Topics in Geom Meas Theory

Kakiya sets / Besicovitch
Baire category thm
Hausdorff distance

Def $E \subset \mathbb{R}^2$ is a Besicovitch set if it contains a unit line segment direction.

Thm (Besicovitch) There is a compact Besicovitch set of zero Lebesgue measure

Q Kakeya's question

Thm (Bes) $\forall \varepsilon > 0$ there is a set of area ε in which we can rotate a needle (unit line segment) by $180°$

Exercise $\varepsilon = 0$ is not possible

Today. Tom Körner's proof
Baire category $\Rightarrow$ $\exists$ Besicovitch sets of zero measure

Hausdorff metric

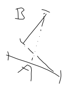
(X, d) metric space

Def $\mathcal{K}(X)$ = family of non-empty compact subsets of X

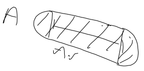
Def Hausdorff metric on $\mathcal{K}(X)$

$$d_H(A, B) = \inf\{\delta > 0 : A \subset B_\delta \text{ and } B \subset A_\delta\}$$

where $A_\delta = \{x \in X : \exists y \in A \; d(x,y) < \delta\}$



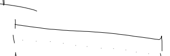
$$d_H(A, B) = \max\left(\sup_{x\in A} \inf_{y \in B} d(x,y), \sup_{x \in B} \inf_{y\in A} d(x,y)\right)$$



Thm (X, d) is complete then $(\mathcal{K}(X), d_H)$ is complete

(X, d) is compact, then $(\mathcal{K}(X), d_H)$ is compact

Example



$A_n = \{\frac{i}{n} : 0 \le i \le n\}$ $\quad A_n \xrightarrow{d_H} [0,1]$

Thm Let $A_n \in \mathcal{K}(\mathbb{R}^n)$, $A_n \xrightarrow{d_H} A$

Then $\lambda(A) \ge \limsup_n \lambda(A_n)$

(λ is upper semi continuous)

Proof

A compact in $\mathbb{R}^n$

$A_\delta \supset A$ open

$A = \bigcap_{\delta > 0} A_\delta = \bigcap_{n=1}^{\infty} A_{1/n}$

$\lambda(A) = \lim_n \lambda(A_{1/n})$

$\Downarrow$

$\forall \varepsilon > 0 \; \exists N$

$\lambda(A_{1/N}) \le \lambda(A) + \varepsilon$

$A^n \xrightarrow{d_H} A$

$\Downarrow$

If n is large enough then $d_H(A^n, A) < \frac{1}{N}$

then $A^n \subset A_{1/N}$

$\lambda(A^n) \le \lambda(A) + \varepsilon$

$\square$

Baire category thm

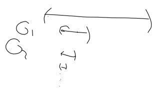
Thm Let (X, d) be a complete metric space

Then the intersection of countably many dense open sets is non-empty (and dense)

def $A \subset X$ is dense if $\bar{A} = X$

eq. $\forall$ non-empty open U, $A \cap U \neq \emptyset$

Proof sketch on $\mathbb{R}$ $\qquad G_1, G_2, \ldots$ dense open



---

def $A \subset X$ is nowhere dense if $\operatorname{int} \bar{A} = \emptyset$

Remark $G \subset X$ is dense open $\iff$ G^c is nowhere dense closed

Thm (Baire cat thm)

(X, d) complete. It is not the union of countably many nowhere dense (closed) sets

Thm (X, d) complete

$Y \subset X$ closed $\iff$ (Y, d) complete

$Y \subset X$ open $\Rightarrow$ (Y, d') complete

G_δ $\qquad$ d and d' are equivalent

def A set is G_δ if it is the intersection of countably many open sets.

def F_σ if it is the union of countably many closed sets.

Thm (Baire cat thm)

Intersection of countably many dense G_δ sets is dense G_δ

Example set of Liouville numbers

$L = \{x \in \mathbb{R} : \forall n \ \exists p, q > 1$

$0 < |x - \frac{p}{q}| < \frac{1}{q^n}\}$

$G_n = \{x \in \mathbb{R} : \exists p, q > 1 \ 0 < |x - \frac{p}{q}| < \frac{1}{q^n}\}$

$L = \bigcap_n G_n$

$G_n = \cup G_{n,p,q}$ $\quad G_{n,p,q} = \{x : \underbrace{0 < |x - \frac{p}{q}| < \frac{1}{q^n}}_{\text{open}}$

$\mathbb{Q}$ is dense $\Rightarrow$ G_n dense open

So L is dense G_δ, so it is non-empty

Example / Exercise

Thm There is NO function $f : \mathbb{R} \to \mathbb{R}$ such that the points of continuity is exactly $\mathbb{Q}$.

Claim $\forall f$ the points of continuity is a G_δ set.

Claim $\mathbb{Q}$ is not G_δ.

Remark $\mathbb{R} \setminus \mathbb{Q}$ is dense G_δ and $\exists f$ ...

