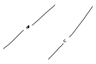
Aimi-Nukodym +

- Cordoba L^2 argument for dimension of Besicovitch sets in the plane
- Mapping sets of positive meas <u>onto</u> balls by Lipschitz maps

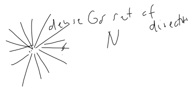
Thm There is a Nikodym set $N \subset \mathbb{R}^2$ (of Leb. measure 0) such that $\forall\, x, x' \in \mathbb{R}^2$ there are parallel lines ℓ, ℓ' through x and x'

$$\ell \setminus \{x\} \subset N$$
$$\ell' \setminus \{x'\} \subset N$$



This is a special case (corollary) of the following

Thm There is a $N \subset \mathbb{R}^2$ of Lebesgue measure 0 such that $\forall\, x \in \mathbb{R}^2$ for a dense G_δ set of directions there are punctured lines in those directions through x in N.



Proof of Thm 2 $\Rightarrow$ Thm 1

"dense $G_\delta \cap$ dense $G_\delta \neq \emptyset$" $\square$

Proof of Thm 2

Step 1 Nikodym set all punctured lines have $\overset{45}{\vee}$ angle $\in (-45^\circ, 45^\circ)$ from vertical

Step 2 Apply affine maps, replace 45° by any fixed α

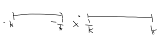
Step 3 By taking the union of countably many (rotated) Nikodym sets of step 2 we obtain punctured lines in a dense set of directions (through every $x \in \mathbb{R}^2$). Obtain N_3

Step 4 Let N_4 be <u>any</u> G_δ set of Lebesgue meas zero such that $N_4 \supset N_3$.

There is such a set N_4 by regularity of Lebesgue measure

Claim N_4 is good: through every x, it contains punctured lines in a dense G_δ set of directions.

Proof Fix x.
Let $N_4 = \bigcap_{n=1}^{\infty} \mathcal{U}_n$, $\mathcal{U}_n$ are open



$S_k = [-k, -\frac{1}{k}] \cup [\frac{1}{k}, k]$

Claim A N_4 contains "dense-G_δ"-many rotated copies of S_k

Proof: Claim A $\Rightarrow$ Claim
1) $\bigcup S_k = \mathbb{R} \setminus \{0\}$
2) Intersection of countably many dense-G_δ set (of directions) is still a dense G_δ set of directions. $\square$

Proof of Claim A

$X_k = \{ \underset{\text{direction}}{\alpha \in S^1} : S_k$ rotated by α is in $N_4 \}$

Aim: X_k is dense G_δ.

We know X_k is dense since $N_4 \supset N_3$ and N_3 contains punctured lines through x in a dense set of directions.

$X_k^n = \{\alpha \in S^1 : S_k$ rotated by α is in $\mathcal{U}_n \}$

Then X_k^n is open ✓ because S_k compact. Recall $N_4 = \bigcap \underset{\text{open}}{\mathcal{U}_i}$

and $X_k = \bigcap X_k^n$

$\square$

Cordoba's L^2 argument
for dimension of Besicovitch
sets in the plane

Recall

Def (Minkowski / Box dimension)

$A \subset \mathbb{R}^n$ bounded.
$N_\delta(A)$ = minimum number of sets of diameter $\leq \delta$ needed to cover A

diam $X = \sup_{x,y \in X} \|x-y\|$

$$\overline{\dim}_M A = \limsup_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}$$

$\underline{\dim}_M A = \liminf$

Claim $N_\delta(A)\,\delta^n \approx Vol^n(\delta\text{-neighb. of } A)$

~~for~~ to be precise

$N_\delta(A)\delta^n \leq C_n Vol^n(\quad)$

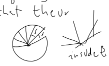
$C_n N_\delta(A) \quad \geq Vol^n(\quad)$

Thm Let E be a Besicovitch set in the plane
(so E contains a unit line segment in every direction)
Then $\dim_M E = 2$.

