mathrm{d}x.$$

The exponential function introduces oscillations in the integral whose behaviour is controlled by the *phase* $\varphi(x)$. As is well known, *e.g.* from the Riemann-Lebesgue lemma and similar results), we can control the integral $I(\lambda)$ as $\lambda \to \infty$.

To see this, suppose that $\psi \in C_c^\infty(\mathbb{R})$ and that φ' does not vanish on $\operatorname{supp}\psi$. Then, clearly,

$$\frac{1}{i\lambda\varphi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{i\lambda\varphi(x)}\right) = e^{i\lambda\varphi(x)}.$$

Define the operators D and its transpose D^t as follows:

$$Df := \frac{f'}{i\lambda\varphi'(x)} \quad \text{and} \quad D^t f = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{i\lambda\varphi'(x)}\right).$$

We then have, for each $N \in \mathbb{N}$,

$$\int_{\mathbb{R}} e^{i\lambda\varphi(x)}\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}} D^N\left(e^{i\lambda\varphi(x)}\right)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}} e^{i\lambda\varphi(x)}\left(D^t\right)^N\psi\,\mathrm{d}x,$$

proving the following

Proposition 2.1. *Let $\psi \in C_c^\infty(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$ such that φ' does not vanish on* $\operatorname{supp}(\psi)$. *Then, for all $N \in \mathbb{N}$ we have*

$$|I(\lambda)| \lesssim \lambda^{-N} \quad \textit{for} \quad \lambda > 0.$$

2.1 Non-stationary phase: van der Corput's Lemma

Of course, if $\varphi' = 0$ somewhere in the support ψ, we need another method.

Lemma 2.2 (van der Corput). *Let $k \in \mathbb{N}$ and $\varphi \in C^\infty(a,b)$ satisfying $\left|\varphi^{(k)}(x)\right| \geqslant 1$ for all $x \in (a,b)$. Then,*

$$\left|\int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \lesssim_k \lambda^{-1/k}$$

holds provided that

(i) $k = 1$ and φ' is monotonic; or

(ii) $k \geqslant 2$.

Proof. We begin with (i). Integration by parts yields

$$\int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x = \frac{e^{i\lambda\varphi(b)}}{i\lambda\varphi'(b)} - \frac{e^{i\lambda\varphi(a)}}{i\lambda\varphi'(a)} - \int_a^b e^{i\lambda\varphi(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{i\lambda\varphi'(x)}\right)\mathrm{d}x.$$

We then absolute values, nothing that φ' and $\frac{\mathrm{d}}{\mathrm{d}x}(1/\varphi')$ do not change sign in (a,b) by assumption. Hence,

$$\left|\int_a^b e^{i\lambda\varphi'(x)}\,\mathrm{d}x\right| \leqslant \frac{2}{\lambda} + \frac{1}{\lambda}\int_a^b \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\varphi'(x)}\right)\right|\mathrm{d}x \lesssim \lambda^{-1},$$

as required.

For (ii) we argue by induction on k. Since $\varphi^{(k)}$ is continuous, the assumption $\left|\varphi^{(k)}\right| \geqslant 1$ implies that $\varphi^{(k)}$ does not change sign on (a,b). Hence, we may assume without loss of generality that

$$\varphi^{(k)}(x) \geqslant 1 \quad \text{for all} \quad x \in [a,b].$$

In particular, $\varphi^{(k-1)}$ is strictly increasing. Let $c \in [a,b]$ be the unique point at which $\left|\varphi^{(k-1)}(x)\right|$ attains its minimum.

Case I: $\left|\varphi^{(k-1)}(c)\right| = 0$. For any δ, apply the Mean Value Theorem at $x \in [a,b] \setminus [c-\delta, c+\delta]$ to see

$$\left|\varphi^{(k-1)}(x)\right| = |x-\delta|\left|\varphi^{(k)}\right| \geqslant \delta.$$

By induction hypothesis on $\lambda\varphi(x) \equiv (\lambda\delta)(\delta^{-1}\varphi(x))$, we have

$$\left|\int_a^{c-\delta} e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \lesssim (\lambda\delta)^{-\frac{1}{k-1}} \quad \text{and} \quad \left|\int_{c+\delta}^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \lesssim (\lambda\delta)^{-\frac{1}{k-1}},$$

so that, since trivially $\left|\int_{c-\delta}^{c+\delta} e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \leqslant 2\delta$, we have

$$\left|\int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \lesssim \frac{2}{(\lambda\delta)^{1/(k-1)}} + 2\delta. \tag{2.1}$$

Case II: $\left|\varphi^{(k-1)}(c)\right| \neq 0$. Since $\varphi^{(k-1)}$ is increasing, it follows that $c = a$ or $c = b$. Without loss of generality we assume that $c = a$. As above, for any $\delta > 0$ we deduce that

$$\left|\varphi^{(k-1)}(x)\right| \geqslant \delta \quad \text{for all} \quad x \in (a, a+\delta)$$

and obtain the bound

$$\left|\int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \leqslant \left|\int_a^{a+\delta} e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| + \left|\int_{a+\delta}^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \lesssim \delta + \frac{1}{(\lambda\delta)^{1/(k-1)}}. \tag{2.2}$$

Taking $\delta = \lambda^{-1/k}$ in either Eq. (2.1) or Eq. (2.2), yields

$$\left|\int_a^b e^{i\lambda\varphi(x)}\,\mathrm{d}x\right| \lesssim \lambda^{-1/k},$$

as required. □

A simple trick allows us to obtain an estimate for $I(\lambda)$:

Corollary 2.3. *Let $\psi \in C^\infty(\mathbb{R})$. Under the same assumptions as Lemma 2.2,*

$$|I(\lambda)| \lesssim_k \left[|\psi(b)| + \int_a^b |\psi'(x)|\,\mathrm{d}x\right]\lambda^{-1/k}.$$

Proof. Letting

$$F(x) := \int_a^t e^{i\lambda\varphi(t)}\,\mathrm{d}t,$$

we have

$$I(\lambda) = \int_a^b F'(x)\psi(x)\,\mathrm{d}x = F(b)\psi(b) - \int_a^b F(x)\psi'(x)\,\mathrm{d}x.$$

By Lemma 2.2, $|F(x)| \lesssim_k \lambda^{-1/k}$ for all $x \in [a, b]$, whence

$$|I(\lambda)| \lesssim_k \lambda^{-1/k}\,|\psi(b)| + \lambda^{-1/k}\int_a^b |\psi'(x)|\,\mathrm{d}x.$$ □

2.2 Application: Bessel Functions and $\widehat{\sigma}$

To illustrate the methods we just discussed, we turn to Bessel functions. (Detailed derivations of the formulae we quote may be found in (Grafakos, 2014, Appendix B).) For $m \in \mathbb{N}$ we have

$$J_m(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\sin(\theta)}e^{-im\theta}\,\mathrm{d}\theta.$$

This is an oscillatory integral with $\lambda = t$ and $\varphi(\theta) = \sin(\theta)$. In $[0, 2\pi]$, the only zeroes of φ' are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Let $1 = \psi_1 + \psi_2 + \psi_3$ be a partition of unity on $[0, 2\pi]$ such that ψ_1 is supported near $\frac{\pi}{2}$ and identically 1 on a smaller neighborhood o $\frac{\pi}{2}$. Choose ψ_2 similarly for $\frac{3\pi}{2}$. Then,

$$\begin{aligned} |J_m(t)| &\lesssim \sum_{j=1}^{3} \left| \int_0^{2\pi} e^{it\sin(\theta)} e^{-im\theta} \psi_j(\theta)\, \mathrm{d}\theta \right| \\ &\lesssim t^{-1/2} + t^{-1/2} + t^{-1} \\ &\lesssim t^{-1/2} \left(1 + \frac{1}{\sqrt{t}}\right), \end{aligned}$$

where we applied Lemma 2.2 to ψ_1 and ψ_2 with $k = 2$, and Corollary 2.3 to ψ_3 with $N = 1$. We thereby obtain the estimate

$$|J_m(t)| \lesssim t^{-1/2} \quad \text{as} \quad t \to +\infty. \tag{2.3}$$

This bound is especially useful since Bessel functions are related to the Fourier transform of the surface measure of the sphere:

$$\widehat{\sigma_{S^{n-1}}}(\xi) = 2\pi \left|\xi\right|^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(2\pi \left|\xi\right|).$$

Thus, Eq. (2.3) suggests the decay

$$\left|\widehat{\sigma_{S^{n-1}}}(\xi)\right| \lesssim \left|\xi\right|^{\frac{n-1}{2}}. \tag{2.4}$$

So far, we have only proved this bound when $\frac{n-1}{2}$ is an integer. Happily, it generalises to all $n \in \mathbb{N}$ as we will now show. We will need to consider higher dimensional oscillatory integrals first.

2.3 Higher Dimensional Oscillatory Integrals

Let $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R}^n, \mathbb{C})$. As before, we define

$$I(\lambda) := \int_{\mathbb{R}^n} e^{i\lambda\varphi(x)} \psi(x)\, \mathrm{d}x.$$

For non-stationary points, *i.e.* $\nabla\varphi(x_0) \neq 0$, we obtain a higher-dimensional Proposition 2.1:

Proposition 2.4. *If $\nabla\varphi(x) \neq$ for all $x \in \operatorname{supp}\psi$, then, for all $N \in \mathbb{N}$,*

$$|I(\lambda)| \lesssim_N \lambda^{-N} \quad \textit{for} \quad \lambda > 0.$$

Proof. We reduce this calculation to the one-dimensional case. For each $x_0 \in \operatorname{supp} \psi$, choose a unit vector θ such that

$$\theta \cdot (\nabla \varphi(x_0)) \geqslant c > 0.$$

By the continuity of $\nabla \varphi(x_0)$, this inequality holds on a ball $B(x_0, r)$. Using a partition of unity, we may write

$$I(\lambda) = \sum_k \int_{B_k} e^{i\lambda\varphi(x)} \psi_k(x)\, \mathrm{d}x$$

where each ψ_k is supported in a ball B_k as above.

By an orthogonal change of coordinates, we may assume that x_1 lies in the direction of θ, so

$$\int_{B_k} e^{i\lambda\varphi(x)} \psi_k(x)\, \mathrm{d}x = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} e^{\lambda\varphi(x_1,x')} \psi_k(x_1, x')\, \mathrm{d}x_1\, \mathrm{d}x'$$

and Proposition 2.1 applies to the inner integral. □

To discuss the non-stationary case, we recall that $x_0 \in \mathbb{R}^n$ is a *non-degenerate critical point* of φ if

$$\nabla \varphi(x_0) = 0 \quad \text{but} \quad \det(H_\varphi(x_0)) \neq 0.$$

(H_φ is the *Hessian* matrix of φ.)

Before we prove a higher-dimensional analogue of van der Corput's lemma, we record a tool from Differential Geometry:

Lemma 2.5 (Morse's Lemma)**.** *Let $U \subset \mathbb{R}^n$ be open and $\varphi \in C^\infty(U, \mathbb{R}^n)$. Suppose $x_0 \in U$ is such that*

$$\varphi(x_0) = 0 \quad \textit{and} \quad \nabla\varphi(x_0) = 0 \quad \textit{but} \quad \det(H_\varphi(x_0)) \neq 0.$$

(That is, x_0 is a non-degenerate critical point and a zero of φ.) Then, there are open set $V, W \subset \mathbb{R}^n$ such that

$$0 \in V \quad \textit{and} \quad x_0 \in W \subset U$$

and a diffeomorphism $G : V \to W$ such that

$$\varphi \circ G(x) = \sum_{j=1}^{k-1} x_j^2 - \sum_{j=k}^{n} x_j^2 \quad \textit{for all} \quad x \in V$$

for some $1 \leqslant k \leqslant n$.

We omit the proof, which can be found, *e.g.* in (Mattila, 2015, Lemma 14.6.).

Theorem 2.6. *If all critical points of φ in* $\operatorname{supp}\psi$ *are non-degenerate, then*

$$|I(\lambda)| \lesssim_{\varphi,\psi} \lambda^{-n/2} \quad \textit{for} \quad \lambda > 0.$$

Proof. Without loss of generality we assume that $\operatorname{supp}\psi \subset B := B(0,1)$,

$$\|\psi\|_{L^\infty(B)} \leqslant 1 \quad \text{and} \quad \|\nabla\psi\|_{L^\infty(B)} \leqslant 1 \quad \text{and} \quad \|H_\psi\|_{L^\infty(B)} \leqslant 1.$$

By Morse's Lemma, it suffices to consider

$$\varphi(x) \equiv Q(x) := \sum_{j=1}^{k-1} x_j^2 - \sum_{j=k}^{n} x_j^2.$$

Claim: If $\|\partial^\alpha\psi\|_{L^\infty(B)} \lesssim_n 1$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant n$, then

$$\left|\int_B e^{i\lambda Q(x)}\psi(x)\right| \lesssim_n \lambda^{-n/2}. \tag{2.5}$$

We establish the claim by induction on n. The case $n = 1$ follows from Lemma 2.2, so assume that the claim holds for $n-1$. By Fubini's Theorem, we may write

$$I(\lambda) = \lambda^{-1/2} \int_{\mathbb{R}^{n-1}} e^{i\lambda(x_2^2+\cdots+x_{k-1}^2-x_k^2-\cdots-x_n^2)}\psi_\lambda('x)\,\mathrm{d}'x$$

where $'x := (x_2, \ldots, x_n)$ and

$$\psi_\lambda('x) := \lambda^{1/2} \int_{\mathbb{R}} e^{i\lambda x_1^2}\psi(x_1, 'x)\,\mathrm{d}x_1.$$

We now observe that $\varphi_\lambda('x)$ is an oscillatory integral in $\mathbb{R}$ of the type already discussed, and that, for $\alpha \in \mathbb{N}_0^{n-1}$,

$$\partial^\alpha\psi_\lambda('x) = \lambda^{1/2} \int_{\mathbb{R}} e^{i\lambda x_1}\partial^{(0,\alpha)}\psi(x_1, 'x),$$

where $(0,\alpha) := (0, \alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^n$. Hence, by the induction hypothesis,

$$\left|\int_{\mathbb{R}^{n-1}} e^{i\lambda' Q('x)}\psi_\lambda('x)\,\mathrm{d}'x\right| \lesssim \lambda^{\frac{1-n}{2}},$$

which proves Eq. (2.5) for n, and thus the theorem. □

2.4 Fourier Transform of Surface Measures

We now apply the results from the last section to obtain decays estimates for the Fourier transforms of surface measures.