Def large/small sets

small: of first category
meagre
def: union of countably many nowhere dense sets

not small:
- of second category
- not meagre

"very large"
- residual
- co-meagre

def: complement is meagre

def: contains a dense G_δ set

Remark Preserved by homeomorphisms

Remark A is Borel
$\Rightarrow \exists\, U$ open
$\exists\, M$ meagre
$A = U \Delta M$

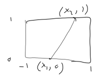
Def (X,d) complete metric
A property P is typical
$\{x \in X : x \text{ has prop } P\}$ is residual
$\{x \in X : x \text{ doesn't have } P\}$ is of first cat.

Exercise
Consider $C[0,1]$ in sup norm.
A Typical continuous function is nowhere differentiable.

Thm A typical compact subset E of $[0,1]$
$(\mathcal{K}([0,1]), d_H)$
$\lambda(E) = 0$
E homeomorphic to the Cantor set

Proof $\mathcal{F} = \{E \subset [0,1] : \lambda(E) = 0\}$ Claim $\mathcal{F}$ is dense G_δ

Proof of Claim
Dense: $\forall$ ⋯⋯ E
$\forall E \subset [0,1]\ \exists$ finite set E_n $d_H(E, E_n) < \frac{1}{n}$
$\mathcal{F}_n = \left\{E \in \mathcal{K}([0,1]) : \lambda(E) < \frac{1}{n}\right\}$
is open.
$\mathcal{F} = \bigcap \mathcal{F}_n$



$[-1,1]\times[0,1]$

$\mathcal{F} = \{ E \subset [-1,1]\times[0,1]$;
compact
E is the union of
line segments
connecting a point
$(x_1, 0)$ to $(x_2, 1)$
And for every $t \in [-1,1]$
there is such a line segment
in E with $x_2 - x_1 = t$

Aim: there is $E \in \mathcal{F}$
with $\lambda(E) = 0$.

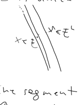
Consider $\mathcal{F}$ in d_H.
Is $\mathcal{F}$ complete metric?

$\mathcal{F} \subseteq \mathcal{K}([-1,1]\times[0,1])$

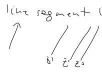
__Claim__ $\mathcal{F}$ is closed in $\mathcal{K}([-1,1]\times[0,1])$

__Proof__ Let $E^n \in \mathcal{F}$. $E^n \to E$. Aim: $E \in \mathcal{F}$.

1. E is union of ~~those~~ such line segments.



2. line segment in every direction

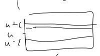


subsequence converges

□

$\mathcal{F}$ compact metric space.

__Def__ $P(u,\varepsilon) = \{ E \in \mathcal{F} : \forall y \in [u-\varepsilon, u+\varepsilon]$
$u \in [0,1]$
$\varepsilon > 0$



$\lambda(\{x \in [-1,1] : (x,y) \in E\}) < 9\varepsilon$

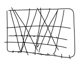
__Claim__ $P(u,\varepsilon)$ is dense open in $\mathcal{F}$.

__Proof__ Let $Q = \mathcal{F} \setminus P(u,\varepsilon)$
If $E \in Q$, then $\exists y \in [u-\varepsilon, u+\varepsilon]$
such that
$\lambda(\{x \in [-1,1] : (x,y) \in E\}) \geq 9\varepsilon$

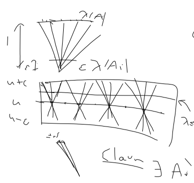
Assume $E^n \in Q$, $E^n \to E$
$\exists y^n \ldots$
$\exists n_i \to \infty$ $y^{n_i} \to y \in [u-\varepsilon, u+\varepsilon]$
Use upper semi continuity
to conclude that
$\lambda(\{x : (x,y) \in E\}) \geq 9\varepsilon$.

__Claim__ $P(u,\varepsilon)$ is dense

__Proof__ $E \in \mathcal{F}$ is fixed.
$\delta > 0$ fixed
Want to show: $\exists E'$ $d_H(E,E') < 10\delta$
$E' \in P(u,\varepsilon)$.



perturb $E \rightsquigarrow E'$
such that all line segments intersect
$[-1,1]\times\{u\}$
in a finite set
say, the δ-grid.



Grid points
$p_1, p_2, \ldots, p_m$
$A_i = \{$ angles of
line segments
going through
$p_i \}$

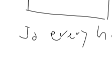
__Claim__ $\exists A_i' \subset A$
$\sum \lambda(A_i') = \lambda([-45°, 45°])$
and
$\bigcup A_i' = \bigcup A_i = [-45°, 45°]$

$\to E' \subset E_\delta$
such that the Leb meas
of horizontal sections
are $\leq \varepsilon\, \lambda([-45°, 45°])$

By adding finitely many
line segments to E'
$E \subset E'_{2\delta}$
$d_H(E,E') \leq 2\delta$

$P(u,\varepsilon)$ dense ~~G_δ~~ open

$E \in \bigcap_{n,i} P(\frac{i}{n}, \frac{1}{n})$ dense G_δ (non-empty)



$\forall y\ \forall n\ \exists i$
$\frac{i}{n} - \frac{1}{n} < y < \frac{i}{n} + \frac{1}{n}$
$\lambda(\{x : (x,y) \in E\}) \leq 9\frac{1}{n}$

So every horizontal section of E
has measure 0

$\Rightarrow \lambda^2(E) = 0$
Fubini