Proof We will estimate

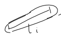
$$Area(\underbrace{\delta\text{-neigh of } E}_{B(E,\delta)})$$

Choose $\frac{1}{\delta}$ many line segments L_i in E such that their angles $\geq \delta$.



inside E

Let $T_i = B(L_i, \delta) \leftarrow \delta$-tube



T_i tube of width 2δ

Then $\forall i$ $T_i \subset B(E,\delta)$

$\cup T_i \subset B(E,\delta)$

Aim: $Area(\cup T_i) \geq \ldots$

Lemma

$$Area(T_i \cap T_j) \leq C \frac{\delta^2}{\theta(T_i,T_j)+\delta}$$

where $\theta(T_i,T_j)$ is the angle between L_i and L_j

Lemma

$$Area(\cup T_i) \geq \frac{(\sum Area(T_i))^2}{\sum_i \sum_j Area(T_i \cap T_j)}$$

(here T_i can be arbitrary)

Proof let $\chi(T_i)$ be the function

$$\chi(T_i)(x) = \begin{cases} 1 & x \in T_i \\ 0 & x \notin T_i \end{cases}$$

$\int \chi(T_i) = Area(T_i)$

Apply Cauchy-Schwartz inequality for $\sum \chi(T_i)$ and $\chi(\cup T_i)$

$$\underbrace{\int (\sum \chi(T_i))^2}\ \underbrace{\int \chi(\cup T_i)^2}_{Area(\cup T_i)} \geq \underbrace{\left(\int \sum \chi(T_i)\chi(\cup T_i)\right)^2}_{T_i \subset \cup_j T_j}$$

$$\int \sum_i \sum_j \chi(T_i)\chi(T_j) \qquad \left(\int \sum \chi(T_i)\right)^2$$

$$\sum_i \sum_j \underbrace{\int \chi(T_i)\chi(T_j)}_{Area(T_i \cap T_j)} \qquad \left(\sum Area(T_i)\right)^2$$

□

By the Lemma

$$Area(\cup T_i) \geq \frac{(\sum Area(T_i))^2}{\sum_i \sum_j Area(T_i \cap T_j)}$$

$$\geq C \frac{\frac{1}{\delta} \cdot 2\delta}{??}$$

$$Area(T_i \cap T_j) \leq C \frac{\delta^2}{\theta(T_i,T_j)+\delta}$$

Fix i.
What are $\theta(T_i,T_j)$?

0 when $j = i$
δ for two js
2δ for two js
3δ

$$\sum_j C \frac{\delta^2}{\delta + \theta(T_i,T_j)} \leq C\delta^2 \left(\frac{1}{\delta} + 2 \cdot \frac{1}{2\delta} + 2 \cdot \frac{1}{3\delta} + \ldots\right)$$

$$\leq C\delta^2 \frac{2}{\delta} \underbrace{\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{1/\delta}\right)}_{\approx \log\frac{1}{\delta}}$$

$$\leq C\delta^2 \frac{2}{\delta} \log\frac{1}{\delta} = C' \delta \log\frac{1}{\delta}$$

$$\sum_i \sum_j C \frac{\delta^2}{\delta + \theta(T_i,T_j)} \leq \frac{1}{\delta} \cdot C' \delta \log\frac{1}{\delta} = C' \log\frac{1}{\delta}$$

So we obtain

$$Area(\cup T_i) \geq \frac{C}{C' \log\frac{1}{\delta}} = C'' \frac{1}{\log\frac{1}{\delta}}$$

So $N_\delta(\overbrace{\cup T_i}^{\subset B(E,\delta)})\,\delta^2 \approx C'' \frac{1}{\log\frac{1}{\delta}}$

$N_\delta(E) \geq \delta^{-2} C'' \frac{1}{\log\frac{1}{\delta}} \quad C'''$

$$\Rightarrow \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \geq 2$$

$\Rightarrow \underline{\dim}_M E \geq 2 \Rightarrow \dim_M E = 2$

What about Hausdorff dimension.

<u>Def</u> $E \subset \mathbb{R}^n$. $\dim_H(E)$ is the infimum of those $s > 0$ for which for every $\varepsilon > 0$ we can cover E by discs $B(x_i, r_i)$ such that $\sum_i r_i^s < \varepsilon$.

