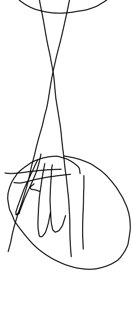
Aim: Nikodym set

Aim:

B



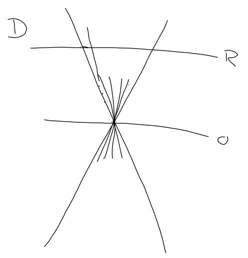
B'

lines cover B'

area of the set covered by lines in B is $< \varepsilon$

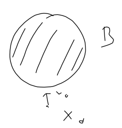
$= 0$.

Lemma 1



We can partition
$D = \cup D_i$
find translates
$D_i' = t_i + D_i$
$t_i \in$ upper half of D_0
$D' = \bigcup_i D_i'$

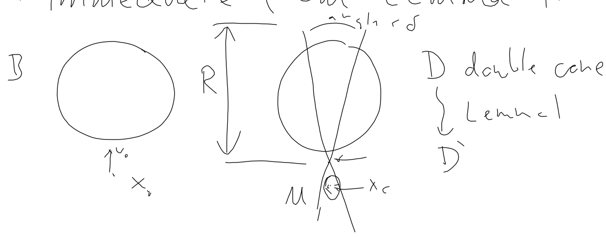
(1) Measure of $D' \cap \{(x,y) : 0 \le y \le R\}$ is less than ε.
(2) D' covers $D \cap \{(x,y) : y \le 0\}$
(3) $D' \cap \{(x,y) : y \le 0\}$ is in the ε-neighb of D.



Lemma 2

Let $(x_0, v_0) \in \mathbb{R}^2 \times S^1$
let $B \subset \mathbb{R}^2$ closed disc, $x_0 \notin B$.
Let $\delta > 0$. Then $\exists$ open neighb. U of x_0 such that:
for each $\varepsilon > 0$ there is a set $D \subset \mathbb{R}^2$ such that for all $x \in U$ there is a line $L \subset D$ through x such that the angle of L and v_0 is $< \delta$
and $\lambda(B \cap D) < \varepsilon$.

Proof: Immediate from Lemma 1.



$\mathcal{K} = \{ K$ compact ~~subset of~~ $[0,1]^2 \times S^1$ such that $\mathrm{proj}_1(K) = [0,1]^2 \}$

Def B closed ball, let $2B$ denote the <u>open</u> ball of the same centre twice the radius.

Def For a closed ball B, let $\mathcal{A}(B)$ be the set of those $K \in \mathcal{K}$ for which

$$\lambda\Big(B \cap \bigcup_{\substack{(x,v)\in K \\ x \notin 2B}} x + \mathbb{R}v\Big) = 0$$

Lemma $\mathcal{A}(B) \subset \mathcal{K}$ is residual, in fact dense G_δ set.

Proof

For $\varepsilon > 0$, $\mathcal{A}(B,\varepsilon) = \dots$ same as before but $\lambda(\ ) < \varepsilon$

Claim: $\mathcal{A}(B,\varepsilon)$ dense, open ?

Claim: $\mathcal{A}(B) = \bigcap_{n=1}^{\infty} \mathcal{A}(B, \frac{1}{n})$ ✓

Proof (open)

$K \in \mathcal{A}(B,\varepsilon)$

$$E = B \cap \bigcup_{\substack{(x,v)\in K \\ x \notin 2B}} x + v\mathbb{R}$$ compact, measure $< \varepsilon$

Then $\exists \delta > 0$

$\lambda(B(E,\delta)) < \varepsilon$ $\lambda(\mathcal{U}) < \varepsilon$

$\mathcal{U} = B(E,\delta) \cup B(\partial B, \delta)$

Then if $d_H(K, K') < \delta/1000$

then $E_{K'} \subset \mathcal{U}$ so $\lambda(E_{K'}) < \varepsilon$

Proof of $\mathcal{A}(B,\varepsilon)$ dense

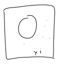
Let $K \in \mathcal{K}$ be arbitrary

For every $x \in [0,1]^2 \setminus 2B$

there is v such that $(x,v) \in K$

By Lemma 2, there is a small open neighb $\mathcal{U}$ of x such that

Since $[0,1]^2 \setminus 2B$ is compact, finitely many of these $\mathcal{U}$ neighbourhoods cover it.



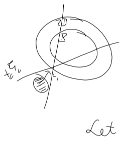
We have finitely many points $x_i \in [0,1]^2 \setminus 2B$

Such that $\bigcup \mathcal{U}_i \supset [0,1]^2 \setminus 2B$

and Lemma 2 holds for $B, (x_i, v_i, \mathcal{U}_i)$

Choose $\varepsilon_i > 0$ such that $\sum \varepsilon_i < \varepsilon$

and consider the union of all lines given by $D = D_i$



$K \in \mathcal{K}$

$\rightsquigarrow$

$K' \subset B(K, \delta)$

$K' \in \mathcal{A}(B, \varepsilon)$

Let $K'' = K' \cup$ a finite δ-dense subset of K

Then $K \subset B(K'', \delta)$

So $d_H(K, K'') \leq \delta$

and we still have $K'' \in \mathcal{A}(B,\varepsilon)$

Theorem For a typical $K \in \mathcal{K}$

$$\bigcup_{(x,v)\in K} x + \mathbb{R}v \setminus \{x\}$$

has Leb. measure zero.