Let S be a smooth $n-1$ dimensional hypersurface in $\mathbb{R}^n$ with measure σ. Pick a bounded open set $U \subset \mathbb{R}^{n-1}$ containing 0 and a local chart $\Phi : C^\infty(U, \mathbb{R}^{n-1})$ such that

$$S = \{(x, \Phi(x)) : x \in U\} \quad \text{and} \quad \Phi(0) = 0 \quad \text{and} \quad \nabla\Phi(0) = 0.$$

We now choose a cutoff function $\zeta \in C_c^\infty(\mathbb{R}^n)$ whose support lies in U. More precisely,

$$\operatorname{supp} \zeta \subset \{(x, t) : x \in U, t \in \mathbb{R}\}.$$

The purpose of ζ is to define the measure $\mu = \zeta\sigma$, also given by

$$\int_{\mathbb{R}^n} g \, \mathrm{d}\mu = \int_U g(x, \Phi(x))\psi(x) \, \mathrm{d}x,$$

where

$$\psi(x) := \zeta(x, \Phi(x))\sqrt{1 + |\Phi(x)|^2}$$

is a smooth function compactly supported in $\mathbb{R}^{n-1}$. Then,

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi(\xi' \cdot x' + \xi_n \Phi(x))}\psi(x) \, \mathrm{d}x.$$

We will rewrite this expression as an oscillatory integral in $\lambda = |\xi|$ to apply Theorem 2.6. Note that, in this case, the condition $\det H_\Phi \neq 0$ has the geometric meaning that the Gaussian curvature of the surface S does not vanish. We will revisit this observation in Section 4.1.

Theorem 2.7. *With μ and U as above, if $\det H_\Phi(0) \neq 0$ on U, then*

$$|\widehat{\mu}(\xi)| \lesssim_{\Phi,\zeta} |\xi|^{\frac{1-n}{2}} \quad \textit{for all} \quad \xi \in \mathbb{R}^n.$$

Proof. Write $\xi = \lambda\theta$ in "polar coordinates":

$$\lambda := |\xi| > 0 \quad \text{and} \quad |\theta| = 1,$$

and set

$$\varphi(x) := -2\pi(\theta' \cdot x' + \theta_n \Phi(x)) \quad \text{for all} \quad x \in U.$$

This will be our phase. Note how

$$\nabla\varphi(x) = -2\pi(\theta' + \theta_n \nabla\Phi(x)) \quad \text{and} \quad H_\varphi(x) = -2\pi\theta_n H_\Phi(x).$$

If $\theta_n = 0$, then $\nabla\varphi(x) \neq 0$, so we apply Proposition 2.4, considering $|\xi|$ in $(0, 1)$ and $[1, \infty)$ separately for our choice of N.

If $\theta_n \neq 0$, then $\det H_\varphi \neq 0$, so we apply Theorem 2.6 for $n-1$. □

In particular, we recover Eq. (2.4) in full generality. This bound will be very useful in Chapter 4.

For further details on oscillatory integrals, we invite the reader to study the masterful exposition in Chapters VIII and IX of the classic text Stein et al. (1993).

Chapter 3

Uncertainty Principles

3.1 Motivational Heuristics

Heuristically, the uncertainty principle is the idea that a function and its Fourier transform cannot both "decay too quickly". Alternatively, we can phrase the principle in terms of "localisation": if a function is "localised" in x space, then its Fourier transform cannot be "localised" in ξ space.

It is easy to convince oneself of this fact by examining the exponential

$$e^{2\pi i\theta} = \cos(2\pi\theta) + i\sin(2\pi\theta).$$

When $|\theta| \approx 1$,

$$\cos(2\pi\theta) \approx 1 \quad \text{and} \quad \sin(2\pi\theta) \approx 0,$$

thus, $f(x)e^{-2\pi\theta} \approx f(x)$.

If, therefore, f is supported at a scale $|x| \ll R$ and we examine frequencies $|\xi| \gg 1/R$, then $|x \cdot \xi| \approx 1$ and we expect something like

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x \approx \int_{|x|\ll R} f(x)\,\mathrm{d}x.$$

Thus, $\widehat{f}$ is essentially constant at scales $|\xi| \gg 1/R$.

Furthermore, by Fourier inversion, we also expect $f(x)$ to be constant at scales $|x| \gg 1/R$ if $\widehat{f}$ is localised to $|\xi| \ll R$. In particular, a non-zero f with compact support should not have a compactly supported Fourier transform.

This can be seen more formally by extending $\widehat{f}$ to the complex plane and showing that it is an entire function when f has compact support. But the zero set of an entire function is discreet, so $\widehat{f}$ cannot have compact support. (This result is part of the Paley-Wiener Theorem.)

3.2 Hardy's Uncertainty Principle

As we discussed above, we cannot simultaneously localise f and $\widehat{f}$. Thus, if f decays fast enough at infinity, then $\widehat{f}$ cannot so decay. The limiting case of this behaviour is observed for the Gaussian $g(x) = e^{-\pi|x|^2}$, which satisfies $\widehat{g} = g$. Thus, g and $\widehat{g}$ *both* decay faster than exponentially This observation prompted Hardy (1933) to prove his uncertainty principle, which effectively formalises that idea.

Theorem 3.1 (Hardy's Uncertainty Principle)**.** *Suppose that f is a measurable function such that*

$$|f(x)| \lesssim e^{-\pi a x^2} \quad \textit{and} \quad \left|\widehat{f}(\xi)\right| \lesssim e^{-\pi \xi^2/a},$$

then $f(x) \equiv A e^{-\pi a x^2}$.

We follow a more modern proof in Tao (2009).

Proof. Without loss of generality, we may assume that

$$|f(x)| \leqslant e^{-\pi x^2} \quad \text{and} \quad \left|\widehat{f}(\xi)\right| \leqslant e^{-\pi \xi^2}.$$

We can extend $\widehat{f}$ to the complex plane by observing that

$$\left|\widehat{f}(\xi + i\eta)\right| \leqslant \int_{\mathbb{R}} e^{-\pi x^2} e^{2\pi \eta x}\, \mathrm{d}x < \infty$$

so that

$$\widehat{f}(\xi + i\eta) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} e^{2\pi \eta x}\, \mathrm{d}x$$

is well-defined for all $\xi + i\eta \in \mathbb{C}$. The fast exponential decay allows us to differentiate under the integral sign and conclude that $\widehat{f}$, so extended, is an entire function.

Thus, returning to our bound on $\left|\widehat{f}\right|$ and applying a standard integral:

$$\left|\widehat{f}(\xi + i\eta)\right| \leqslant e^{\pi\eta^2} \int_{\mathbb{R}} e^{-\pi(x-\eta)^2}\, \mathrm{d}x = e^{\pi\eta^2}, \tag{3.1}$$

whence $F(z) := e^{\pi z^2} \widehat{f}(z)$ is entire and bounded in modulus by 1 on the imaginary axis. From the decay assumption on $\widehat{f}$, we directly conclude the same on the real axis.

All that remains is to apply some Complex-Analytical magic to extend the bound to $\mathbb{C}$ that we may use Liouville's Theorem.

Fix an aperture $\theta \in (0, \frac{\pi}{2})$ and define the sector

$$\Gamma_\theta := \left\{re^{i\alpha} : r > 0, 0 \leqslant \alpha \leqslant \theta\right\}.$$

From Eq. (3.1) we see

$$|F(\xi + i\eta)| \leqslant e^{\pi(\xi^2-\eta^2)} \left|e^{2\pi i\xi\eta}\right| e^{\pi\eta^2} = e^{\pi\xi^2}.$$

Now, for $\delta > 0$ and θ close to $\frac{\pi}{2}$,

$$\left|e^{i\delta z^2} F(z)\right| \leqslant e^{\pi(\xi^2-2\xi\eta)\delta} = e^{\pi\xi\eta\delta(\tan(\theta)-2)} \leqslant 1 \quad \text{on} \quad \partial\Gamma_\theta.$$

(See Fig. 3.1.) Hence, for all small $\varepsilon > 0$, $e^{i\varepsilon e^{i\varepsilon} z^{2+\varepsilon}} e^{i\delta z^2} F(z)$ is also bounded by 1 in modulus on $\partial\Gamma_\theta$ and goes to zero as $|z| \to \infty$ in Γ_θ°. By the maximum modulus principle, the function is bounded by 1 on Γ_θ. Hence, sending $\varepsilon \to 0$, $\theta \to \frac{\pi}{2}$ and $\delta \to 0$, we obtain $|F(z)| \leqslant 1$ on the first quadrant. Similar arguments cover the rest of the plane. □

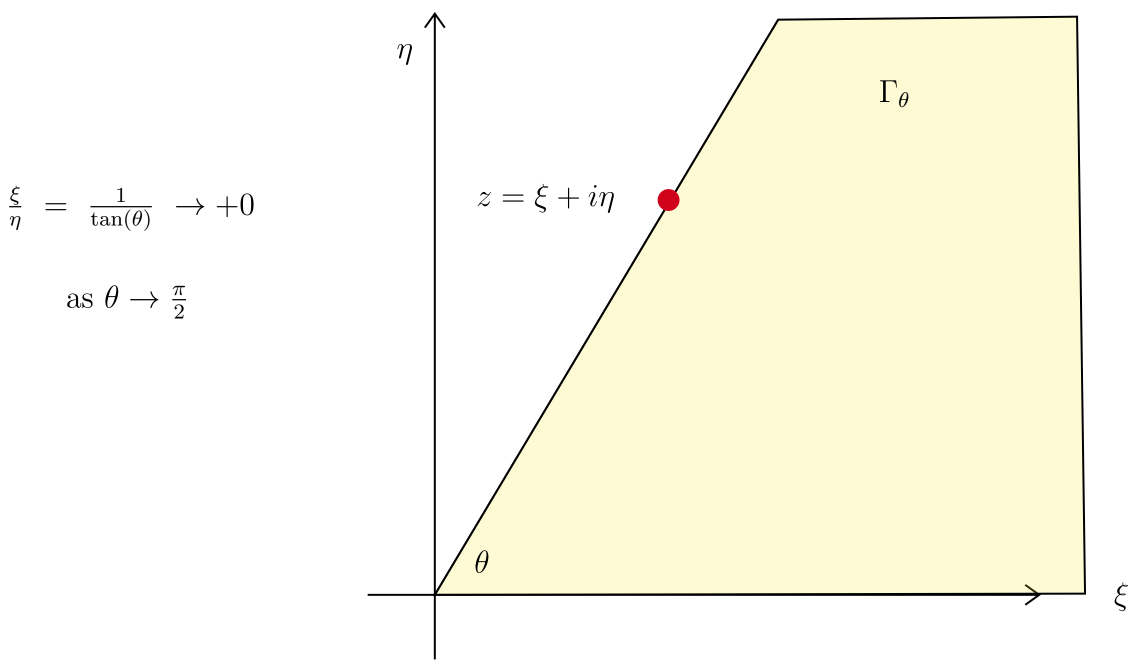

Figure 3.1: The contour Γ_θ and the behaviour of its boundary points.

3.3 The Amrein-Berthier Theorem

We already saw that a function and its Fourier transform cannot *both* be compactly supported in $\mathbb{R}^n$. In this section we refine this observation to arbitrary sets of finite measure.

Question: Given sets $E, F \subset \mathbb{R}^n$ of finite measure, can there be a non-zero $f \in L^2(\mathbb{R}^n)$ such that $\operatorname{supp} f \subset E$ and $\operatorname{supp} \widehat{f} \subset F$?

Suppose that there were. By defining the double projection $Tg := \chi_E(\chi_F \widehat{g})^\vee$ we have $Tf = f$. Hence: $\|T\| \geqslant 1$.

On the other hand, T is an integral operator:

$$(Tg)(x) = \int_{\mathbb{R}^n} \chi_E(x)\check{\chi}_F(x-y)g(y)\,\mathrm{d}y$$

with kernel $K(x,y) = \chi_E(x)\check{\chi}_F(x-y)$. Its norm, therefore, is bounded by

$$\|T\| \leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y = |E|\,|F| =: \sigma.$$

Hence, if E and F are such that $\sigma < 1$, then there cannot be an f as required.

