Lemma $f: S^1 \to \mathbb{R}$ continuous, $E \subset S^1$ compact

Assume $\forall x \in S^1 \setminus E$

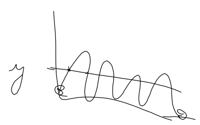
f is orientation preserving ~~we~~ in a neighb. of x

Then $f(S^1) = f(E)$.

'Proof': $f: [0,1] \to \mathbb{R}$ cont.

$f(0) = f(1)$. Assume $\forall x \in (0,1) \setminus E$

f is monotone incr. around x.



exercise

Lemma (topology)

$f: \underset{\text{sphere}}{S^2} \to \mathbb{R}^2$ continuous,

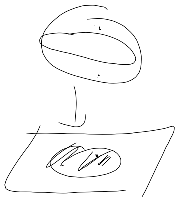
$E \subset S^2$ compact

Assume $\forall x \in S^2 \setminus E$

f is orient. preserving around x

Then $f(S^1) = f(E)$

(connected to Brouwer's definition of degree of maps between manifold.)



Thm (Ugy, Khrushchev)

$E \subset \mathbb{C}$ compact, positive Leb. meas.

Then $\exists f: \mathbb{C} \to \mathbb{C}$

Lipschitz

analytic on $\hat{\mathbb{C}} \setminus E$

and f is not constant

(so $f: \hat{\mathbb{C}} \to \mathbb{C}$ continuous)

Lemma Ugy's thm implies a solution to Laczkovich's question in the plane.

That is, if $A \subset \mathbb{R}^2$, meas, $\lambda(A) > 0$, then $\exists f: A \to \mathbb{R}^2$ Lipschitz such $\operatorname{int} f(A) \neq \emptyset$

Proof Take $E \subset A$ compact, $\lambda(E) > 0$, totally disconnected

Then apply Ugy's thm for E.

$f: \hat{\mathbb{C}} \to \mathbb{C}$

f is not constant

Exercise $\operatorname{int} f(\hat{\mathbb{C}}) \neq \emptyset$

By the topological lemma,

$f(\hat{\mathbb{C}}) = f(E)$

since $\forall x \in \hat{\mathbb{C}} \setminus E$ f is analytic around x

so orientation preserving

$\square$

Aim: Proof of [illegible]'s thm

Hilbert's tranform on $\mathbb{R}$

$\varphi : \mathbb{R} \to \mathbb{R}$ smooth, bounded support.

$$H\varphi(z) = \text{p.v.} \int_{-\infty}^{\infty} \frac{\varphi(t)}{z-t}\, dt$$

$$= \lim_{\delta \to 0} \int_{(-\infty, z-\delta) \cup (z+\delta, +\infty)} \frac{\varphi(t)}{z-t}\, dt$$

$$= K_1 * \varphi \quad \text{where} \quad K_1(z) = \frac{1}{z}$$

Theorem

$$\|H\varphi\|_p \leq C_p \|\varphi\|_p \quad \forall p > 1$$

Weak type (1,1)

$$\lambda\left(\left\{x \in \mathbb{R} : |H\varphi(x)| \geq N\right\}\right) \leq \frac{C}{N} \|\varphi\|_1$$

In the complex plane,

Beurling's transform

$K(z) = \frac{1}{z^2}$ $\quad K : \mathbb{C} \to \mathbb{C}$

For nice φ, let

$$B\varphi = K * \varphi$$

$$= \iint_{\mathbb{C}} \frac{\varphi(t)}{(z-t)^2}\, d\lambda^2(t)$$

Known: (bounded in L^p)

Weak type (1,1) $\Leftarrow$ this we will use

$f: \mathbb{C} \to \mathbb{C}$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, $\quad \partial \overset{\text{def}}{=} \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$

$$\bar{\partial} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$$

Cauchy-Riemann equations

f is analytic $\iff \bar{\partial} f = 0$

$\bar{f}$ is analytic $\iff \partial f = 0$

Claim

$f: \mathbb{R} \to \mathbb{R}$ Lipschitz $\iff$ the weak derivative $\frac{\partial}{\partial x} f \in L^\infty$

$f: \mathbb{R}^2 \to \mathbb{R}^2$ Lipschitz $\iff$

$(f: \mathbb{C} \to \mathbb{C})$ $\quad \left(\frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f\right) \in L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$

$f: \mathbb{C} \to \mathbb{C}$ Lipschitz if and only if the generalised gradient $(\partial f, \bar{\partial} f) \in L^\infty(\mathbb{C}) \times L^\infty(\mathbb{C})$

