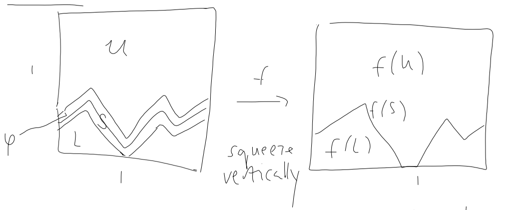
Thm (Matousek - Alberti - Csörnyei - Preiss)

If $A \subset \mathbb{R}^2$, measurable, $\lambda(A) > 0$,
then there is a lipschitz map f
of the plain onto $Q = [0,1]^2$
such $f(A) = Q$.

Proof will be a countable iteration of
finitely many iterations of the
following lemma.

Lemma



"Contracting along φ by width w"

$$L = \{(x,y) \in [0,1]^2 : y \le \varphi(x) - w\}$$
$$U = \{(x,y) \in [0,1]^2 : y \ge \varphi(x) + w\}$$
$$S = \{(x,y) \in [0,1]^2 : \varphi(x) - w \le y \le \varphi(x) + w$$

$$f : [0,1]^2 \to [0,1] \times [0, 1-2w]$$

$$f(x,y) = \begin{cases} (x, \min(y, 1-2w)) & (x,y) \in L \\ (x, \max(y-2w, 0)) & \text{if } (x,y) \in U \\ (x, \max(0, \min(1-2w, \varphi(x)-w))) & \text{if } (x,y) \in S \end{cases}$$

Then this $f = f_{\varphi, w}$ is
1-Lipschitz in d_∞

$$d_\infty((x_1,y_1),(x_2,y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)$$

$\square$

Remark We will actually show that
$f(A) = Q$
$f(\mathbb{R}^2 \setminus A)$ is of zero Lebesgue measure.

Stronger result:
$f(\mathbb{R}^2 \setminus A)$ is "rectifiable"
in fact, can be covered
by countably many
Lipschitz curves

Proposition

Let $Q=[0,a]^2$ be an axis parallel square, let $K \subseteq Q$, compact, with $\lambda(K) \geq \lambda(Q)\,(1-\varepsilon)$ $0<\varepsilon<\frac{1}{100}$. Then we can find a 1-Lipschitz (in d_∞) map $g: Q \to Q'$ $Q'=[0,a']^2$ such that

(i) $a' \geq a(1-\sqrt{\varepsilon})$

(ii) $d_\infty(p, g(p)) \leq a\sqrt{\varepsilon}$ $\forall p \in Q$

(iii) $\lambda(g(K)) \geq \lambda(Q')(1-0.9\varepsilon)$

Proof of Prop $\Rightarrow$ Thm.

$A \subset \mathbb{R}^2$, $\lambda(A)>0$

By Leb density thm, we can find an axis-parallel square Q in which the density of A is $\geq 1-\frac{1}{1000}$

Let $K \subset A \cap Q$ be compact with $\lambda(K) \geq \left(1-\frac{2}{1000}\right)\lambda(Q)$

We may assume that $Q=[0,a]^2$ (by applying a translation)

We can apply the Proposition.

$K_1 \subset Q_1 = Q = [0,a_1]$ $(a_1 = a)$ $K_1 = K$

$\exists f_1$ by Prop

$K_2 = f_1(K_1)$, $Q_2=[0,a_2]^2$

apply Prop again for K_2, Q_2

$K_3 = f_2(K_2)$, $Q_3=[0,a_3]^2$

and so on

We also have the sequence ε_n.

$\varepsilon_{n+1} = 0.9\,\varepsilon_n$

We know: $a_{n+1} \geq a_n(1-\sqrt{\varepsilon_n})$

$a_{n+1} \geq a_n\left(1-\sqrt{0.9^{n-1}\varepsilon_1}\right)$

So $a_n \geq \frac{1}{10} a_1$

$d_\infty(p, f_n(p)) \leq a_n\sqrt{\varepsilon_n}$

$\leq a_1\sqrt{0.9^{n-1}\varepsilon_1}$

So $\sum_{n=1}^{\infty} \sup_{p\in Q_n} d_\infty(p, f_n(p)) \leq$

$\leq \sum_{n=1}^{\infty} a_1\sqrt{0.9^{n-1}\varepsilon_1}$ convergent

Consider

f_1

$f_2 \circ f_1$

$f_3 \circ f_2 \circ f_1$, and so on

Take limit

$F = \lim_{n\to\infty} f_n \circ f_{n-1} \circ \dots \circ f_1$

Does it exist?

$\forall n$ $f_n \circ \dots \circ f_1$ is 1-Lipschitz and F exists since that $\sum d_\infty \dots$ converges ✓

Now $F: Q_1 \longrightarrow [0, \underbrace{\lim a_n}_{\geq 0.1 a_1 > 0}]^2$

1-Lipschitz

$\to \dots$ $F(K) \subset [0, a_\infty]^2$

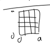
compact

and it is dense

since $\lambda(K_n)/\lambda(Q_n) \to 1$. $\square$

It is left to prove the Proposition.

Proof of Prop



$K \subset [0,a]^2$, $\lambda(K) \geq \lambda(Q)(1-\varepsilon)$.
Lay the $n \times n$ grid on $[0,a]^2$
of squares
of side a/n.