In a sense, the Amrein-Berthier Theorem will show that the remaining case, $\sigma \geqslant 1$ equally leads to a contradiction, thereby answering our question negatively in general. More precisely, we obtain an estimate on the norm of f by "discarding" the portions of f and $\widehat{f}$ that lie on E and F:

$$\|f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(E^c)} + \left\|\widehat{f}\right\|_{L^2(F^c)}. \tag{3.2}$$

Clearly Eq. (3.2) implies a negative answer to our question. Interestingly, both statements are in fact equivalent, as we now show. Hence, to prove Eq. (3.2), it suffices to argue about the operator T as above.

Lemma 3.2. *Let $E, F \subset \mathbb{R}^n$ be measurable sets. The following are equivalent:*

(i) $\|f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(E^c)} + \left\|\widehat{f}\right\|_{L^2(F^c)}$ *for all* $f \in L^2(\mathbb{R}^n)$*;*

(ii) there is an $\varepsilon > 0$ *such that* $\|f\|^2_{L^2(E)} + \left\|\widehat{f}\right\|^2_{L^2(F)} \leqslant (2-\varepsilon)\,\|f\|^2_{L^2(\mathbb{R}^n)}$ *for all* $f \in L^2(\mathbb{R}^n)$*;*

(iii) if $\operatorname{supp}\widehat{f} \subset F$*, then* $\|f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(E^c)}$*;*

(iv) if $\operatorname{supp} f \subset E$*, then* $\left\|\widehat{f}\right\|_{L^2(\mathbb{R}^n)} \lesssim \|\widehat{g}\|_{L^2(F^c)}$*;*

(v) there is a $\rho \in (0,1)$ such that $\|Tf\|_{L^2(\mathbb{R}^n)} \leqslant \rho \|f\|_{L^2(\mathbb{R}^n)}$ for all $f \in L^2(\mathbb{R}^n)$.

Proof. Suppose (i) holds. Then, by Plancherel's Theorem,

$$\|f\|^2_{L^2(E)} + \left\|\widehat{f}\right\|_{L^2(F)} = 2\|f\|^2_{L^2(\mathbb{R}^n)} - \|f\|^2_{L^2(E^c)} - \left\|\widehat{f}\right\|^2_{L^2(F^c)}$$
$$\leqslant (2 - (2C^2)^{-1}) \|f\|^2_{L^2(\mathbb{R}^n)},$$

which is (ii) with $\varepsilon = 1/4C^2$.

Assume (ii) and $\operatorname{supp} \widehat{f} \subset F$. Then, $\left\|\widehat{f}\right\|_{L^2(F)} = \left\|\widehat{f}\right\|_{L^2(\mathbb{R}^n)}$, so (ii) implies

$$\|f\|^2_{L^2(E)} \leqslant (2-\varepsilon)\|f\|^2_{L^2(\mathbb{R}^n)} - \left\|\widehat{f}\right\|^2_{L^2(\mathbb{R}^n)} = (1-\varepsilon)\|f\|_{L^2(\mathbb{R}^n)},$$

i.e. $\|f\|^2_{L^2(\mathbb{R}^n)} \leqslant \varepsilon^{-1}\|f\|_{L^2(E^c)}$.

Define the projections

$$P_F f := (\chi_F \widehat{f})^\vee \quad \text{and} \quad Q_{E^c} f := \chi_{E^c} f.$$

Clearly $f = P_F f + P_{F^c} f$. Assuming (iii) and since $\operatorname{supp} P_F f \subset F$,

$$\begin{aligned}
\|f\|_{L^2(\mathbb{R}^n)} &\leqslant \|P_F f\|_{L^2(\mathbb{R}^n)} + \|P_{F^c} f\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|P_F f\|_{L^2(E^c)} + \|P_{F^c} f\|_{L^2(\mathbb{R}^n)} \\
&= \|f - P_{F^c} f\|_{L^2(E^c)} + \|P_{F^c} f\|_{L^2(\mathbb{R}^n)} \\
&\leqslant \|f\|_{L^2(E^c)} + \|P_{F^c} f\|_{L^2(E^c)} + \|P_{F^c} f\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|f\|_{L^2(E^c)} + \|P_{F^c} f\|_{L^2(\mathbb{R}^n)} \\
&= \|f\|_{L^2(E^c)} + \left\|\widehat{f}\right\|_{L^2(F^c)}.
\end{aligned}$$

This proves the equivalence between (i), (ii) and (iii). Exchanging f and $\widehat{f}$, we obtain the equivalence with (iv).

Assuming (iii) once again,

$$\|P_F f\|^2_{L^2(\mathbb{R}^n)} = \|P_F f\|^2_{L^2(E)} + \|P_F f\|^2_{L^2(E^c)}$$
$$\geqslant \|P_F f\|^2_{L^2(E)} + C^2 \|P_F f\|^2_{L^2(\mathbb{R}^n)},$$

so rearranging,

$$\|Tf\|^2_{L^2(\mathbb{R}^n)} = \|P_F f\|^2_{L^2(E)} \leqslant (1 - C^2)\left\|\widehat{f}\right\|^2_{L^2(F)} \leqslant \rho^2 \|f\|^2_{L^2(\mathbb{R}^n)},$$

by Plancherel and setting $\rho = (1 - C^2)^{1/2}$.

Finally, (v) clearly implies (iii). For if $\operatorname{supp}\widehat{f} \subset F$, then

$$\|f\|_{L^2(E^c)} = \|Tf\|_{L^2(\mathbb{R}^n)} \leqslant \rho \|f\|_{L^2(\mathbb{R}^n)},$$

whence $\|f\|_{L^2(\mathbb{R}^n)} \leqslant \rho^{-1} \|f\|_{L^2(E^c)}$. □

To prove Eq. (3.2) we need the following "spectral inequality":

Lemma 3.3. *For T and σ as above and any $\lambda \in \sigma_p(T)$,*

$$\dim\ker(T - \lambda I) \leqslant \lambda^{-2}\sigma^2.$$

Proof. Let $\{f_j\}_{j=1}^m \subset \ker(T - \lambda I)$ be an orthonormal sequence. By Bessel's inequality,

$$m\lambda^2 = \sum_{j=1}^m |(Tf_j, f)_{L^2}|^2 = \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y) f_j(x)\overline{f_j(y)}\,\mathrm{d}x\,\mathrm{d}y\right|^2 \leqslant \|K\|^2_{L^2(\mathbb{R}^{2n})} = \sigma^2,$$

as required. □

Theorem 3.4 (Amrein-Berthier)**.** *Let $E, F \subset \mathbb{R}^n$ have finite measure. Then,*

$$\|f\|_{L^2(\mathbb{R}^n)} \lesssim_{E,F,d} \|f\|_{L^2(E^c)} + \left\|\widehat{f}\right\|_{L^2(F^c)}$$

for all $f \in L^2(\mathbb{R}^n)$.

Proof. Note that $\|T\| \leqslant 1$ as T is a product of projections. If $\|T\| < 1$, then we are done by Lemma 3.2 and our opening argument in this section. Hence, suppose $\|T\| = 1$. As T is compact, $\pm 1 \in \sigma_p(T)$, so there is a non-zero $f \in L^2(\mathbb{R}^n)$ such than $\|Tf\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$. In particular, $\operatorname{supp} f \subset E$ and $\operatorname{supp}\widehat{f} \subset F$. (These are essential supports as f and $\widehat{f}$ are only defined a.e.)

We will use f to generate an infinite sequence in the eigenspace of a slightly different operator $\widetilde{T}$, contradicting Lemma 3.3.

Define, inductively,

$$S_0 := \operatorname{supp} f \quad \text{and} \quad S_{k+1} := S_k \cup (S_k - x_k)$$

where the points x_k ($k \geqslant 0$) are chosen such that

$$|S_k| < |S_{k+1}| < |S_k| + 2^{-k}.$$

We then set $f_k(x) := f(x + x_k)$. By taking Fourier transforms,

$$\widehat{f_k}(\xi) = e^{-2\pi i \xi\cdot x_k}\widehat{f}(\xi),$$

so the set $\{f_k\}_{k\in\mathbb{N}_0}$ is linearly independent in $L^2(\mathbb{R}^n)$. Futhermore,

$$\operatorname{supp}\widehat{f}_k \subset F \quad \text{and} \quad \operatorname{supp} f_k \subset S_\infty := \bigcup_{j\geqslant 0} S_j$$

for all $k \geqslant 0$. By construction, $|S_\infty| < \infty$, so $\widetilde{T}$ defined as T with S_∞ replacing E still satisfies the assumptions of Lemma 3.3, but clearly contradicts the conclusion. Hence, no such f exists, and we conclude that $\|T\| \neq 1$. □

Chapter 4

The Fourier Restriction Problem

This material is based on (Muscalu and Schlag, 2013, Chap. 11), (Mattila, 2015, Chap. 19) and (Wolff, 2003, Chap. 7).

4.1 Introduction and Equivalent Formulations

In this chapter we consider the question of restricting the Fourier transform $\widehat{f}$ on $\mathbb{R}^n$ to the sphere S^{n-1}. More precisely, given $f \in L^p(\mathbb{R}^n)$, when is $\widehat{f}|_{S^{n-1}} \in L^q(S^{n-1})$?

Proposition 4.1. *Let $f \in L^p(\mathbb{R}^n)$ for some $1 \leqslant p \leqslant 2$, and let $1 \leqslant q \leqslant \infty$. Then, $\widehat{f}|_{S^{n-1}} \in L^q(S^{n-1})$ if, and only if,*

$$\left\| \widehat{f}|_{S^{n-1}} \right\|_{L^q(S^{n-1})} \lesssim_{n,p,q} \|f\|_{L^p(\mathbb{R}^n)} . \tag{4.1}$$

Proof. Uniform Boundedness Principle. □

For $p = 1$ and $q = \infty$ we have the usual estimate for the Fourier transform

$$\left\| \widehat{f}|_{S^{n-1}} \right\|_{L^\infty(S^{n-1})} \leqslant \|f\|_{L^1(\mathbb{R}^n)} .$$

However, for $p = 2$, $\widehat{f} \in L^2(\mathbb{R}^n)$, thus defined only a.e., so we cannot restrict $\widehat{f}$ to the zero-measure set S^{n-1}.

Estimate (4.1) is sensitive to the curvature of the surface:

Proposition 4.2. *Suppose S is a bounded subset of a hyperplane in $\mathbb{R}^n$. If*

$$\left\| \widehat{f}|_{S} \right\|_{L^1(S)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \textit{for all} \quad f \in \mathscr{S}(\mathbb{R}^n),$$

then $p = 1$.

Proof. Without loss of generality we may assume

$$S := \{(x', 0) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| \leqslant 1\}.$$

Now, choose a smooth cutoff function $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leqslant \eta \leqslant 1$ and $\eta \equiv 1$ on S. Let $f = \check{\eta} \in \mathscr{S}(\mathbb{R}^n)$ and rescale:

$$f_\lambda(x) := f(\lambda x) \quad \text{and} \quad \widehat{f_\lambda}(\xi) = \lambda^{-n}\eta(\lambda^{-1}\xi)$$

On the one hand, if $\lambda > 1$, then

$$\left\| \widehat{f_\lambda}|_S \right\|_{L^1(S)} = \lambda^{-n} \int_{|\xi'|\leqslant 1} \eta(\lambda^{-1}\xi', 0)\,\mathrm{d}\xi' = \lambda^{-n}.$$

On the other hand,

$$\|f_\lambda\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |f(\lambda x)|^p \,\mathrm{d}x = \lambda^{-n} \|f\|_{L^p(\mathbb{R}^n)}^p .$$

So, from the assumed inequality we have

$$\lambda^{-n} = \left\| \widehat{f_\lambda}|_S \right\|_{L^1(S)} \lesssim \lambda^{-n/p} \|f\|_{L^p(\mathbb{R}^n)} \quad \text{or} \quad \lambda^{n/p-n} \lesssim \|f\|_{L^p(\mathbb{R}^n)} .$$

The latter inequality holds for all $\lambda > 1$, implying $p \geqslant 1$. A similar argument with reversed scaling shows that $p \leqslant 1$. □

We will work, for simplicity, on the surface S^{n-1} and prove the following, sharp estimate for $p = 2$:

Theorem 4.3 (Thomas-Stein)**.** *For every $n \geqslant 2$ we have the estimate*

$$\left\| \widehat{f}|_{S^{n-1}} \right\|_{L^2(S^{n-1})} \lesssim_n \|f\|_{L^p(\mathbb{R}^n)} \tag{4.2}$$

for all $f \in L^p(\mathbb{R}^n)$, provided that

$$p \leqslant p_n := \frac{2n+2}{n+3}.$$

Furthermore, Eq. (4.2) fails if $p > p_n$.