Proof

Recall: $E \subset \mathbb{C}$

Def $L^\infty(E) = \{ f \in L^\infty(\mathbb{C}) : f$ is a.e. zero in $\mathbb{C}\setminus E \}$ (complex valued)

For a fixed $\varphi \in L^\infty(E)$, $\varphi \not\equiv 0$, consider

$$f = K_0 * \varphi \quad \text{where } K_0(z) = -\frac{1}{z}$$

$$= \iint_E \frac{\varphi(t)}{t-z} \, d\lambda(z)$$

Claim f is in L^∞ and continuous

Proof $\left.\begin{array}{l} K_0 \in L^1(\mathbb{C}) \\ \varphi \in L^\infty(\mathbb{C}) \end{array}\right\} \Rightarrow K_0 * \varphi$ is in L^∞ (Young's ineq)

and also, $K_0 * \varphi$ is continuous (exercise)

Claim f is analytic in $\mathbb{C} \setminus E$

Proof

$$f = \underbrace{\underbrace{\frac{-1}{z}}_{\text{analytic outside } 0} * \varphi}_{\text{average (or integral) of analytic functions outside } E}$$

Claim f is analytic at $\infty \in \hat{\mathbb{C}}$ [...]

(exercise: decide what this means)

(because E is bounded)

(the $-\frac{1}{z}$ is analytic at ∞)

Aim: Find $\varphi \in L^\infty(E)$ such that
$f = -\frac{1}{z} * \varphi$ is Lipschitz and not constant.

Claim: If $\varphi \not\equiv 0$, f is not constant.
(later)

Claim: $\bar{\partial} f = \varphi$

Proof: (f is analytic outside E so $\bar{\partial} f(z) = 0$ $\forall z \notin \bar{E}$)

$$\bar{\partial} f = \bar{\partial}\left(-\frac{1}{z} * \varphi\right)$$
$$= \underbrace{\bar{\partial}\left(-\frac{1}{z}\right)}_{\text{(in the distr. sense)}} * \varphi$$

Exercise: δ_0 Dirac measure at zero.
(or maybe constant times Dirac measure)

$$= \delta_0 * \varphi = \varphi \in L^\infty(E) \subseteq L^\infty(\mathbb{C})$$

Enough to show:
$\exists\, \varphi \in L^\infty(E)$ that is not a.e. zero, such that
$$\partial f \in L^\infty(\mathbb{C})$$
$$\underbrace{\partial}\left(-\frac{1}{z} * \varphi\right) \in L^\infty(\mathbb{C})$$

(by our claim on gen. gradients)

What is ∂f?
$$\partial\left(-\frac{1}{z} * \varphi\right) = \underbrace{\partial\left(-\frac{1}{z}\right)}_{\substack{\text{Exercise} \\ C \cdot \frac{1}{z^2}}} * \varphi$$

(Claim: If g is analytic, then $\partial g = C \cdot g'$)

So $K(z) = \frac{1}{z^2}$,
then $\partial f = K * \varphi = \underbrace{B\varphi}_{\substack{\text{Beurling's} \\ \text{transform} \\ \text{(up to a constant)}}}$

Aim: $\exists\, \varphi \in L^\infty(E),\ \varphi \neq 0,\ \underbrace{K}_{z \mapsto \frac{1}{z^2}} * \varphi \in L^\infty(\mathbb{C})$

$$L^1(E) \times L^1(\mathbb{C}) \overset{\text{def}}{=} \left\{ (f, g) : \|(f,g)\|_1 = \iint_E |f| + \iint_{\mathbb{C}} |g| < \infty \right\}$$

$$L^\infty(E) \times L^\infty(\mathbb{C}) = \left\{ (\varphi, \psi) : \|(\varphi,\psi)\|_\infty = \max\left(\operatorname*{ess\,sup}_{t \in E} \varphi(t),\ \operatorname*{ess\,sup}_{t \in \mathbb{C}} \psi(t) \right) \right\}$$

Duality:

$$\langle (f, g), (\varphi, \psi) \rangle = \iint_E f\varphi + \iint_{\mathbb{C}} g\,\psi$$

(We will use Hahn-Banach)

<u>Def</u> $M = \left\{ \left((K * f)\big|_E,\ f \right) : f \in L^1(\mathbb{C}),\ (K*f)\big|_E \in L^1(E) \right\}$

So $M \subseteq L^1(E) \times L^1(\mathbb{C})$

<u>Claim</u> M is a closed subspace.

Clear: M is a subspace.

Is it closed?

