Question (Laczkovich)

Let $A \subset \mathbb{R}^d$ be a measurable set of positive Lebesgue measure.
Is there a Lipschitz map $f: A \to \mathbb{R}^d$ such that $f(A)$ has non-empty interior?

Observations

1. We could ask for a Lipschitz map $f: \mathbb{R}^d \to \mathbb{R}^d$ such that $f(A)$ has non-empty interior
(equivalent question, exercise)

2. We can replace Lipschitz by contraction

3. Non-empty interior can be replaced by ball / cube.

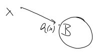
Proof. Assume $\operatorname{int} f(A) \neq \emptyset$

Let $B \subset \operatorname{int} f(A)$ be a ball/cube

Let $h: \mathbb{R}^d \to B$

For $x \in \mathbb{R}^d$, let $h(x)$ be the closest point of B to x.

If B was convex, this h exists and 1-Lipschitz.



Now $h \circ f$ maps A onto the ball/cube B.

Question (Laczkovich, another form)

Let $A \subset \mathbb{R}^d$ be meas., $\lambda(A) > 0$.
Is there a contraction $f: \mathbb{R}^d \to \mathbb{R}^d$ that maps A onto a ball.

What's known?

If $d=1$, this is true and an exercise.

If $d=2$, then it is true. (Preiss, Matoušek, Ung)

If $d \geq 3$, then unsolved

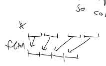
$d=1$. $A \subset \mathbb{R}$, $\lambda(A) > 0$. (measurable)

Let $K \subset A$ be compact, $\lambda(K) > 0$.
Then there is a "best" way to map K onto an interval.
Let
$$f(x) = \lambda\big((-\infty, x] \cap K\big).$$

Claim: 1-Lipschitz, since
$$\lambda\big((x, x'] \cap K\big) \leq |x'-x|$$

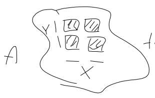
Claim: $\underbrace{f(K)}_{\substack{\text{cont. image} \\ \text{of compact set,} \\ \text{so compact}}} = [0, \lambda(K)]$.

and $f(K)$ is dense in this interval.



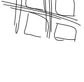
$d=2$, $A \subset \mathbb{R}^2$, $\lambda(A) > 0$

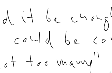
If we had that
$$A \supset X \times Y$$
where X, Y have positive Leb. measure, then we could repeat the previous argument in both coordinates.



Problem is, for most A there are no such X, Y.

$X \times Y$ complement is
$$(X^c \times \mathbb{R}) \cup (\mathbb{R} \times Y^c)$$

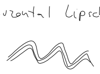




"Would it be enough if A^c could be covered by "not too many" almost horizontal/vertical curves?"

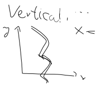
<u>Thm</u> Let $E \subset [0,1]^2$ be compact,
let $\lambda(E) = \varepsilon$.
Then E can be covered by finitely many horizontal/vertical Lipschitz strips of total width $\leq C\sqrt{\varepsilon}$

<u>Def</u> Horizontal Lipschitz strip of width w is



a w-neighbourhood of the graph of a $y = f(x)$ 1-Lipschitz function.

Vertical ... $x = f(y)$



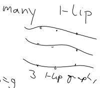
<u>Thm</u> $E \subset [0,1]^2$, $\lambda(E) = \varepsilon$, E is measurable
Then E can be covered by countably many Lip strips of total width $C\sqrt{\varepsilon}$.
(Alberti, Csörnyei, Preiss)

Connection to Laczkovich's question:
$A \subset \mathbb{R}^2$, $\lambda(A) > 0$, we can choose a small square S such that

$$\lambda(A \cap S) \geq (1-\varepsilon)\lambda(S)$$

Then $E = S \setminus (A \cap S) \subset S$.

<u>Thm</u> (Matousek + Alberti/Csörnyei/Preiss)
$E \subset [0,1]^2$, finite set of n points
Then E can be covered by $C\sqrt{n}$ many 1-Lip graphs (horizontal/vertical)



n=9 3 1-Lip graphs

<u>Lemma</u> Let $E \subset \mathbb{R}^2$, finite, of n points.
Then there is a horizontal or vertical 1-Lip graph ($y = f(x)$ or $x = f(y)$) that covers $\geq \sqrt{n}$ points.

<u>Proof that this lemma implies previous:</u>
Use iteratively.
$E_0 = E$, cover $\geq \sqrt{n}$ points, the rest is E_1
$|E_1| \leq n - \sqrt{n}$.
We cover $\geq \sqrt{|E_1|}$. The rest E_2, and so on

If $|E_k| \geq \frac{n}{2}$, then we cover at least $\sqrt{\frac{n}{2}}$ point by the next 1-Lip graph.

So if $k = \frac{n/2}{\sqrt{n/2}}$, then $|E_k| \leq \frac{1}{2}|E_0|$

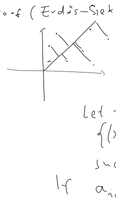
$E_0, E_1, E_2, \ldots, E_k$ $\quad k = \sqrt{\frac{n}{2}}$
knocking $\geq \sqrt{\frac{n}{2}}$ points

$$n \xrightarrow[\text{steps}]{\sqrt{\frac{n}{2}}} \frac{n}{2} \xrightarrow[\text{steps}]{\sqrt{\frac{n}{2^2}}} \frac{n}{2^2} \longrightarrow \cdots$$

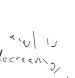
So $\sqrt{\frac{n}{2}} + \sqrt{\frac{n}{2^2}} + \sqrt{\frac{n}{2^3}} + \ldots = C\sqrt{n}$ steps are enough.

<u>Lemma</u> (Erdős–Szekeres)
Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers.
There is a monotone subsequence of length $\geq \sqrt{n}$.