We will derive this theorem by proving a dual formulation. But first, a technical identity:

Lemma 4.4. *For any measure μ on $\mathbb{R}^n$ and any $f, g \in \mathscr{S}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,\mathrm{d}\mu(\xi) = \int_{\mathbb{R}^n} f(x)(g * \widehat{\mu})(x)\,\mathrm{d}x.$$

Proof. Recall the Convolution Theorem for $\varphi \in \mathscr{S}(\mathbb{R}^n)$:

$$\widehat{\varphi\mu} = \widehat{\varphi} * \widehat{\mu}. \tag{4.3}$$

We now compute

$$\begin{aligned}\int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,\mathrm{d}\mu(\xi) &= \int_{\mathbb{R}^n} f(x)\left[\overline{\widehat{g}}\mu\right]^{\wedge}(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(x)\left(\widehat{\overline{\widehat{g}}} * \widehat{\mu}\right)(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(x)(\overline{g} * \widehat{\mu})(x)\,\mathrm{d}x,\end{aligned}$$

using $\widehat{\overline{\widehat{g}}} = \widehat{\widehat{\overline{g}}} = \overline{g}$. □

We can now prove our duality result:

Lemma 4.5. *Let μ be a finite measure on $\mathbb{R}^n$ and $n \geqslant 2$. The following are equation:*

(i) $\left\|\widehat{f\mu}\right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mu)}$ *for all* $f \in \mathscr{S}(\mathbb{R}^n)$.

(ii) $\|\widehat{g}\|_{L^2(\mu)} \lesssim \|g\|_{L^{q'}(\mathbb{R}^n)}$ *for all* $g \in \mathscr{S}(\mathbb{R}^n)$.

(iii) $\|\widehat{\mu} * f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^n)}$ *for all* $f \in \mathscr{S}(\mathbb{R}^n)$.

Proof. From the duality expression for the norm and Lemma 4.4,

$$\begin{aligned}\|\widehat{g}\|_{L^2(\mu)} &= \sup_{f\in\mathscr{S}, \|f\|_{L^2(\mu)}=1} \left|\int_{\mathbb{R}^n} \widehat{g}(\xi) f(\xi)\,\mathrm{d}\mu(\xi)\right| \\ &= \sup_{f\in\mathscr{S}, \|f\|_{L^2(\mu)}=1} \left|\int_{\mathbb{R}^n} \widehat{g}(x)\widehat{f\mu}(x)\,\mathrm{d}x\right|.\end{aligned}$$

Thus Hölder's inequality implies the equivalence between (i) and (ii).

To show (iii) from the joint assumption of (i) and (ii), let $g \in \mathscr{S}(\mathbb{R}^n)$ and set $f = \widehat{g}$ in (i) to find

$$\|(\widehat{g}\mu)^{\wedge}\|_{L^q(\mathbb{R}^n)} \lesssim \|\widehat{g}\|_{L^2(\mu)} \lesssim \|g\|_{L^{q'}(\mathbb{R}^n)}.$$

But $(\widehat{g}\mu)^{\wedge} = \widetilde{g} * \widehat{\mu}$, by Eq. (4.3), thereby proving (iii).

Conversely, for any $f, g \in \mathscr{S}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} g(x)(\widehat{\mu} * f)(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} g(x)(\mu\check{f})^{\wedge}(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \widehat{g}(\xi)\check{f}(\xi)\,\mathrm{d}\mu(\xi)$$

by Eq. (4.3) and Parseval's identity. Thus, (iii) and Hölder's inequality imply

$$\left|\int_{\mathbb{R}^n} \widehat{g}(\xi)\check{f}(\xi)\,\mathrm{d}\mu(\xi)\right| \lesssim \|g\|_{L^{q'}(\mathbb{R}^n)} \|f\|_{L^{q'}(\mathbb{R}^n)}.$$

Taking $f = \widetilde{g}$ so that $\check{f} = \overline{g}$, yields (ii). □

In particular, when $\mu = \sigma$, the surface measure on S^{n-1}, we arrive at

Corollary 4.6. *The following are equivalent:*

(i) The Thomas-Stein "restriction" theorem:

$$\left\| \widehat{f}|_S \right\|_{L^2(\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for $p = p_n = (2n+2)/(n+3)$ and all $f \in \mathscr{S}(\mathbb{R}^n)$.

(ii) The Thomas-Stein "extension" theorem:

$$\left\| \widehat{f\sigma} \right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\sigma)}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$, provided $q = (2n+2)/(n-1)$.

(iii) For all $f \in \mathscr{S}(\mathbb{R}^n)$,

$$\|f * \widehat{\sigma}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^n)},$$

provided $q = (2n+2)/(n-1)$.

In the next section we provide a "natural" interpolation argument that proves the Thomas-Stein bound for all $p < p_n$. (Section 4.2.) This argument, however, will fail at the endpoint. Afterwards, in Section 4.3, we will offer one and a half proofs of the endpoint case. Finally, Section 4.4 is devoted to proving the optimality of the exponent $p = p_n$ using a beautiful geometrical construction, before looking at an application to PDE in Section 4.5.

4.2 The "Simple" (Non-Endpoint) Proof

By Corollary 4.6, it suffices to prove

$$\|\widehat{\sigma} * f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^n)} \tag{4.4}$$

for $q > (2n+2)(n-1)$. To this end we will use a Littlewood-Paley partition of unity to break up $\widehat{\sigma}$ over dyadic blocks, obtain interpolation estimates on each block separately and then sum over the blocks to recover the bound.

Let $\varphi \in C^\infty(\mathbb{R}^n)$ be a Littlewood-Paley partition of unity supported in $\frac{1}{4} \leqslant |x| \leqslant 1$ and with

$$\sum_{j=0}^{\infty} \varphi(2^{-j}x) = 1 \quad \text{whenever} \quad |x| \geqslant 1.$$

Setting

$$K_j(x) = \varphi(2^{-j}x)\widehat{\sigma} \quad (j \geqslant 0) \quad \text{and} \quad K_{-\infty} := \widehat{\sigma} - \sum_{j=0}^{\infty} K_j$$

we clearly have

$$\widehat{\sigma} = K_{-\infty} + \sum_{j=0}^{\infty} K_j.$$

And, as $K_{-\infty} \in C_c^\infty(\mathbb{R}^n)$, we obtain the bound

$$\|K_{-\infty} * f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^n)}$$

by Young's inequality for convolutions.

For the remaining terms we will establish bounds

$$\|K_j * f\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-j(n-1)/2} \|f\|_{L^1(\mathbb{R}^n)} \tag{4.5}$$

$$\|K_j * f\|_{L^2(\mathbb{R}^n)} \lesssim 2^{j} \|f\|_{L^2(\mathbb{R}^n)} . \tag{4.6}$$

By interpolating between Eqs. (4.5) and (4.6), we obtain a summable series of $L^{q'} \to L^q$ bounds for the K_j convolution operators.

Eq. (4.5) Recall the decay of $\widehat{\sigma}$ given by Eq. (2.4)

$$|\widehat{\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{n-1}{2}}.$$

Thus, using the fact that $\varphi(2^{-j}x)$ is supported on $2^{j-2} \leqslant |x| \leqslant 2^j$,

$$|K_j(\xi)| \lesssim_\varphi \sup_{2^{j-2}\leqslant|x|\leqslant 2^j} (1+|\xi|)^{-\frac{n-1}{2}} \lesssim 2^{-j(n-1)/2}$$

and we conclude Eq. (4.5) from the simple estimate

$$\|K_j * f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|K_j\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)} .$$

Eq. (4.6) For the L^2 bound, we recall another bound on σ:

$$\sigma(B(x,r)) \lesssim r^{n-1}.$$

(See (Mattila, 2015, Chap. 19) for a discussion of a generalisation where they consider arbitrary measures with this property.) We will use Plancherel's theorem as it is more convenient to work now with $\widehat{K_j}$. Since σ is reflection invariant and $\varphi \in \mathscr{S}$, we have, by Eq. (4.3),

$$\widehat{K_j}(\xi) = (\varphi(2^{-j}x)\widehat{\sigma})^\wedge(\xi) = 2^{jn}\widehat{\varphi}(2^j\xi) * \sigma \equiv \psi_j * \sigma,$$

upon setting $\psi_j(\xi) := 2^{nj}\widehat{\varphi}(2^j\xi) \in \mathscr{S}(\mathbb{R}^n)$, and therefore, for any $N \in \mathbb{N}$,

$$\left|\widehat{K_j}(\xi)\right| \lesssim_N 2^{jn} \int_{\mathbb{R}^n} (1 + 2^j |\xi - \eta|)^{-N} \, \mathrm{d}\sigma(\eta).$$

We now estimate the integral on the right by breaking up the space into dyadic annuli. (See Fig. 4.1.)

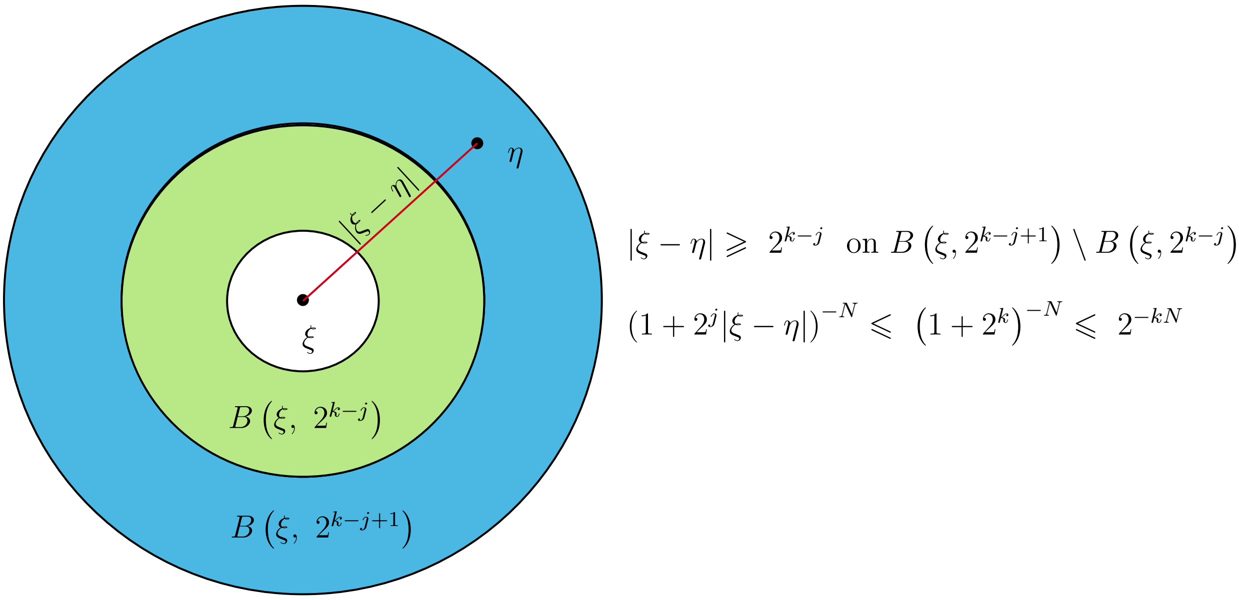

Figure 4.1: Estimates for anuli $B(\xi, 2^{k+1-j}) \setminus B(\xi, 2^{k-j})$ for the bounds on $\widehat{K_j}$.

$$\begin{aligned}
\left|\widehat{K_j}(\xi)\right| &\lesssim_N 2^{jn} \left(\int_{B(\xi, 2^{-j})} \underbrace{(1 + 2^j |\xi - \eta|)^{-N}}_{\leqslant 1} \, \mathrm{d}\sigma(\eta) + \sum_{k=0}^{\infty} \int_{B(\xi, 2^{k+1-j}) \setminus B(\xi, 2^{k-j})} \underbrace{(1 + 2^j |\xi - \eta|)^{-N}}_{\leqslant 2^{-kN}} \, \mathrm{d}\sigma(\eta) \right) \\
&\lesssim_N 2^{jn} \left(\sigma(B(\xi, 2^{-j})) + \sum_{k=0}^{\infty} 2^{-kN} \sigma(B(\xi, 2^{k+1-j}) \setminus B(\xi, 2^{k-j})) \right) \\
&\lesssim_N 2^{jn} \left(2^{-j(n-1)} + \sum_{k=0}^{\infty} 2^{-kN + (n-1)(k-j)} \right).
\end{aligned}$$

By choosing $N = n$, we can simplify the expression in parenthesis to a geometric series:

$$2^{-j(n-1)} + 2^{-j(n-1)} \sum_{k=0}^{\infty} 2^{-k} = 3 \times 2^{-j(n-1)},$$

so we have

$$\left\|\widehat{K_j}\right\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^j.$$

We make use of this bound by the Plancherel and Convolution Theorems:

$$\|K_j * f\|_{L^2(\mathbb{R}^n)} = \left\|\widehat{K_j}\widehat{f}\right\|_{L^2(\mathbb{R}^n)} \leqslant \left\|\widehat{K_j}\right\|_{L^\infty(\mathbb{R}^n)} \left\|\widehat{f}\right\|_{L^2(\mathbb{R}^n)} \lesssim 2^j \|f\|_{L^2(\mathbb{R}^n)},$$

which is Eq. (4.6).