Assume $\left(K * f_n\big|_E,\ f_n\right) \in M$

converges to some

$(g, f) \in L^1(E) \times L^1(\mathbb{C})$

We want to show this limit is in M

So we know $\|f_n - f\|_1 \to 0$

and $\left\| K * f_n\big|_E - g \right\|_1 \to 0$

Let $h = f_n - f$.

Apply weak type $(1,1)$ ineq for h.

$$\lambda\left(\{ z \in \mathbb{C} : |K * h(z)| \geq N \}\right) \leq \frac{C}{N}\|h\|_1$$

$$|K*f_n - K*f|(z) \geq \delta \quad \leq \frac{C}{\delta}\|f_n - f\|$$

So $K * f_n$ converges <u>in measure</u> to $K * f$ (on the set E)

But $K * f_n\big|_E \to g$ in L^1

so $K * f_n\big|_E \to K * f\big|_E$ in measure

$\Rightarrow g = K * f\big|_E$ $\square$

<u>Def</u> Let $\mathcal{D}$ be the set of all infinitely diff. functions on the plane having compact support.

<u>Def</u> $M_0 = \overline{\left\{ \left((K*f)\big|_E,\ f \right) \in M : f \in \mathcal{D} \right\}}$

Clearly, $M_0 \subseteq M$.

<u>Def</u> $M_0^\perp$ = annihilator of the subspace M_0 in the space $L^\infty(E) \times L^\infty(\mathbb{C})$

So $M_0^\perp = \left\{ (\varphi, \psi) \in L^\infty(E) \times L^\infty(\mathbb{C}) : \forall (f, g) \in M_0\ \ \langle (f,g), (\varphi,\psi) \rangle = 0 \right\}$

<u>Lemma</u> $M_0^\perp = \left\{ (\varphi, \psi) \in L^\infty(E) \times L^\infty(\mathbb{C}) : \psi = -K * \varphi \right\}$

<u>Proof</u> If $f \in \mathcal{D}$, then

$$0 \overset{?}{=} \underbrace{\iint_E \varphi\,(K*f) + \iint_{\mathbb{C}} \psi f}_{\langle (\,), (\,) \rangle} = \iint_{\mathbb{C}} f\,(\psi + K * \varphi)$$

Check: $\iint_E \varphi\,(K*f) = \iint_{\mathbb{C}} f\,(K * \varphi)$

(φ is zero outside E)

$\iint_{\mathbb{C}} \varphi\,(K*f)$

usual Fubini for convolution

So if $\psi = -K * \varphi$, then these integrals are obviously zero.

If $\psi \neq -K * \varphi$, then

$\exists f \in L^1,\ \iint f(\psi + K*\varphi) \neq 0$

(duality between L^1 and L^∞)

$\square$

<u>Claim</u> $(1,0) \notin M$

(In the paper $(1,0) \in M$)

<u>Proof</u>

Obs $\underbrace{K * 0}_{0}\big|_E \neq 1\big|_E$ $\square$

So $(1,0) \notin M_0 \leftarrow$ closed subspace of M

So by Hahn-Banach thm

there is $(\varphi, \psi) \in M_0^{\perp}$

such that (it does not annihilate $(1,0)$)

$$\langle (1,0), (\varphi,\psi) \rangle \neq 0.$$

So

$$\iint_E 1 \cdot \varphi + \iint_{\mathbb{C}} 0 \cdot \psi \neq 0$$

So

$$\iint_E \varphi \neq 0 \quad (\text{So } \varphi \not\equiv 0)$$

And, by the Lemma,

$$(\varphi, \psi) \in M_0^{\perp} \Rightarrow \psi = -K * \varphi$$

Observe that $\psi \in L^{\infty}(\mathbb{C})$!

So we have found $\varphi \in L^{\infty}(E)$,

$\varphi \not\equiv 0$, such that

$\psi = -K * \varphi \in L^{\infty}(\mathbb{C})$

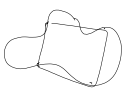
So $K * \varphi \in L^{\infty}(\mathbb{C})$

So $f = \frac{1}{z} * \varphi$ is Lipschitz $\square$

<u>Open problem</u>

If $E \subset \mathbb{R}^3$, $\lambda(E) > 0$
compact

Is there a Lip map mapping E to a set of non-empty interior in $\mathbb{R}^3$



γ smooth curve $\overset{\text{known}}{\Longrightarrow}$ there is a square such that all 4 vertices are on γ

<u>Open question</u>

What if γ is (only) continuous injective curve

$\gamma \; \underset{\text{circle}}{S^1} \to \mathbb{R}^2$

cont, inj.