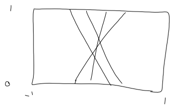
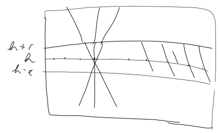
Recall



Complete metric space

Besicovitch sets

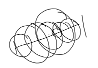
Thm: Typical has measure zero

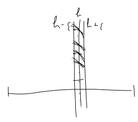


Same proof works in other settings.

<u>Thm</u> (Talagrand)

There is a compact set $E \subset \mathbb{R}^2$ of zero Lebesgue measure that contains a circle centred at every point in a fixed line segment.





<u>Thm</u> There is a nullset in the plane containing circles of every radius.

Same proof techniques works: union of 1-dimensional family of 1-dim objects.

<u>Remark</u> In $\mathbb{R}^n$, ~~of~~ k-dim family of n-k dimensional objects.

For example:

Spheres of every radius in $\mathbb{R}^n$

There is a Lebesgue nullset in $\mathbb{R}^n$ containing a sphere of every radius.

<u>Theorem</u> (Bourgain, Marstrand)

If $E \subset \mathbb{R}^2$, contains circles centred at every point in $\mathbb{R}^2$

Then E has positive Lebesgue measure.

<u>Remark</u> Besicovitch sets in $\mathbb{R}^n$

compact set containing a line segment in every direction

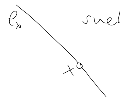
<u>Thm</u> There are B sets in $\mathbb{R}^n$ of $\mathcal{L}$ measure zero.

<u>Proof</u> Our typical proof works.

$B_2 \subset \mathbb{R}^2$ Besicovitch

Then $B_2 \times \mathbb{R}^{n-k}$ is a Besicovitch set in $\mathbb{R}^n$

Nikodym set in $\mathbb{R}^2$



Thm There is a set $E \subset \mathbb{R}^2$ of Lebesgue measure zero such that for every $x \in \mathbb{R}^2$ there is a line ℓ_x through x such that $\ell_x \setminus \{x\} \subset E$.

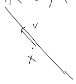
Def Nikodym set: E as in theorem.

Thm (Davies)
Let $A \subset \mathbb{R}^2$ measurable.
Then we can cover A by lines such that $\lambda(\text{union of lines} \setminus A) = 0$

Aim: there is a Nikodym set, in fact, a "typical" construction works
(Davies, Falconer, Chang-Csörnyei-Héra-Keleti)

Def $K \subset [0,1]^2 \times S^1$

Then $\varphi(K) \overset{\text{def}}{=} \bigcup_{(x,v) \in K} (x + \mathbb{R}v) \setminus \{x\}$



is a subset of the plane containing a punctured line through every $x \in \operatorname{proj}_1(K)$

Def Let $\mathcal{K}$ denote the space of all compact sets $K \subset [0,1]^2 \times S^1$ for which $\operatorname{proj}_1 K = [0,1]^2$.

Claim $(\mathcal{K}, d_H)$ is a complete metric space.

Proof Enough to show: $\mathcal{K}$ is a closed subset of $\mathcal{K}([0,1]^2 \times S^1)$.

We show complement is open.
Take $K \subset [0,1]^2 \times S^1$ non-empty compact such that $\underbrace{\operatorname{proj}_1 K}_{\text{closed}} \neq [0,1]^2$

Then $\exists \delta > 0$ open δ-neigh of $\operatorname{proj}_1 K$ is not $[0,1]^2$.

Then, if K' $d_H(K, K') < \delta/2$
then $K' \subset B(K, \delta)$
$\operatorname{proj}_1 K' \subset B(\operatorname{proj}_1 K, \delta) \neq [0,1]^2$ $\square$

Claim $K \in \mathcal{K}$. Then $\varphi(K) = \bigcup_{(x,v) \in K} (x + \mathbb{R}v) \setminus \{x\}$ is measurable subset of $\mathbb{R}^2$.
It is F_σ.