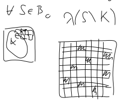
By Leb density theorem,
we can choose n so large
that: - in $n^2\left(\frac{\lambda(K)}{\lambda(Q)} - \delta\right)$ many
squares S
$\lambda(K \cap S) \geq (1-\delta)\lambda(S)$

- in $n^2\left(1 - \frac{\lambda(K)}{\lambda(Q)} - \delta\right)$ many
grid squares S
$\lambda(K \cap S) < \delta \lambda(S)$.

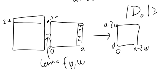
(Exercise: prove this from Leb density thm)
(that is, $\forall \delta > 0 \; \exists n$ ..)

So now fix n that works $\delta = \frac{1}{16}$.

Let B_0 denote the family of
those grid squares S
for which $\lambda(K \cap S) < \frac{1}{16} \underbrace{\lambda(S)}_{a^2/n^2}$
If n is large enough:
$|B_0| \geq \frac{\varepsilon}{2} \cdot n^2$.
assuming $\lambda(K) = (1-\varepsilon)\lambda(Q)$
$\forall S \in B_0 \quad \lambda(S \setminus K) \geq \frac{15}{16} \lambda(S)$.



Now consider the centres of the
squares of B_0. ($\varepsilon n^2/2$ points)
Find a horizontal/vertical 1-Lipschitz
graph φ that goes through
$\geq \sqrt{\varepsilon n^2/2}$ many centres
(see ~~the~~ Erdős-Szekeres lemma
from last week)
Let $w = a/n$.
Now apply Lemma "contracting
along φ by w"
(Let D_0 be those grid squares
whose centre is covered by φ)
$|D_0| \geq \sqrt{\varepsilon n^2/2}$.



Lemma $f_{\varphi,w}$

Observation When we apply $f_{\varphi,w}$
this maps each grid square S
into one grid square.

Why: recall $w = a/n$

Indeed.

Let g_0 be this $f_{\varphi,w}$
composed with the map
that contract the
last two columns of
grid squares onto
the vertical line
segment.

Observation
g (and $f_{\varphi,w}$) maps squares
of D_0 onto Lipschitz curves

Let $B_1 = \{$ grid squares S:
$\exists S' \in B_0 \quad g_0(S') \subset S$
and $\lambda(g_0(S')) > 0 \}$

If $S' \in D_0$ then
$\lambda(g_0(S')) = 0$.

So $|B_1| \leq |B_0 \setminus D_0| = |B_0| - |D_0|$
Recall: $|D_0| \geq \sqrt{|B_0|}$.

Iterate this argument, now for
B_0 replaced by B_1

$B_0, B_1, B_2, \ldots, B_t$ families of grid squares

$|B_{n+1}| \leq |B_n| - \sqrt{|B_n|}$

Let t be minimal for which
$|B_t| \leq \underbrace{|B_0|}_{\geq \frac{\varepsilon n^2}{2}} - \frac{\varepsilon n^2}{5}$

Claim $t < n\sqrt{\varepsilon}/2$

Proof $\forall k = 0, \ldots, t-1$
$|B_k| \geq \frac{\varepsilon n^2}{2} - \frac{\varepsilon n^2}{5} > \frac{\varepsilon n^2}{4}$
So $\sqrt{|B_k|} > \sqrt{\varepsilon}\, n/2$
So $|B_{k+1}| \leq |B_k| - \sqrt{\varepsilon}\, n/2$
So $|B_t| \leq |B_0| - t\sqrt{\varepsilon}\, n/2$
exercise ☐

So do $t \leq n\sqrt{\varepsilon}/2$ iterations.

Now $F = g_t \circ g_{t-1} \circ \cdots \circ g_0$.
Claim: this is the F we were
looking for.
1-Lip,
$d_\infty(p, F(p))$ is small
Q_t is large

check (exercise)...
check Matoušek's paper.

$$F : Q_0 \longrightarrow Q_n \qquad Q_n = [0, a_n]^2 \quad a_n \geq 0.9\, a_0$$

$$\lambda\big(F(Q_0 \setminus K)\big) < \tfrac{1}{2}\lambda(Q_0 \setminus K)$$

$$\boxed{F(Q_0) = Q_n}$$

$$\lambda\big(\underbrace{F(K)}_{\supseteq Q_t \setminus F(Q_0 \setminus K)}\big) \geq \lambda(Q_t) - \lambda\big(F(Q_0 \setminus K)\big)$$

$$Q_t = \underbrace{F(K) \cup F(Q_0 \setminus K)}_{F(Q_0)}$$

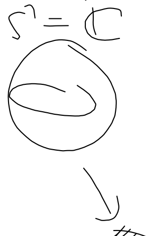
$$|Q_t| = a_t^2$$

$$a_t = a_0 - 2\frac{a_0}{n} \cdot t \qquad t \leq 2\sqrt{\varepsilon}\, n$$

$$\Rightarrow a_t \geq a_0 - 2\frac{a_0}{n} 2\sqrt{\varepsilon}\, n$$

$$a_t \geq a_0(1 - 2\sqrt{\varepsilon})$$

$S^2 = \hat{\mathbb{C}}$



$f(\hat{\mathbb{C}}) = $ squ

f on $\hat{\mathbb{C}} \setminus E$ is area preserving