We now interpolate between Eqs. (4.5) and (4.6) using the Riesz-Thorin Theorem, choosing θ precisely such that

$$\frac{\theta}{2} + \frac{1-\theta}{\infty} = \frac{1}{q},$$

whence

$$\|K_j * f\|_{L^q(\mathbb{R}^n)} \lesssim 2^{j\theta} 2^{-j\frac{n-1}{2}(1-\theta)} \|f\|_{L^{q'}(\mathbb{R}^n)}.$$

Since $\theta = 2/q$, we have

$$\|K_j * f\|_{L^q(\mathbb{R}^n)} \lesssim 2^{j\left(\frac{n+1}{q} - \frac{n-1}{2}\right)} \|f\|_{L^{q'}(\mathbb{R}^n)}, \tag{4.7}$$

which is summable in j if, and only if,

$$\frac{n+1}{q} - \frac{n-1}{2} < 0 \quad i.e. \quad q > \frac{2n+2}{n-1},$$

precisely as required of q.

4.3 The Endpoint

Unfortunately, the above proof breaks down at the critical value

$$q = \frac{2n+2}{n-1} \quad \text{or} \quad p = \frac{2n+2}{n+3}$$

since then the wieghts in Eq. (4.7) are no longer summable in j.

To work around this issue, we discuss two proofs for the endpoint value.

4.3.1 Proof I: Complex Interpolation (Sketch)

In this section we sketch a natural extension of the proof in Section 4.2. Rather than interpolate for K_j and *then* sum over blocks, we try to sum before interpolating. This is achieved by using the Stein Interpolation Theorem from Section 1.2 instead of Riesz-Thorin.

In order to do so, we define a suitable family of operators, T_z such that

$$T_z : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \quad \text{when} \quad \Re z = 0;$$
$$T_z : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \quad \text{when} \quad \Re z = 1,$$

Provided all the technical details are taken care of, we can apply Theorem 1.3 with θ such that

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2},$$

to obtain

$$\|T_\theta f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

So the main challenge is to carefully choose T_z in such a way that $T_\theta f = \widehat{\sigma} * f$ when $q = \frac{2n+2}{n-1}$ and $p = q'$.

Locally, we may write S^{n-1} as the graph of some $h \in C^\infty(\mathbb{R}^{n-1})$:

$$S^{n-1} = \{\xi \in \mathbb{R}^n : \xi_n = h(\xi')\}.$$

Choosing two suitable cutoff functions $\chi_1 \in C_c^\infty(\mathbb{R}^{n-1})$ and $\chi_2 \in C_c^\infty(\mathbb{R})$, we define

$$m_z(\xi) := \frac{1}{\Gamma(z)} \left(\xi_d - h(\xi')\right)_+^{z-1} \chi_1(\xi')\chi_2(\xi_d - h(\xi')) \quad \text{for all} \quad \Re z > 0.$$

The operators T_z are defined by

$$T_z f := (m_z \widehat{f})^\vee$$

by means of an analytic continuation to $\Re z \leqslant 0$.

The necessary estimates for Theorem 1.3 are of the form

$$\begin{aligned} \|T_z f\|_{L^2(\mathbb{R}^n)} &\leqslant M_0(y) \|f\|_{L^2(\mathbb{R}^n)} \quad \text{when} \quad \Re z = 1; \\ \|T_z f\|_{L^\infty(\mathbb{R}^n)} &\leqslant M_1(y) \|f\|_{L^1(\mathbb{R}^n)} \quad \text{when} \quad \Re z = -\frac{n-1}{2}, \end{aligned} \tag{4.8}$$

provided M_0 and M_1 satisfy the growth condition (1.5). (Note that we are "shifting things around" a bit in z and θ, but the calculations reduce to the theorem on the strip $\overline{S}$.) Assuming this is so, we want the critical line to land on $\Re z = 0$ as then

$$m_0(\xi) = \chi_1(\xi')\delta_0(\xi_n - h(\xi'))\, \mathrm{d}\xi',$$

which, for suitable cutoff functions, is the surface measure of S^{n-1}, so that

$$T_0 f = (\sigma \widehat{f})^\vee \equiv \widehat{\sigma} * f.$$

This operator is bounded $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ provided

$$\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2} \quad \text{and} \quad \Re z = 0 = -\frac{n-1}{2}(1-\theta) + \theta,$$

which implies

$$q = \frac{2}{\theta} = \frac{2n+2}{n-1},$$

as desired.

The calculations required to justify the assumptions of the Stein Interpolation Theorem are fairly involved. We will only consider the growth condition for M_0 in Eq. (4.8). We refer the interested reader to (Muscalu and Schlag, 2013, Chap. 11, Sect. 11.2.1) for the details.

By Plancherel's Theorem,

$$\|T_z f\|_{L^2(\mathbb{R}^n)} = \left\| m_z \widehat{f} \right\|_{L^2(\mathbb{R}^n)} \leqslant \|m_z\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)},$$

so we consider $M_0(y) = \|m_{1+iy}\|_{L^\infty(\mathbb{R}^n)}$ in Eq. (4.8).

Recall that $\frac{1}{\Gamma(z)}$ has simple poles at $z \in \mathbb{Z}_{\leqslant 0}$ and admits the product representation

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{\nu=1}^{\infty} \left(1 + \frac{z}{\nu}\right) e^{-z/\nu} \quad \text{for all} \quad z \in \mathbb{C}.$$

(See (Grafakos, 2014, Appendix A) fore details on the Γ function.) Hence,

$$\begin{aligned} \left| \frac{1}{\Gamma(z)} \right|^2 &\leqslant |z|^2 e^{2\gamma x} \prod_{\nu=1}^{\infty} \left(\left(1 + \frac{x}{\nu}\right)^2 + \frac{y^2}{\nu^2} \right) e^{-2x/\nu} \\ &= |z|^2 e^{2\gamma x} \prod_{\nu=1}^{\infty} \left(1 + \frac{2x}{\nu} + \frac{|z^2|}{\nu^2}\right) e^{-2x/\nu} \\ &\leqslant |z|^2 e^{2\gamma x} \prod_{\nu=1}^{\infty} \left(e^{2x/\nu + |z|^2/\nu^2} e^{-2x/\nu} \right) \\ &= |z|^2 e^{2\gamma x} e^{|z|^2 \sum_\nu \frac{1}{\nu^2}} \\ &= |z|^2 e^{2\gamma x} e^{|z|^2 \pi^2/6}, \end{aligned}$$

where we used $1 + t \leqslant e^t$ for $t \geqslant 0$ in the central inequality. In particular, for $x = \Re z = 1$ as in Eq. (4.8), we have

$$|m_z(\xi)| \leqslant (1+y^2) e^{2\gamma} e^{(1+y^2)\pi^2/6} \chi_1(\xi') \chi_2(\xi_n - h(\xi')) \lesssim (1+y^2) e^{cy^2}.$$

Taking $\tau_1 = c$ in Eq. (1.5) we have

$$e^{-c|y|} \log(|m_z(\xi)|) \lesssim (\log(1+y^2) + y^2) e^{-c|y|}$$

which remains bounded in $y \in \mathbb{R}$.

4.3.2 Proof II: Fractional Integration

In this section we give a full proof of the endpoint case by using the fractional integration method from Section 4.6. This real-variable technique provides a fast and clean way of obtaining the relevant bounds while avoiding the complex interpolation theorems.

Define $\mu := \zeta\sigma$ as in Section 2.4. We want to show that

$$\|f * \widehat{\mu}\|_{L^{p'}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^p(\mathbb{R}^n)}, \quad \text{for all} \quad f \in \mathscr{S}(\mathbb{R}^n)$$

where $p = p_n = (2n+2)/(n+3)$.

Writing $x = (x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have

$$f * \widehat{\mu}(x) = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \widehat{\mu}(x'-y', t-s) f(y, s) \, \mathrm{d}y' \, \mathrm{d}s,$$

By Theorem 2.7,

$$|\widehat{\mu}(x'-y', t-s)| \lesssim |t-s|^{-\frac{n-1}{2}}.$$

Defining

$$(U(t)g)(x') := \int_{\mathbb{R}^{n-1}} \widehat{\mu}(x'-y', t) g(y') \, \mathrm{d}y' \quad \text{for all} \quad g \in \mathscr{S}(\mathbb{R}^{n-1}),$$

we have

$$f * \widehat{\mu}(x', t) = \int_{\mathbb{R}} (U(t-s) f(\cdot, s))(x') \, \mathrm{d}s.$$

Observe the estimates

$$\|U(t)g\|_{L^\infty(\mathbb{R}^{n-1})} \lesssim |t|^{-\frac{n-1}{2}} \|g\|_{L^1(\mathbb{R}^{n-1})},$$

and

$$\|U(t)g\|_{L^2(\mathbb{R}^{n-1})} = \|\widehat{\mu}(\cdot, t) * g\|_{L^2(\mathbb{R}^{n-1})} = \|\widetilde{\mu}(\widehat{\cdot}, t)\widehat{g}\|_{L^2(\mathbb{R}^{n-1})} \lesssim \|g\|_{L^2(\mathbb{R}^{n-1})},$$

using the fact that $\widetilde{\mu} = \widetilde{\zeta}\sigma \in L^\infty$. (In the above, $(\widehat{\cdot}, t)$ means that we took the Fourier transform in the "first", *i.e.* $\mathbb{R}^{n-1}$, variable, x', and left t as a parameter.)

Interpolating between the L^2 and L^∞ bounds above with

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2} \quad \text{or} \quad \theta = \frac{2}{p'}$$

we arrive at

$$\|U(t)g\|_{L^{p'}(\mathbb{R}^{n-1})} \lesssim |t|^{-\alpha(p)} \|g\|_{L^p(\mathbb{R}^{n-1})},$$

where

$$\alpha(p) = (1-\theta)\left(\frac{n-1}{2}\right) = \frac{n-1}{2}\left(1-\frac{2}{p'}\right) = \frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{p'}\right). \tag{4.9}$$

With this bound, we proceed to estimate the convolution

$$\begin{aligned}\|f * \widehat{\mu}\|_{L^{p'}(\mathbb{R}^n)} &\leqslant \left\| \int_{\mathbb{R}} \|U(t-s)f(\cdot,s)\|_{L^{p'}(\mathbb{R}^{n-1})} \,\mathrm{d}s \right\|_{L^{p'}(t\in\mathbb{R})} \\ &\lesssim \left\| \int_{\mathbb{R}} |t-s|^{-\alpha(p)} \|f(\cdot,s)\|_{L^{p}(\mathbb{R}^{n-1})} \,\mathrm{d}s \right\|_{L^{p'}(t\in\mathbb{R})}.\end{aligned}$$

For the final push, we apply fractional integration (see Section 4.6) in the inner integral. Thus, Proposition 4.7 applies provided

$$\frac{1}{p'} = \frac{1}{p} + \alpha(p) \quad \text{and} \quad 0 < \alpha(p) < 1,$$

which is precisely Eq. (4.9) when $p = (2n+2)/(n+3)$, and we conclude

$$\|f * \widehat{\mu}\|_{L^{p'}(\mathbb{R}^n)} \lesssim \left\| \|f(\cdot,t)\|_{L^{p}(\mathbb{R}^{n-1})} \right\|_{L^{p}(t\in\mathbb{R})} = \|f\|_{L^{p}(\mathbb{R}^n)}.$$

4.4 Knapp's Example

We know discuss a geometric construction by Knapp which will allow us to show that the exponent p_n is optimal.

For $n \geqslant 2$, let $e_n = (0,\dots,0,1) \in \mathbb{R}^n$ and $\delta \in (0,1)$. Define the cap

$$C_\delta := \{x \in S^{n-1} : 1 - x\cdot e_n \leqslant \delta^2\}$$

(see Fig. 4.2) and the rectangle

$$R_\delta := \left\{\xi \in \mathbb{R}^n : |\xi_j| \leqslant \frac{c}{\delta} \text{ for } 1 \leqslant j \leqslant n-1, |\xi_n| \leqslant \frac{c}{\delta^2}\right\},$$

where $c = 1/12n$.

We now consider the function $f = \chi_{C_\delta}$ and show that is satisfies

$$\left|\widehat{f}(\xi)\right| \geqslant \frac{1}{2}\sigma(C_\delta) \quad \text{for all} \quad \xi \in R_\delta. \tag{4.10}$$

Note that, if $x \in C_\delta$, then

$$x_1^2 + \cdots + x_{n-1}^2 = 1 - x_n^2 = (1+x_n)(1-x_n) \leqslant 2\delta^2.$$

Hence,

$$|x_j| \leqslant \sqrt{2}\delta \quad \text{for all} \quad 1 \leqslant j \leqslant n-1, x \in C_\delta.$$

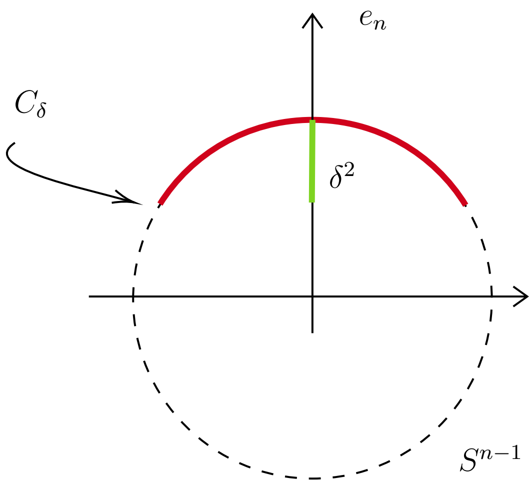

Figure 4.2: Knapp's construction of the cap C_δ.