<u>Remark</u>
If $s < \dim_H E$, then $\sum_i r_i^s \geq c_s > 0$ for any family of discs covering E.

<u>Thm</u> let E be a Bes. set in the plane. Then $\dim_H E = 2$.

<u>Proof</u>
Cover E by discs $B(x_i, r_i)$
we may assume that every r_i is of the form 2^{-k}.

Let $E_k = E \cap \bigcup_{r_i = 2^{-k}} B(x_i, r_i)$

Then $E = \bigcup_{k=1}^{\infty} E_k$

For $t \in \mathbb{R}$, let $L_t \subset E$ be a unit line segment of slope t

Let $S_k = \left\{ t \in [0,1] : \lambda_1(E_k \cap L_t) \geq \frac{6}{\pi^2 k^2} \right\}$

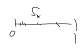
Then $\bigcup_k S_k = [0,1]$. Proof: $\sum_{k=1}^{\infty} \frac{6}{\pi^2 k^2} = 1$

and $\bigcup (E_k \cap L_t) = L_t$ (length 1)

Choose k such that

$$\lambda_1(S_k) \geq \frac{6}{\pi^2 k^2}$$

Now we will use the L^2 (Cauchy-Schwartz) argument for line segments L_t with $t \in S_k$ and the set $E_k' = B(\underbrace{E_k}_{E \cap \bigcup B(x_i, 2^{-k})}, 2^{-k})$

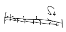


$\lambda(S_k) \geq \frac{6}{\pi^2} \frac{1}{k^2}$

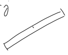
Choose about $\frac{\lambda(S_k)}{2} 2^k$ points in S_k that are 2^{-k}-separated

$t_1 < t_2 < \dots < t_N \in S_k$ and $|t_{i+1} - t_i| \geq 2^{-k}$

$N \doteq \frac{\lambda(S_k)}{2} 2^k$



$$T_j = E_k' \cap B(L_{t_j}, 2^{-k})$$



$j = 1, \dots, N \approx \frac{\lambda(S_k)}{2} 2^k$

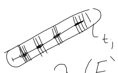
$\text{Area}(T_j) = ?$

Recall: $S_k = \left\{ t \in [0,1] : \lambda_1(E_k \cap L_t) \geq \frac{6}{\pi^2} \frac{1}{k^2} \right\}$

So $\forall j \quad t_j \in S_k$

$$\lambda_1(E_k \cap L_{t_j}) \geq \frac{6}{\pi^2} \frac{1}{k^2}$$

$E_k' = B(E_k, 2^{-k})$



$$\lambda_2(E_k' \cap B(L_{t_j}, 2^{-k})) \geq \frac{6}{\pi^2} \frac{1}{k^2} \frac{2}{2^k}$$

$\text{Area}(T_j) \geq \frac{2 \cdot 6}{\pi^2} \frac{1}{k^2} 2^{-k}$

$\geq c \, 2^{-k} / k^2$

<u>Lemma</u>
$$\text{Area}(T_i \cap T_j) \leq C \frac{\delta^2}{\theta(T_i, T_j) + \delta}$$

as this holds for the full tubes

By the C-S Lemma $(\delta = 2^{-k})$

$\log \frac{1}{2^{-k}} \cdot \text{Area}(E_k') \geq (c/k^2)^2$

$\underbrace{\text{Area}(E_k')}$ $\geq c' \frac{1}{k^4} \frac{1}{\log 2^{+k}} = c'' \frac{1}{k^5}$

$E_k = E \cap \bigcup B(x_i, 2^{-k})$

$\text{Area}(\bigcup B(x_i, 2^{-k})) \geq c'''' \frac{1}{k^5}$

discs of radius 2^{-k} is $\geq c \frac{1}{k^5} 4^k$

So $\sum_i r_i^s \geq c \frac{1}{k^5} 4^k (2^{-k})^s = c \frac{1}{k^5} 2^{k(2-s)}$

If $s < 2$, then this is $\geq c_s > 0$ not depending k

$\square$