Proof

$$\varphi_N(K) = \bigcup_{(x,v) \in K} x + v \cdot \left([-N, -\tfrac{1}{N}] \cup [\tfrac{1}{N}, N]\right)$$

1. $\varphi_N(K) \subset \varphi(K)$
2. $\varphi(K) = \bigcup_{N=1}^{\infty} \varphi_N(K)$
3. $\forall N$ φ_N is compact

φ_N is a continuous image of $K \times \left([-N, -\tfrac{1}{N}] \cup [\tfrac{1}{N}, N]\right)$

Thm For a typical $K \in \mathcal{K}$,
$\varphi(K)$ has Lebesgue measure zero.

Corollary There is a Nikodym set.

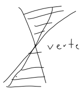
Proof Take one $K \in \mathcal{K}$ with $\varphi(K)$ has measure zero.

$$\text{Nikodym set} = \underbrace{\varphi(K) + \mathbb{Z}^2}_{\text{union of countably many translates}}$$

$\square$

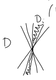
Geometric construction

Double cone (not containing a horizontal line)

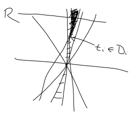


vertex

Lemma 1
Let D be a double cone centred at 0.
Let $R > 0$. Let $\varepsilon > 0$.
Then we can partition D into finitely many double cones D_i (with same centre)



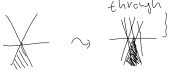
$D = \cup D_i$
and find $t_i \in D_i$
$D_i' = t_i + D_i$
$D' = \cup_i D_i'$
such that
(1) $\lambda(D' \cap \{(x,y) : 0 \le y \le R\}) < \varepsilon$.
(2) D' covers the bottom half of D, that is
$D' \supset D \cap \{(x,y) : y \le 0\}$
(3) $D' \cap \{(x,y) : y \le 0\}$ is in the ε-neighbourhood of $D \cap \{(x,y) : y \le 0\}$



R
$t_i \in D_i$
measure here $< \varepsilon$
here we are close to D.

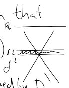
Remark D' covers bottom half of D
$\Rightarrow \forall p \in$ bottom half of D there is a line in D' through p.

Idea:



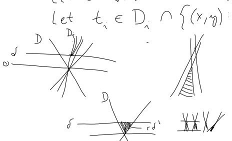
$\} \lambda < \varepsilon$

Lemma 0
D double cone centred at 0.
Let $\delta > 0$. Let $\varepsilon > 0$.
Then we can choose a partition $D = \cup_i D_i$ and translates $D_i' = t_i + D_i$ such that



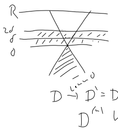
(1) $\lambda(D' \cap \{(x,y) : 0 \le y \le \delta\}) \le \varepsilon\delta^2$
(2) bottom half of D is covered by D'
(3) $t_i \in \{(x,y) : y = \delta\}$
(4) $D' \cap \{(x,y) : y \le 0\}$ is in the ε-neighb of D.

Proof Let $D = \cup_i D_i$ be ε-fine partition.
Let $t_i \in D_i \cap \{(x,y) : y = \delta\}$



$\square$

Proof of lemma 1
Fix N large integer.
Apply lemma 0 for $\delta = \frac{R}{N}$, repeat this N times



$\rightsquigarrow$ after "N steps" all new cones are at height $N\delta = R$.

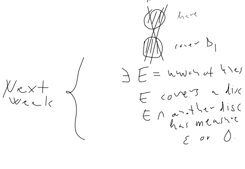
$D \xrightarrow{\text{lemma } 0} D' = D^{(1)} \rightsquigarrow D^{(2)} \to \dots$

$D^{(n)}$ has small measure in $\{(x,y) : \delta n \le y \le \delta(n+1)\}$

and $D^{(n)}$ is <u>very close</u> to $D^{(n-1)}$ below the line $y = \delta n$

$N\delta = R$

$D^{(1)}$ small meas



area is small

have

cover D_1

Next week $\{$
$\exists E$ = union of lines
E covers a disc
$E \cap$ another disc has measure ε or 0.