On to the Fourier transform: for any $\xi \in \mathbb{R}^n$,

$$\begin{aligned}\left|\widehat{f}(\xi)\right| &= \left|\int_{C_\delta} e^{-2\pi i\xi\cdot x}\,\mathrm{d}\sigma(x)\right| \\ &= \left|e^{2\pi i\xi_n}\right| \left|\int_{C_\delta} e^{-2\pi i(\xi'\cdot x'+(x_n-1)\xi_n)}\,\mathrm{d}\sigma(x)\right| \\ &\geqslant \int_{C_\delta} \cos\left(2\pi\xi\cdot(x-e_n)\right)\,\mathrm{d}\sigma(x),\end{aligned}$$

using $|z| \geqslant \Re(z)$ for $z \in \mathbb{C}$. Now, if $x \in C_\delta$ and $\xi \in R_\delta$, we have

$$\begin{aligned}2\pi\left|\xi\cdot(x-e_n)\right| &\leqslant 2\pi\sum_{j=1}^{n-1}|\xi_j|\,|x_j| + 2\pi(1-x_n)\,|\xi_n| \\ &\leqslant 2\pi\left(\sum_{j=1}^{n-1}\frac{\sqrt{2}\delta}{12n\delta} + \frac{\delta^2}{12n\delta^2}\right) \\ &< \frac{2\pi\times 2}{12} = \frac{\pi}{3}\end{aligned}$$

proving that

$$\cos(2\pi\xi\cdot(x-e_n)) > \frac{1}{2} \quad \text{for all} \quad x\in C_\delta, \xi\in R_\delta,$$

whence

$$\widehat{f(\xi)} \geqslant \frac{1}{2}\int_{C_\delta}\mathrm{d}\sigma,$$

as claimed.

We now use Eq. (4.10) to show that Eq. (4.2) fails when $p > p_n$. With f as above, note that

$$\|f\|_{L^2(S^{n-1})} = \sigma(C_\delta)^{1/2} \sim \delta^{\frac{n-1}{2}}. \tag{4.11}$$

On the other hand,

$$|R_\delta| = 2^n c^n \delta^{-n-1} \quad \text{where} \quad c = \frac{1}{12n},$$

so Eq. (4.10) implies

$$\left\|\widehat{f}\right\|_{L^q(\mathbb{R}^n)} \geqslant \frac{1}{2}\sigma(C_\delta)\,|R_\delta|^{1/q} \sim \delta^{n-1-\frac{n+1}{q}}. \tag{4.12}$$

If Corollary 4.6(ii) holds for f, then

$$\left\|\widehat{f}\right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^2(S^{n-1})},$$

which implies

$$\delta^{n-1-\frac{n+1}{q}} \lesssim \delta^{\frac{n-1}{2}} \quad \text{for all} \quad \delta \in (0,1).$$

This is only possible if

$$n-1-\frac{n+1}{q} \geqslant \frac{n-1}{2},$$

i.e. if $q \geqslant \frac{2n+2}{n-1}$, the (dual) critical exponent.

4.5 Application: Strichartz Estimate

To conclude this chapter, we present an application of the end-point case of Corollary 4.6 to Strichartz estimates for the Schrödinger equation in $\mathbb{R}^{n+1}$:

$$\begin{cases} \dfrac{1}{i}\partial_t u + \dfrac{1}{2\pi}\Delta_x u = 0; \\ \qquad\qquad u|_{t=0} = f, \end{cases} \tag{4.13}$$

for $f \in \mathscr{S}(\mathbb{R}^n)$. (Here $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.) Taking Fourier transforms in Eq. (4.13) yields

$$u(t,x) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi + t|\xi|^2)}\widehat{f}(\xi)\,\mathrm{d}\xi \equiv (\widehat{f}\mu)^\vee(x,t),$$

where the measure μ is defined by

$$\int_{\mathbb{R}^{n+1}} g(\xi,\tau)\,\mathrm{d}\mu(\xi,\tau) = \int_{\mathbb{R}^n} g(\xi,|\xi|^2)\,\mathrm{d}\xi \quad \text{for all} \quad g \in C^0(\mathbb{R}^{n+1}).$$

We will apply Corollary 4.6(iii) to this measure.

Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ such that

$$\varphi(\xi,\tau) \equiv 1 \quad \text{on} \quad |\xi| + |\tau| \leqslant 1.$$

Then, for $q = \frac{2(n+1)+2}{(n+1)-1} = 2 + \frac{4}{n}$,

$$\left\| (\widehat{f}\varphi\mu)^{\vee} \right\|_{L^q(\mathbb{R}^{n+1})} \lesssim \left\| \widehat{f} \right\|_{L^2(\varphi\mu)}.$$

In particular, if $\operatorname{supp} \widehat{f} \subset B(0,1)$, we have

$$\|u\|_{L^{2+\frac{4}{n}}(\mathbb{R}^{n+1})} = \left\| (\widehat{f}\mu)^{\vee} \right\|_{L^{2+\frac{4}{n}}(\mathbb{R}^{n+1})} \lesssim \left\| \widehat{f} \right\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}. \tag{4.14}$$

Te remove the restriction on the support of $\widehat{f}$, we rescale f and u to ensure that Eq. (4.14) applies:

$$f_\lambda(x) := f(x/\lambda) \quad \text{and} \quad u_\lambda(x,t) := u(x/\lambda, t/\lambda^2).$$

Clearly u_λ and f_λ satisfy Eq. (4.13). Assuming $\widehat{f}$ has compact support in $\mathbb{R}^n$, then for λ large enough $\operatorname{supp}(\widehat{f_\lambda}) \subset B(0,1)$. Hence, Eq. (4.14) implies

$$\|u_\lambda\|_{L^q(\mathbb{R}^{n+1})} \lesssim \|f_\lambda\|_{L^2(\mathbb{R}^n)}.$$

But

$$\begin{aligned} \|f_\lambda\|_{L^2(\mathbb{R}^n)} &= \lambda^{n/2} \|f\|_{L^2(\mathbb{R}^n)} \quad \text{and} \\ \|u_\lambda\|_{L^q(\mathbb{R}^{n+1})} &= \lambda^{\frac{n+2}{q}} \|u\|_{L^q(\mathbb{R}^{n+1})} = \lambda^{n/2} \|u\|_{L^q(\mathbb{R}^{n+1})}, \end{aligned}$$

so we can conclude the Strichartz bound

$$\|u\|_{L^{2+\frac{4}{n}}(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$ whose Fourier tranform has compact support. But this set is dense in $L^2(\mathbb{R}^n)$, as it is $\mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^n))$, so the Strichartz bound holds for all $f \in L^2(\mathbb{R}^n)$.

4.6 Appendix: Fractional Integration

To motivate this topic, we recall that differentiation corresponds to multiplication by ξ in Fourier space. More precisely, in $f \in \mathscr{S}(\mathbb{R}^n)$, then

$$\widehat{\partial^\alpha f}(\xi) = (2\pi i)^{|\alpha|} \xi^\alpha \widehat{f}(\xi),$$

where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. In particular,

$$-\widehat{\Delta f}(\xi) = 4\pi^2 |\xi|^2 \widehat{f}(\xi). \tag{4.15}$$

(Indcidentaly, this makes it obvious that $-\Delta$ is a positive operator!) While differentiation *per se* is restricted to integer orders, $\alpha \in \mathbb{N}_0^n$, the right hand side of Eq. (4.15) makes sense even if we replace the exponent 2 by more exotic choices:

$$-\widehat{\Delta^s f}(\xi) := (2\pi\,|\xi|)^{2s}\widehat{f}(\xi).$$

We can thus *define* a "fractional laplacian" by the multiplier $(2\pi\,|\xi|)^{2s}$. Note how this agrees perfectly with $-\Delta^k = -\Delta \circ \overbrace{\cdots}^{k} \circ \Delta$ when $k \in \mathbb{N}_0$.

So if differentiation corresponds to *positive* powers of $|\xi|$, by analogy we can expect integration to correspond with *negative* powers $|\xi|$. This is exactly the definition of the *Riesz Potential* or *fractional integral*:

$$\mathscr{I}^s f := \left((2\pi i\,|\xi|)^{-s}\widehat{f}(\xi)\right)^{\vee}.$$

Note that $\mathscr{I}^s = -\Delta^{-s/2}$, so that $-\Delta = \mathscr{I}^{-2}$, *i.e.* the Laplacian, a second order derivative, is two "negative order integrals". If we apply the convolution theorem, we can write this explicitly in x-space as

$$\mathscr{I}^s f(x) = C_n \int_{\mathbb{R}^n} |x-y|^{s-n} f(y)\,\mathrm{d}y.$$

Since we are only interested in the boundedness properties of this operator, we will drop the dimensional constant and simply write:

$$I^s f(x) = \int_{\mathbb{R}^n} |x-y|^{s-n} f(y)\,\mathrm{d}y.$$

A naïve application of Young's inequality would suggest

$$\|I^s f\|_{L^q(\mathbb{R}^n)} \lesssim_{p,q,n} \|f\|_{L^p(\mathbb{R}^n)} \quad \text{when} \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{r}$$

provided $|x|^{s-n} \in L^r(\mathbb{R}^n)$. This, however, is *not* true for any r, so a different approach is required.

Proposition 4.7. *If* $1 < p < q < \infty$ *and* $s \in (0,n)$ *such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}, \tag{4.16}$$

then

$$\|I^s f\|_{L^q(\mathbb{R}^n)} \lesssim_{p,q,n} \|f\|_{L^p(\mathbb{R}^n)} \quad \textit{for all} \quad f \in \mathscr{S}(\mathbb{R}^n).$$

Proof. By the Marcinkiewicz interpolation theorem (Folland, 2013, Thoerem 6.28), it suffices to check the weak-type estimate:

$$|\{x \in \mathbb{R}^n : |I^s f(x)| > \lambda\}| \lesssim \lambda^{-q} \|f\|_{L^p(\mathbb{R}^n)}^q . \tag{4.17}$$

Fix $f \in \mathscr{S}(\mathbb{R}^n)$ with $\|f\|_{L^p(\mathbb{R}^n)} = 1$ and $\lambda > 0$. We break up the kernel $k^s(x) := |x|^{s-n}$ into

$$k_1^s(x) := \chi_{\{|x|<\varepsilon\}} |x|^{s-n} \quad \text{and} \quad k_2^s(x) := \chi_{\{|x|\geqslant\varepsilon\}} |x|^{s-n} .$$

By Hölder's inequality,

$$\|k_2^s * f\|_{L^\infty(\mathbb{R}^n)} \leqslant \|k_2^s\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} = C\varepsilon^{s-n/p}$$

using the normalisation of f and

$$\int_{|x|\geqslant\varepsilon} |x|^{p'(s-n)} \,\mathrm{d}x = C \int_\varepsilon^\infty r^{p's-p'n} r^{n-1} \,\mathrm{d}r = C\varepsilon^{p's-n(p'-1)},$$

while noting that

$$\frac{p's - n(p'-1)}{p'} = s - n\frac{p'-1}{p'} = s - n/p.$$

Choosing ε so that $C\varepsilon^{s-n/p} = \lambda/2$,

$$\begin{aligned} |\{x \in \mathbb{R}^n : |I^s(x)| > \lambda\}| &\leqslant |\{x \in \mathbb{R}^n : |k_1^s * f(x)| > \lambda\}| \\ &\lesssim \lambda^{-p} \|k_1 * f\|_{L^p(\mathbb{R}^n)}^p \\ &\lesssim \lambda^{-p} \|k_1^s\|_{L^1(\mathbb{R}^n)}^p \\ &\lesssim \lambda^{-p}\varepsilon^{sp}. \end{aligned}$$

By our choice of ε and Eq. (4.16):

$$\lambda^{-p}\varepsilon^{sp} = C\lambda^{-p+sp/(s-n/p)} \quad \text{and} \quad p - \frac{sp}{s-n/p} = \frac{-n}{s-n/p} = \frac{1}{1/p - s/n} = q,$$

whence Eq. (4.17) follows. We conclude by taking $p = p_0 = p_1$, $q = q_0 = q_1$ and $t = 1/2$ in the Marcinkiewicz interpolation theorem. □

Chapter 5

The Whitney Extension Theorem

In this chapter we consider the reverse question: given a closed set $F \subset \mathbb{R}^n$ and a function $f : F \to \mathbb{R}$, how can we extend it to $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ while preserving its smoothness?

5.1 A review of Hölder and Lipschitz Functions

We briefly recall the definition of the Hölder spaces which we will use to quantify the measure of *smoothness* of f and its extension.

Definition 5.1. A function $f : F \to \mathbb{R}$ is called *Lipschitz* if

$$[f]_1 := \sup_{x,y \in F : x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < \infty.$$

We define the space

$$\mathrm{Lip}(F) := \{f : [f]_1 < \infty\}.$$

Clearly $\mathrm{Lip}(F) \subset C^0(F)$.

Note that the map $[\cdot]_1 : \mathrm{Lip}(F) \to [0, \infty)$ does *not* define a norm!

Definition 5.2. Let $\gamma \in (0, 1]$. A function $f : F \to \mathbb{R}$ is γ-*Hölder continuous* if

$$[f]_\gamma := \sup_{x,y \in F : x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma} < \infty.$$

Defining the norm

$$\|f\|_{C^{0,\gamma}(F)} := \|f\|_{C^0(F)} + [f]_\gamma$$

allows us to define the space

$$C^{0,\gamma}(F) = \{f : \|f\|_{C^{0,\gamma}(F)} < \infty\}.$$

Proposition 5.3. *The space* $(C^{0,\gamma}(F), \|\cdot\|_{C^{0,\gamma}(F)})$ *is a Banach space.*

Note that the Hölder spaces are trivial for $\gamma > 1$, as they only contain constant functions; hence the restriction $\gamma \in (0, 1]$.

