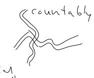
We have seen

<u>Thm</u> $A \subset \mathbb{R}^2$, measurable, $\lambda^2(E) = \varepsilon$

Then we can cover A by countably many 1-Lipschitz strips of total width $C\sqrt{\varepsilon}$.



Trivial
<u>Corollary</u>
If $A \subset \mathbb{R}^2$, $\lambda(E) = 0$

Then for every $\varepsilon > 0$ we can cover A by Lip strips of total width $\leq \varepsilon$.

(Non-trivial)
<u>Corollary</u>
Let $A \subset \mathbb{R}^2$, $\lambda(A) = 0$ (Lebesgue meas zero)

Then $A = A_1 \cup A_2$

such that $\forall \varepsilon > 0$,

we can cover A_1 by countably many "horizontal" 1-Lip strips $(f(x) - w \leq y \leq f(x) + w)$ of total width $\leq \varepsilon$

and

A_2 by $\underbrace{\text{"vertical"}}_{x - w \leq f(y) \leq x + w}$ Lip strips of total width $\leq \varepsilon$.

<u>Remark</u> A_1 and A_2 does not depend on ε.

<u>Proof of Corollary from Thm</u>

Thm: $\forall \varepsilon > 0$ we can cover A by "horizontal" and "vertical" strips.

Let $A_1^\varepsilon = A \cap$ (union of "horizontal" strips)

$A_2^\varepsilon = A \cap$ ("vertical")

Then let.

$$A_1 = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_1^{2^{-n}}.$$

$A_2 = A \setminus A_1$.

$\forall N \quad A_1 \subset \bigcup_{n=N}^{\infty} A_1^{2^{-n}}$, so A_1 can be covered by countably many horizontal Lip strips of total width $\sum_{n=N}^{\infty} 2^{-n} = 2^{-N+1} \xrightarrow{N\to\infty} 0$

and

$A_2 = A \setminus A_1 \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_2^{2^{-n}}$ $\square$

Weak tangent fields

<u>Def</u> $E \subset \mathbb{R}^2$ Borel

Let $\tau : E \to \underbrace{G(2,1)}_{\text{space of lines through origin in the plane}}$ Borel

(it's like fixing one line through each point of E)

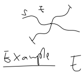
This τ is a weak tangent field to E if

for every C^1-smooth curve S

for $\mathcal{H}^1$-almost all points $p \in S \cap E$

$\tau(p)$ is the tangent of S at p.

<u>Example</u>
If E is itself a smooth curve, then τ = tangent line is a weak tangent field.



<u>Example</u> $E = [0,1]^2$. Then there is no weak tangent field.

Proof.

Consider horizontal line curves. Then τ must be horizontal at almost every point of <u>every</u> line so τ is horizontal Lebesgue-a.e in $[0,1]^2$.

And vertical as well. Contradiction. $\square$

Same argument gives:

<u>Thm</u> weak tangent field exists for E $\Rightarrow$ E has Lebesgue measure zero

<u>Thm</u> If $E \subset \mathbb{R}^2$, Borel, $\lambda(E) = 0$

Then there is a weak tangent field.

<u>Def</u> Weak tangent cone field:

$\tau : E \to$ cones

such that $\forall C^1$-curve S, at $\mathcal{H}^1$-a.e point of $S \cap E$ the tangent line of S is in the cone given by τ

<u>Lemma</u> Let E be a Borel nullset in the plane.
Let $E = E_1 \cup E_2$ such that $\forall \varepsilon > 0$ E_1 can be covered by "horizontal" 1-lip strips of total width $\leq \varepsilon$
and E_2 ... "vertical" .....

Let $\tau(x) = \begin{cases} & \text{if } x \in E_1 \\ & \text{if } x \in E_2 \end{cases}$ ← we may assume $E_2 = E \setminus E_1$

the cone containing lines of slope between -1 and $+1$

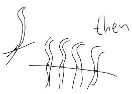
Claim: Then this τ is a weak tangent cone field to E.

<u>Proof</u> Assume S is a C^1-curve
we may assume that S is actually a graph of a $y = f(x)$ 1-Lipsch (C^1) function.

It is now enough to show that

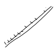
$$\mathcal{H}^1(S \cap E_2) = 0$$

if S = graph of a $1-\delta$ lip function
and E_2 is covered by "vertical" lip strips of total width ϑ

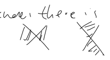


then $\mathcal{H}^1(S \cap E_2) \leq \underbrace{C_\delta}_{\approx \frac{1}{\delta}} \vartheta$

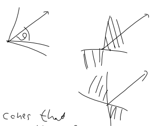
□



Now we know there is cone field



for every ϑ there is a cone field using cones

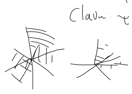


$\tau_\vartheta : E \to$ cones that are the ϑ rotated copy of or

Let Θ = countable dense set of angles
Let

$$\tau(p) = \underbrace{\bigcap_{\vartheta \in \Theta} \tau_\vartheta(p)}_{\text{countable intersection of cones}}$$

Claim: $\tau(p) =$ is a line (or a point)



<u>Lemma</u> τ is a tangent field, because intersection of cone fields is still a cone field

<u>Proof</u>
Fix S C^1-curve
We know that for $\mathcal{H}^1$-a.e point of $S \cap E$
the cone $\tau_\vartheta(p)$ contains the tangent of S at p.

So at a.e point of $S \cap E$
$\forall \vartheta \in \underset{\text{countable}}{\Theta}$ $\tau_\vartheta(p)$ contains the tangent of S at p

so $\tau = \bigcap \tau_\vartheta$ equals (contains) the tangent of S at p.