Proof Let B_i be a countable collection of balls such that every point in $\mathbb{R}^2$ is covered by a ball of arbitrarily small diameter.

B_i

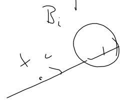
Take $K \in \overbrace{\bigcap_i \mathcal{A}(B_i)}^{\text{residual}}$

Let $E = \bigcup_i \left(B_i \cap \underbrace{\bigcup_{\substack{(x,v)\in K \\ x \notin 2B_i}} (x+\mathbb{R}v)\setminus\underbrace{\{x\}}_{\text{unnecessary}}}_{\text{measure is zero since } K \in \mathcal{A}(B_i)} \right)$

So $\lambda(E) = 0$

Claim E contains all the punctured lines $x + v\mathbb{R} \setminus \{x\}$ where $(x,v) \in K$

Proof

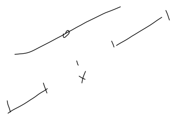


Take any $(x,v) \in K$

Let $y \in (x + v\mathbb{R}) \setminus \{x\}$

Take B_i such that $y \in B_i$, $x \notin 2B_i$

Then from def. of E we see that $y \in E$



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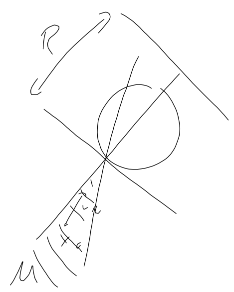
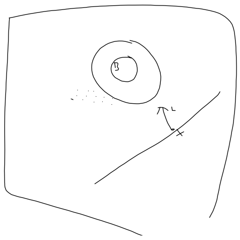




$K'' \in \underline{\underline{\mathcal{K}}}$

$\cup U_i \supset [0,1]^2 \setminus 2B$

Nikodym sets in higher dimensions?

Theorem

There is $N \subset \mathbb{R}^n$ that is measurable, Lebesgue measure zero, and for every $x \in \mathbb{R}^n$ there is a "punctured" hyperplane going through x inside N.

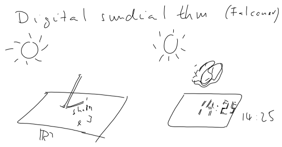
Proof $\cancel{A}$ $\mathcal{K} = \underset{[0,1]^n}{\mathbb{R}^n} \times S^{n-1}$

Keep Lemma 1, 0 in the plane.
After that $\mathbb{R}^2 \rightsquigarrow \mathbb{R}^n$

cones in plane $\times \mathbb{R}^{n-2}$

$\square$

Digital sundial thm (Falconer)



<u>Thm</u> For every projection $\mathbb{R}^3 \xrightarrow{P} \mathbb{R}^2$ we are given a prescribed set $A_P \subset \mathbb{R}^2$.

Then $\exists E \subset \mathbb{R}^3$ such that

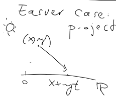
$$\mathrm{proj}_P E \supset A_P$$

$$\lambda^2(\mathrm{proj}_P E \setminus A_P) = 0$$

for (almost) all projections P.

Easier case:

projections from $\mathbb{R}^2$ to $\mathbb{R}$



$$\mathrm{proj}_t : (x,y) \mapsto x + yt$$

<u>Thm</u> Assume that prescribed "projections" (sets) $A_t \subset \mathbb{R}$ are given such that

$$A = \{(t,x) : x \in A_t\} \subset \mathbb{R}^2$$

is measurable.

Then there is a Borel set $E \subset \mathbb{R}^2$ such that $\mathrm{proj}_t E \supset A_t$

$$\lambda(\mathrm{proj}_t E \setminus A_t) = 0$$

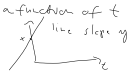
$\forall t$ (or almost every t)

<u>Proof using Nikodym sets</u>

$$A = \{(t,x) \in \mathbb{R}^2 : x \in A_t\}$$

(x,y) $\quad \mathrm{proj}_t(x,y) = x + yt$

as a function of t

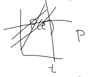


line slope y through $(0,x)$

We can get any non-vertical line on the plane

$E \subset \mathbb{R}^2$

$$P(E) = \{(t, x+yt) : (x,y) \in E\}$$



$P(E) \cap$ vertical line at position t is exactly $\mathrm{proj}_t E$.

Aim: E such that $P(E) \approx A$

We need

$$P(E) \supset A$$

For (almost) every t,

$$\lambda(P(E)_t \setminus A_t) = 0$$

where $X_t = \{x : (t,x) \in X\}$

For almost every t, by Fubini, this is equivalent to

$$\lambda^2(P(E) \setminus A) = 0$$

and $P(E) \supset A$.

Davies's thm finishes the proof

<u>Thm (Davies)</u> $A \subset \mathbb{R}^2$ measurable.

Then $\exists P \subset \mathbb{R}^2$ $\quad P \supset A$

non-vertical

P is a union of lines

$$\lambda^2(P \setminus A) = 0.$$

<u>Proof</u> $P = A \cup \{$punctured lines in a fixed Nikodym set that go through a point of $A\}$

projective plane

<u>Remark</u> In $\mathbb{R}^2$, there is point–line duality

How to prove digital sundial thm $\mathbb{R}^3 \to \mathbb{R}^2$?

1. Read Falconer's paper.
2. Similar to our proof $\mathbb{R}^2 \to \mathbb{R}$ & Replace Nikodym set by something else.
   You might end up using the following.

Thm There is a Nikodym set N in the plane such that

$\forall x, x' \in \mathbb{R}^2$

$\exists v \in S^1$