Before moving on to the extension theorems, we mention a generalisation of Hölder spaces. The quantity $[f]_\gamma$ is obviously of interest, since it allows us to control the difference $f(x) - f(y)$ by controlling $x - y$. However, we can consider more general functions of $\delta = |x - y|$ than δ^γ.

Definition 5.4. A positive increasing function $\omega : [0, \infty) \to (0, \infty)$ satisfying

(i) $\delta \mapsto \frac{\omega(\delta)}{\delta}$ is increasing as $\delta \to 0$, and

(ii) $\omega(2\omega) \lesssim \omega(\delta)$

is called a *modulus of continuity.* Setting

$$[f]_\omega := \sup_{x,y \in F : x \neq y} \frac{|f(x) - f(y)|}{\omega(|x - y|)},$$

we define the space

$$C^0(\omega; F) := \left\{ f : \|f\|_{C^0(\omega;F)} < \infty \right\},$$

where $\|f\|_{C^0(\omega;F)} := \|f\|_{C^0(F)} + [f]_\omega$.

In particular, we have

$$C^{0,\gamma} = C^0(\delta^\gamma; F).$$

Condition (i) serves to rule out the degenerate case δ^γ for $\gamma > 1$, while condition (ii) is handy in estimates involving the triangle inequality, as we shall soon see. In fact, it is equivalent to

$$\text{for every } c_1 > 0 \text{ there is a } c_2 > 0 \text{ such that } \omega(c_1\delta) \leqslant c_2\omega(\delta) \text{ for all } \delta \in (0, \infty). \tag{5.1}$$

5.2 A Dydadic Partition of Unity

For the rest of this chapter, F will be a fixed non-empty, closed set in $\mathbb{R}^n$, and $\Omega = F^c$ its (open) complement.

As is often of interest in Harmonic Analysis, we want to work with dyadic cubes (hereafter: "cubes"). (This idea should be familiar from Calderón-Zygmund Theory;

see (Muscalu and Schlag, 2013, Chap. 7).) In this section we construct an appropriate cover of Ω by cubes and define a partition of unity which we will later exploit to extend functions beyond F into Ω.

To define cubes, note that $\mathbb{Z}^n$ introduces a lattice of evenly spaced points in $\mathbb{R}^n$. The segments parallel to the standard axes thus partition n-space into cubes of side length 1 and diameter $\sqrt{n}$. We scale these cubes "dyadically" by considering the lattices $2^{-k}\mathbb{Z}^n$ where $k \in \mathbb{Z}$, which yields cubes of side length 2^{-k} and diameter $2^{-k}\sqrt{n}$.

We take all cubes to be closed, but agree to call two such cubes *disjoint* if they have *disjoint interiors*. Thus, adjacent cubes overlapping on a boundary are disjoint for our purposes.

The main theorem concerning cubes is the following:

Theorem 5.5. *With F as above, there is a collection $\mathscr{Q} = \{Q_j\}_j \in \mathbb{Z}$ of cubes such that*

(i) $\Omega = \bigcup_j Q_j$;

(ii) the $Q_j \in \mathscr{Q}$ are mutually disjoint; and

(iii) $d(Q, F) \sim \operatorname{diam}(Q)$ *for each* $Q \in \mathscr{Q}$.

In particular, the bound (iii) is independent of Q, F and even n.

Proof. We layer Ω by "dyadic sectors" whose "widths" are proportional to their distance to F:

$$\Omega_k := \{x \in \Omega : c2^{-k} < d(x, F) < c2^{-k+1}\}.$$

We cover each "sector" Ω_k by cubes Q in the lattice $2^{-k}\mathbb{Z}^n$ which intersect Ω_k; see Fig. 5.1. (Note that this may involve some *over* covering.) Then take $\mathscr{Q}_0$ to be the collection of all such Q. Clearly

$$\Omega = \bigcup_{Q \in \mathscr{Q}_0} Q.$$

We now prove (iii) by an appropriate choice of c. More precisely, we show that

$$\operatorname{diam}(Q) \leqslant d(Q, F) \leqslant 4 \operatorname{diam}(Q) \quad \text{for all} \quad Q \in \mathscr{Q}_0. \tag{5.2}$$

Indeed, each such Q will have a side length 2^{-k} and meet Ω_k, say at x. The clearly

$$d(Q, F) \leqslant d(x, F) \leqslant c2^{-k+1},$$

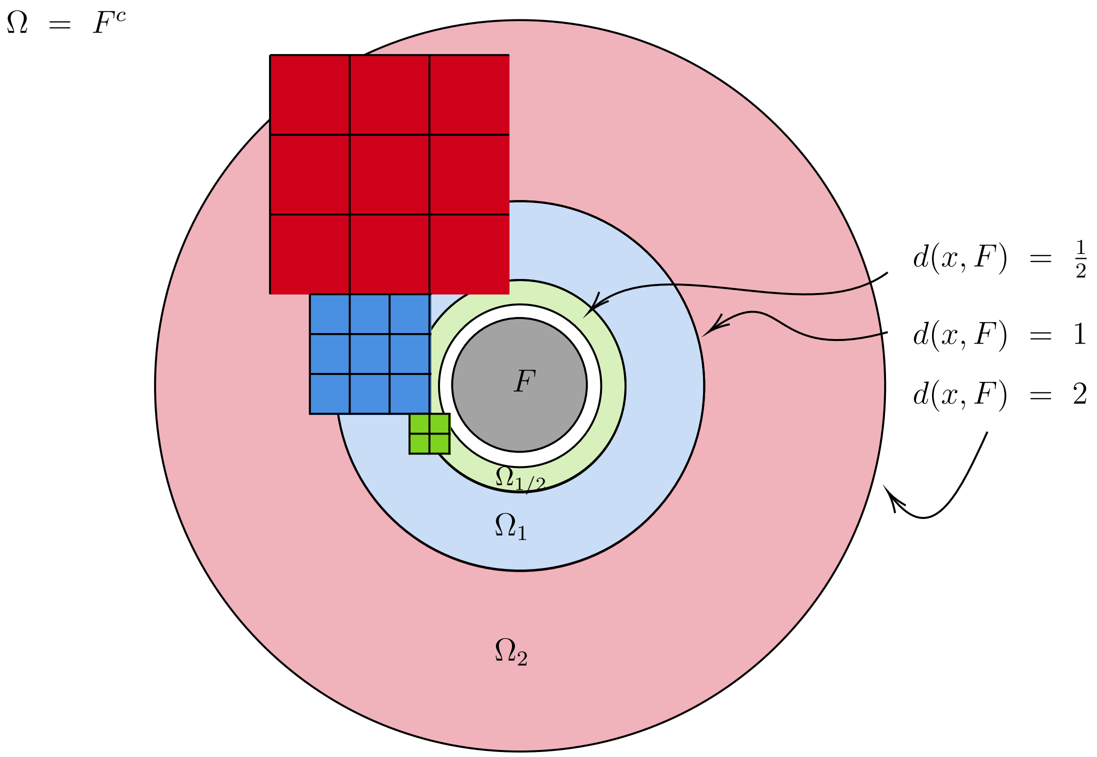

Figure 5.1: Cube decomposition for a disk (F) with $c = 1$. Each shaded annulus is a "sector" Ω_k. A sample of cubes chosen for $\mathscr{Q}_0$ is shown for $k = 1, 0, -1$.

and geometrically we can see that

$$d(Q, F) \geqslant d(x, F) - \operatorname{diam}(Q) > c2^{-k} - \sqrt{n}2^{-k}.$$

Thus, Eq. (5.2) follows upon choosing $c = 2\sqrt{n}$.

Furthermore, Eq. (5.2) implies that the cubes in $\mathscr{Q}_0$ are disjoint from F and therefore cover Ω. Thus is remains to address the over counting alluded to earlier, as we cannot conclude that the cubes in $\mathscr{Q}_0$ are necessarily disjoint.

Note that, by construction, any two cubes must be either disjoint, or one is contained in another.

Given any $Q \in \mathscr{Q}_0$, consider the maximal cube in $\mathscr{Q}_0$ that contains it. If, say, $Q \subset Q' \in \mathscr{Q}_0$, then Eq. (5.2) implies

$$\operatorname{diam}(Q') \leqslant d(Q', F) \leqslant d(Q, F) \leqslant 4 \operatorname{diam}(Q),$$

so a cube of maximal diameter exists. Furthermore, if Q' and Q'' are two such cubes, then $Q \subset Q' \cap Q''$, so $Q' = Q''$. Thus, Q is contained in a *unique maximal cube* in $\mathscr{Q}_0$. Define $\mathscr{Q}$ to be the subset of maximal cubes in $\mathscr{Q}_0$. These cubes are necessarily mutually disjoint, and clearly still satisfy (i). $\square$

We say that two *distinct* cubes $Q_1, Q_2 \in \mathscr{Q}$ *touch* if

$$\partial Q_1 \cap \partial Q_2 \neq \varnothing.$$

(Recall that the cubes in $\mathscr{Q}$ are closed with disjoint interiors.)

Proposition 5.6. *If $Q_1, Q_2 \in \mathscr{Q}$ touch, then*

$$\frac{1}{4}\,\mathrm{diam}(Q_1) \leqslant \mathrm{diam}(Q_2) \leqslant 4\,\mathrm{diam}(Q_1).$$

Proof. By Eq. (5.2),

$$d(Q_1, F) \leqslant 4\,\mathrm{diam}(Q_1).$$

Hence,

$$d(Q_2, F) \leqslant d(Q_1, F) + \mathrm{diam}(Q_1) \leqslant 5\,\mathrm{diam}(Q_1),$$

so, again by Eq. (5.2),

$$\mathrm{diam}(Q_2) \leqslant 5\,\mathrm{diam}(Q_1).$$

Now,

$$\mathrm{diam}(Q_2) = 2^k\,\mathrm{diam}(Q_1) \quad \text{for some} \quad k \in \mathbb{Z}$$

implying $2^k \leqslant 5$, *i.e.* $2^k \leqslant 4$. The result now follows by reversing the roles of Q_1 and Q_2. □

An important and remarkable feature of the decomposition in Theorem 5.5 is that there is a uniform bound on the number of neighboring cubes to any given cube, *independent of* F. This fact follows from a simple geometrical observation:

Lemma 5.7. *If Q is a cube at scale 2^{-k}, then there are 3^n cubes of the same scale touching Q.*

Proof. The observation is that we count an adjacent cube for each m-dimensional cubes contained in Q; see Fig. 5.2. From (Coxeter, 2012, Eq. 7.25) we learn that the number of m-cubes in an n-cube is

$$2^{n-m}\binom{n}{m}.$$

Thus, by the Binomial Formula,

$$\sum_{k=0}^{n}\binom{n}{m}2^{n-m} = (1+2)^n = 3^n.$$ □

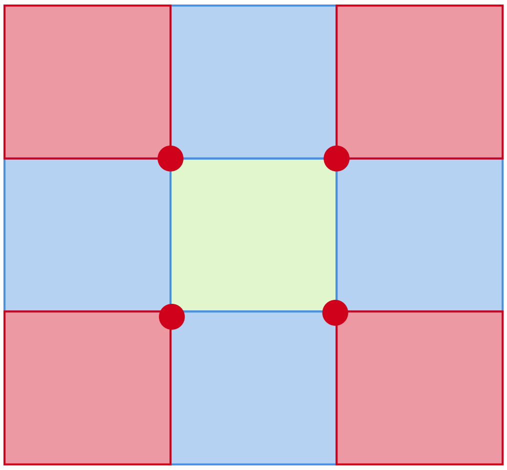

Figure 5.2: Counting adjacent cubes in $n = 2$: there are one 2-cube (the square itself, green), four 1-cubes (edges, blue) and four 0-cubes (vertices, red), and each counts an adjacent 2-cube for a total of $9 = 3^2$.

Proposition 5.8. *For each $Q \in \mathscr{Q}$ there are at most 12^n cubes in $\mathscr{Q}$ that touch Q.*

Proof. By Lemma 5.7, there are at most 3^n adjacent cubes as the same scale. On the other hand, by Proposition 5.6, each cube Q contains at most 4^n cubes $Q' \in \mathscr{Q}$ such that

$$\operatorname{diam}(Q') \geqslant \frac{1}{4}\operatorname{diam}(Q).$$

For if N is the number of cubes Q' as above, then

$$\begin{aligned} V(Q) &\geqslant \sum_{j=1}^{N} V(Q'_j) \\ \implies 2^{-kn} &\geqslant N \times 2^{-kn-2n} \\ \implies N &\leqslant 2^{2n}. \end{aligned}$$

Hence, there are at most 3^n adjacent cubes each containing at most 4^n cubes from $\mathscr{Q}$, for a total upper bound of 12^n. □

Finally, we consider "fattenings" of the cubes $Q_k \in \mathscr{Q}$ as follows. Fix $\varepsilon \in (0, \frac{1}{4})$. Let x_k denote the centre of Q_k and l_k its side-length. Then Q_k^* is the cube centred at x with side length $(1+\varepsilon)l \in (l, \frac{5}{4}l)$. (See Fig. 5.3.)