<u>Remark</u>
The defined τ is either a line or $\{0\}$ at every $p \in E$

<u>Remark</u>
If $E \subset \mathbb{R}^2$ is purely unrectifiable (so intersects every C^1-curve in $\mathcal{H}^1$-measure zero)
then <u>every</u> τ is a weak tangent field to E.
(Example
E = Cantor set $\times$ Cantor set) □

Things to read for this topic are:

Alberti - Csörnyei - Preiss:
Structure of null sets
in the plane and applications

Matoušek's paper

Paper by Ng or Khrushchev

Simple proof of a
thm on removable
singularities of analytic
functions satisfying a
Lipschitz condition

Remark

If $E \subset \mathbb{R}^2$ is a Kakeya set
(Besicovitch)
of zero Leb measure
there is still a weak tangent
field.
(What does it look like?)

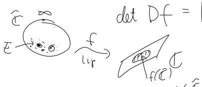
<u>Thm</u> (Uy 1979)

Let $E \subset \mathbb{C}$ compact
of positive Lebesgue measure.
Then there is a nonconstant
function $f : \mathbb{C} \to \mathbb{C}$
that is analytic in $\mathbb{C} \setminus E$
and Lipschitz in $\mathbb{C}$.
and bounded.
(And in fact, $f : \hat{\mathbb{C}} \to \mathbb{C}$,
analytic at ∞ as well.)

Observation (P. Jones)
This thm implies that E
is mapped onto a set
of non-empty interior
by this Lipschitz map f.

Why?
$\hat{\mathbb{C}}$ is a sphere topologically
Analytic maps are orientation preserving

<u>Remark.</u> $f : \mathbb{C} \to \mathbb{C}$, analytic at z,
then $f'(z) \in \mathbb{C}$
and $Df = 2 \times 2$ matrix
$\det Df = |f'(z)|^2 \geq 0$



$f(\hat{\mathbb{C}})$ is compact
(continuous image
of a compact set)

<u>Claim 1</u> $f(\hat{\mathbb{C}})$ has non-empty interior

<u>Proof</u>
f is not constant and
analytic at $\underbrace{\hat{\mathbb{C}} \setminus E}_{\text{open}}$
If $f'(z) \neq 0$ for some $z \in \hat{\mathbb{C}} \setminus E$
then $f(B(z, \varepsilon))$ has non-empty
interior.
If $f'(z) = 0 \ \forall z \in \underbrace{\hat{\mathbb{C}} \setminus E}_{\text{open}}$
then f is
constant (on connected
components of
$\hat{\mathbb{C}} \setminus E$)

Assume:
E is totally disconnected,
then $\hat{\mathbb{C}} \setminus E$ is connected.

Then $f|_{\hat{\mathbb{C}} \setminus E}$ is constant
so $f|_{\overline{\hat{\mathbb{C}} \setminus E}}$ is constant
and $\overline{\hat{\mathbb{C}} \setminus E} = \hat{\mathbb{C}}$. (as E has
empty
interior)

So Claim 1 holds if E is totally disconnected
(that we can assume)

<u>Claim 2</u> $f(\hat{\mathbb{C}}) = f(E)$.
(So Claim 1 + Claim 2 $\Rightarrow$ $f(E)$ has non-empty
interior)

<u>Claim 2b</u>
$f : \underbrace{S^2}_{\text{sphere}} \longrightarrow \mathbb{R}^2$ continuous
and f is orientation preserving
at all points of $S^2 \setminus E$ ← compact
then $f(S^2) = f(E)$.

f is smooth in $S^2 \setminus E$ and $\det Df(p) > 0$
for $p \in S^2 \setminus E$
(sign is independent of chart used.

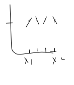
Why is this Claim true?
Let's look at it for $f : S^1 \to \mathbb{R}$.

<u>Claim</u> Let $f : S^1 \to \mathbb{R}$ continuous,
$E \subset S^1$ compact,
f is orient. preserving on $S^1 \setminus E$
then $f(S^1) = f(E)$.

<u>Proof</u> (Assume f is smooth in $S^1 \setminus E$.)
Let $g : [0,1] \to \mathbb{R}$, $g(0) = g(1)$.
$E \subset (0,1)$ compact
g is orient. preserving in $[0,1] \setminus E$
that is $g'(p) > 0$ for $p \in [0,1] \setminus E$
(or g is locally monotone
increasing in $[0,1] \setminus E$.

$\forall y \in f([0,1])$
$\exists x \in E \quad y = f(x)$

If f is ~~...~~ smooth $[0,1] \to \mathbb{R}$,
for ~~...~~ almost every $y \in \mathbb{R}$
for every $x \in f^{-1}(y)$
$f'(x) \neq 0$.



$p = \#\{i : f'(x_i) > 0\}$
$n = \#\{i : f'(x_i) < 0\}$
$p - n$ = constant zero
(for every y)

Degree (Brouw

<u>Def</u> Degree (Brouwer's degree)

for smooth maps between

continuous

closed oriented manifold (of

~~so~~ the same dimension)

$$f : M^n \to N^n$$

closed oriented manifolds

Most $y \in N^n$ has the property that

$f^{-1}(y)$ is finite and

$\forall x \in f^{-1}(y)$ "$f'(x)$" has full rank

sign det $Df(x)$ is ± 1

$$\text{degree } f = P - n$$

$\#\{i : \text{sign at } x_i = +1\}$

$\#\{i : \text{sign at } x_i \text{ is } -1\}$