Proof (Erdős–Szekeres $\Longrightarrow$ lemma about 1-Lip graph with $\sqrt{n}$ points)



Rotate E by $45°$ clockwise.
Let the obtained set be $\{(x_1, a_1), (x_2, a_2), \ldots, (x_n, a_n)\}$ such that $x_1 < x_2 < \ldots < x_n$.
If $a_{i_1}, a_{i_2}, \ldots, a_{i_N}$ is monotone increasing



then the resp. points on E are on 1-Lip graph $x = f(y)$

If $(a_{i_1}, \ldots, a_{i_N})$ is monotone decreasing, then $(x_{i_1}, a_{i_1}) \ldots$ rotated back $45°$ anticlockwise are on a 1-Lip graph $y = f(x)$.

Proof of Erdös-Szekeres

Claim $n = km+1$

Let $a_1, \ldots, a_n \in \mathbb{R}$

then there is a monotone increasing subsequence of length $k+1$

OR there is a monotone decreasing subsequence of length $m+1$

Proof

Let l_i be the length of the longest monotone increasing subsequence starting with a_i

Then $1 \leq l_i$. If $\exists i$ $l_i \geq k+1$, the we are done

So $\forall i \in \{1, \ldots, n\}$ $l_i \in \{1, 2, \ldots, k\}$

Recall $n = mk+1$, so $\exists j \in \{1, \ldots, k\}$

$$\left| \{ i : l_i = j \} \right| \geq m+1$$

Claim These indices give a monotone decreasing subsequence (of length $\geq m+1$)

$l_i = j$ $l_{i'} = j$

(If $a_i \leq a_{i'}$ and $i < i'$, then $l_i \geq 1 + l_{i'}$)

□

Remark

This Erdös-Szekeres lemma is a special case of Dilworth's theorem

Dilworth's theorem

Let P be a finite partially ordered set. Then the size of longest antichain equals the minimum number of chains that cover P.

Given $a_1, a_2, \ldots, a_n$ let the partial order be $i \leq_P j$ if $i \leq j$ and $a_i \leq a_j$

From here, exercise to show Dilworth $\Rightarrow$ Erdös-Szekeres

Recall

<u>Thm</u> (Alberti–Csörnyei–Preiss–Matoušek)

$E \subset [0,1]^2$, compact, $\lambda(E) = \varepsilon$.

The E can be covered by 1-Lip strips of total width $C\sqrt{\varepsilon}$. (finitely many).

<u>Proof</u>

Let n be a large integer.

Cover $[0,1]^2$ by $n \times n$ squares of side $\frac{1}{n}$.

Let E_n be the union of these squares that intersect E.

So $E \subset E_n$, and $E = \bigcap_{n=2^k} E_n$

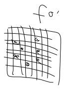
$E_2 \supset E_{2^2} \supset E_{2^3} \supset E_{2^4} \supset \dots$

So $\lambda(E_{2^k}) \to \lambda(E) = \varepsilon$.

So take a large n such that

$$\lambda(E_n) < 2\varepsilon.$$

Let F be the centres of the squares forming E_n.



$$|F| < n^2 2\varepsilon$$

Apply an earlier lemma to cover F by $C\sqrt{|F|}$ many 1-Lip graphs.

If we replace these 1-Lip graphs by 1-Lip strips of width $1/n$ then they cover all the squares forming E_n.

So $C\sqrt{|F|} \le C\sqrt{n^2 2\varepsilon}$ many strips of width $\frac{1}{n}$, so total width is

$$C\sqrt{2}\,\underline{\underline{\sqrt{\varepsilon}}}. \qquad \square$$

<u>Thm</u> $E \subset [0,1]^2$, measurable, $\lambda(E) = \varepsilon$, then E can be covered by countably many Lip strips of total width $C\sqrt{\varepsilon}$.

<u>Proof</u>

We can cover E by an open set of measure 2ε.

Choose a sequence $n_1 < n_2 < n_3 < \dots$ (powers of 2)

Let E_1 = union of n_1-grid squares in E.

let E_2 = union of n_2-grid squares that are in E but not covered by E_1.

E_3 = n_3-grid E_2.

and so on.

<u>Claim</u> $E = \cup E_i$, these are not disjoint but

$$2\varepsilon \ge \lambda(E) = \sum \lambda(E_i) \qquad \square$$

Use previous thm ~~(or its proof)~~,

We can cover E_i by $\uparrow C\sqrt{\lambda(E_i)}$ 1-Lip strips of total width

So E is covered by countably many Lip-strips of total width $\sum_i C\sqrt{\lambda(E_i)} \le ?$

Choose n_1 such that

$$\lambda(E_1) \ge \frac{1}{2}\lambda(E).$$

Choose n_2 large enough to have

$$\lambda(E_2) \ge \frac{1}{2}\left(\lambda(E) - \lambda(E_1)\right)$$

$$\vdots$$

$$\lambda(E_k) \ge \frac{1}{2}\Big(\underbrace{\lambda(E) - \sum_{i=1}^{k-1}\lambda(E_i)}_{\lambda(E \setminus \bigcup_{i=1}^{k-1} E_i)}\Big)$$

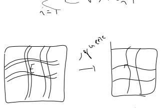
then

$$\lambda(E_k) \le \frac{1}{2^{k-1}}\lambda(E).$$

Then $\sum_{i=1}^{\infty} C\sqrt{\lambda(E_i)} \le C'\sqrt{\lambda(E)}$.

$\square$

Next week:



A, $E = A^c$, $f(E)$ = union of ~~finitely~~ countably many Lip curves

$f(A) \supset \underbrace{f([0,1]^2)}_{\text{but smaller squares}} \setminus f(E)$

dense

$f(A)$ compact if A is compact.