Note that $Q_k \subset Q_k^*$ and that the new cubes Q_k^* need not be disjoint. In the following, we assume all cubes Q_k to be in $\mathscr{Q}$.

Proposition 5.9. *Any $x \in \Omega$ is contained in at most 12^n cubes Q_k^*.*

Proof. We begin with the following

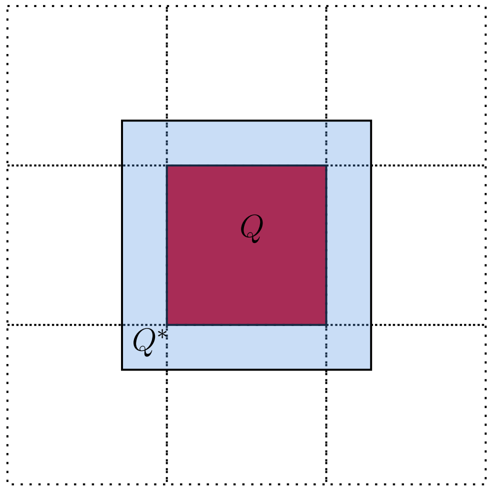

Figure 5.3: A cube Q and its fattening Q^*.

Claim: Q_k^* intersects Q only if Q_k and Q touch.

To see this, let $\widetilde{Q}$ the union of Q_k with all its adjacent cubes Q'. By Proposition 5.6,

$$\operatorname{diam}(Q') \geqslant \frac{1}{4} \operatorname{diam}(Q_k),$$

Thus, counting Q_k and one other cube Q' meeting Q_k at a vertex,

$$\operatorname{diam}(\widetilde{Q}) \geqslant (1 + 1/4) \operatorname{diam}(Q_k) > (1 + \varepsilon) \operatorname{diam}(Q_k) = \operatorname{diam}(Q_k^*),$$

so we conclude that $Q_K^* \subset \widetilde{Q}$.

Now, if Q intersects Q_k^*, then Q intersects $\widetilde{Q}$, so Q must meet one of the Q' that touch Q_k. But, by construction of $\mathscr{Q}$, this means $Q = Q'$, proving the claim.

Finally, if $x \in \Omega$, then x lies in some Q. By the claim, any Q_k such that Q_k^* contains x intersects Q, so Q_k touches Q, and there are at most 12^n adjacent cubes. □

Corollary 5.10. *Any $x \in \Omega$ lies in a neighborhood meeting at most 12^n Q_k^*'s.*

Armed with these results about the cubes $\mathscr{Q}$, we can define the promised partition of unity.

Let $Q_0 := [-1/2, 1/2]^n$, the unit cube centred at the origin and select a $\varphi \in C^\infty(\mathbb{R}^n)$ such that

(i) $0 \leqslant \varphi \leqslant 1$;

(ii) $\varphi \equiv 1$ on Q_0; and

(iii) $\varphi \equiv 0$ on $\mathbb{R}^n \setminus Q_0^*$.

By rescaling and translating, we can obtain further functions

$$\varphi_k(x) := \varphi\left(\frac{x - x_k}{l_k}\right)$$

with the properties (i)-(iii) adapted to Q_k.

Clearly,

$$|\partial^\alpha \varphi_k| \lesssim_\alpha (\mathrm{diam}(Q_k))^{-|\alpha|} \quad \text{for all} \quad \alpha \in \mathbb{N}_0^n,$$

and the Leibniz rule then implies

$$|\partial^\alpha \varphi_k^*| \lesssim_\alpha (\mathrm{diam}(Q_k))^{-|\alpha|} \quad \text{for all} \quad \alpha \in \mathbb{N}_0^n. \tag{5.3}$$

The partition of unity is formed by

$$\varphi_k^*(x) = \frac{\varphi_k(x)}{\sum_k \varphi_k(x)} \quad \text{for all} \quad x \in \Omega.$$

Note that, by Corollary 5.10, the sum on the denominator is really finite and non-zero, so the φ_k^* are well-defined and satisfy

$$\sum_k \varphi^*(x) = 1 \quad \text{for all} \quad x \in \Omega.$$

5.3 The Whitney Extension Theorem for Continuous Functions

As always, let $F \subset \mathbb{R}^n$ be a closed set and $\{Q_k\}$ the cubes from the Whitney decomposition (Theorem 5.5). Consider a function $f : F \to \mathbb{R}$. We will construct an extension $\mathscr{E}_0 f : \mathbb{R}^n \to \mathbb{R}$ agreeing with f on F.

Since F is closed, for each Q_k we may pick a $p_k \in F$ such that

$$d(Q_k, F) = d(Q_k, p_k).$$

We record two useful estimates regarding the sets Q_k:

Lemma 5.11. *With the notation above,*

(i) if $x \in Q_k^*$, then $|x - p_k| \sim \mathrm{diam}(Q_k)$;

(ii) $d(Q_k^*, F) \sim \mathrm{diam}(Q_k)$.

Proof. For (i) we apply Eq. (5.2):

$$|x - p_k| \leqslant (1 + \varepsilon)\operatorname{diam}(Q_k) + d(Q_k, F) \lesssim \operatorname{diam}(Q_k),$$

and, conversely,

$$|x - p_k| \geqslant d(F, Q_k) - d(x, Q_k) \gtrsim \operatorname{diam}(Q_k).$$

For (ii), clearly

$$d(Q_k^*, F) \leqslant d(Q_k, F) \lesssim \operatorname{diam}(Q_k).$$

Further, if $x \in Q_k^*$, then

$$d(x, F) \geqslant d(Q_k, F) - \frac{1}{4}\operatorname{diam}(Q_k) \geqslant \frac{3}{4}\operatorname{diam}(Q_k),$$

whence $d(Q_k^*, F) \gtrsim \operatorname{diam}(Q_k)$. □

Set

$$\mathscr{E}_0 f(x) := \begin{cases} f(x), & \text{if } x \in F; \\ \sum_k f(p_k)\varphi_k^*(x), & \text{if } x \in \Omega. \end{cases} \tag{5.4}$$

As before, we note that each $x \in \Omega$ belongs to finitely many Q_k^*, so that the sum over cubes is really finite at each point.

Proposition 5.12. *Let $f : F \to \mathbb{R}$. Then $\mathscr{E}_0 f$ defined as in Eq. (5.4) is an extension of f to $\mathbb{R}^n$. Furthermore, if $f \in C^0(F)$, then $\mathscr{E}_0 f \in C^0(\mathbb{R}^n)$, and, in fact, C^∞ on Ω.*

Proof. Clearly $\mathscr{E}_0 f$ extends f to $\mathbb{R}^n$ by definition. Thus, we assume that f is continuous on F and prove the second statement.

If $y \in F$ and $x \in Q_k^*$, then, combining (i) and (ii) in Lemma 5.11,

$$\begin{aligned} |y - p_k| &\leqslant |y - x| + |p_k - x| \\ &\lesssim |y - x| + \operatorname{diam}(Q_k) \\ &\lesssim |y - x| + d(Q_k^*, F). \end{aligned}$$

Since $|y - x| \geqslant d(Q_k^*, F)$, we arrive at

$$(y \in F \text{ and } x \in Q_k^*) \implies |y - p_k| \leqslant c\,|y - x|\,, \tag{5.5}$$

for some constant $c > 0$ independent of x, y *and* Q_k.

On Ω, $\mathscr{E}_0 f$ is defined (pointwise) as a finite sum of the C^∞ functions φ_k^*, and is therefore itself C^∞. It remains to check continuity on $\mathbb{R}^n$.

Fix $\varepsilon > 0$ and $y \in F$. For each $x \in \mathbb{R}^n$ we have

$$\mathscr{E}_0 f(y) - \mathscr{E}_0 f(x) = \begin{cases} f(y) - f(x), & \text{if } x \in F; \\ \sum_k (f(y) - f(p_k))\varphi_k^*(x), & \text{if } x \in \Omega. \end{cases} \tag{5.6}$$

By the continuity of f on F, select a $\delta_F > 0$ such that

$$|x - y| < \delta_F \implies |f(x) - f(y)| < \frac{\varepsilon}{12^n}$$

whenever $x \in F$.

For $x \in \Omega$, x belongs to up to 12^n cubes Q_k^*, and is therefore only in finitely many supports $\operatorname{supp}(\varphi_k^*)$. Provided that $|y - x| < \frac{1}{c}\delta_F$, Eq. (5.5) implies

$$|f(y) - f(p_k)| < \frac{\varepsilon}{12^n}.$$

Thus, if $|y - x| < \min(\delta_F, \frac{\delta_F}{c})$, then Eq. (5.6) yields

$$|\mathscr{E}_0 f(y) - \mathscr{E}_0 f(x)| < \varepsilon$$

whether $x \in F$ or $x \in \Omega$, proving the continuity of $\mathscr{E}_0 f$ on F too. □

We now move on to the main theorem of this section:

Theorem 5.13. *The linear extension operator $\mathscr{E}_0$ is bounded $C^0(\omega; F) \to C^0(\omega; \mathbb{R}^n)$.*

Proof. Without loss of generality, we assume that $\|f\|_{C^0(\omega;F)} = 1$. Write $\delta(x) := d(x, F)$.

We prove

$$|\partial_j \mathscr{E}_0 f(x)| \lesssim \frac{\omega(\delta(x))}{\delta(x)} \quad \text{for all} \quad 1 \leqslant j \leqslant n, x \in \Omega. \tag{5.7}$$

Since $\sum_k \varphi_k^*(x) \equiv 1$ on Ω, any derivative of the sum vanishes and we have

$$\partial_j \mathscr{E}_0 f(x) = \sum_k f(p_k) \partial_j \varphi_k^*(x) \equiv \sum_k (f(p_k) - f(y)) \partial_i \varphi_k^*(x).$$

For $x \in \Omega$ choose $y \in F$ such that $\delta(x) = d(x, F) = |x - y|$. The above sum ranges over the cubes Q_k^* containing x, and Eq. (5.5) implies

$$|y - p_k| \leqslant c\,|y - x| = c\delta(x).$$

Therefore, from Eq. (5.3) we deduce

$$|\partial_j \mathscr{E}_0 f(x)| \lesssim_j \sum_{Q_k^* \ni x} |f(p_k) - f(y)| \, |\mathrm{diam}(Q_k)|^{-|\alpha|}.$$

Since, by Lemma 5.11, $\delta(x) \sim \mathrm{diam}(Q_k)$ when $x \in Q_k^*$, we have

$$|\partial_j \mathscr{E}_0 f(x)| \lesssim_j \sum_{Q_k^* \ni x} \omega(|p_k - y|)\delta(x)^{-1} \lesssim_j \frac{\omega(\delta(x))}{\delta(x)},$$

using the montonicity of ω and Eq. (5.1) with $c_1 = c$. This proves Eq. (5.7).

Next, recall that for $y \in F$ and $x \in \Omega$ we have

$$\mathscr{E}_0 f(y) - \mathscr{E}_0 f(x) = \sum_k (f(y) - f(p_k))\varphi_k^*(x).$$

Hence, Eqs. (5.1) and (5.5) again imply that

$$|\mathscr{E}_0 f(y) - \mathscr{E}_0 f(x)| \leqslant \sup_{Q_k^* \ni x} |f(y) - f(p_k)| \lesssim \omega(|x - y|) \quad \text{for all} \quad x \in \Omega, y \in F. \tag{5.8}$$

Finally, we consider $x, y \in \Omega$. Denote by "L" the line segment joining x and y.

Case I: $d(L, F) > |x - y|$**.** This is a straightforward consequence of Eq. (5.7) and the mean value inequality (recall that $\mathscr{E}_0 f$ is C^∞ on Ω!):

$$\begin{aligned} |\mathscr{E}_0 f(y) - \mathscr{E}_0 f(x)| &\leqslant |y - x| \sup_{z \in L} |\nabla \mathscr{E}_0 f(z)| \\ &\lesssim |y - x| \sup_{z \in L} \frac{\omega(\delta(z))}{\delta(z)}. \end{aligned}$$

Note that, by assumption, $\delta(z) \geqslant |y - x|$ for each $z \in L$, so, by the montonicity of $\frac{\omega(\eta)}{\eta}$, we have

$$|\mathscr{E}_0 f(y) - \mathscr{E}_0 f(x)| \lesssim \omega(|y - x|).$$

Case II: $d(L, F) \leqslant |x - y|$**.** Pick $z \in L$ and $w \in F$ such that

$$|z - w| \leqslant |x - y|.$$

Then

$$|w - x| \leqslant 2|x - y| \quad \text{and} \quad |w - y| \leqslant 2|x - y|,$$

so, by Eq. (5.8) and the triangle inequality,

$$|\mathscr{E}_0 f(y) - \mathscr{E}_0 f(x)| \lesssim \omega(|y - x|).$$

Having covered all possible cases, we conclude that $\mathscr{E}_0 f \in C^0(\omega; F)$. Furthermore,

$$\|\mathscr{E}_0 f\|_{C^0(\omega;F)} \lesssim 1 = \|f\|_{C^0(\omega;F)}\,,$$

so $\mathscr{E}_0$ is a bounded operator. □

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