TOPICS IN GEOMETRIC MEASURE THEORY

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1. Possible problems for credit

(Get in touch in email if you haven't already done that; reading papers and writing something is also an option.)

Problem 1. Prove the digital sundial theorem for $\mathbb{R}^3 \to \mathbb{R}^2$ projections using the existence of various Nikodym sets as we covered here in these lectures.

Problem 2. Show that there is a Lebesgue nullset $E \subset \mathbb{R}^2$ that contains circles centred at every point of a fixed line segment. Copy Körner's proof on the existence of Kakeya sets of measure zero via Baire category, as we did in the lectures (also covered in these notes), or as Körner proved it in his paper.

Problem 3. Let $E \subset \mathbb{R}^3$ be a Lebesgue nullset in space. Prove that for every $\varepsilon > 0$, there are numbers $\varepsilon_i > 0$ and countably many Lipschitz curves L_i of length 1 such that

$$\sum \varepsilon_i^2 < \varepsilon$$

and

$$E \subset B(L_i, \varepsilon_i)$$

(here $B(L_i, \varepsilon_i)$ is the ε_i neighbourhood of the Lipschitz curve L_i).

(Hint: enough to consider graphs of Lipschitz functions; and you may want to show that E is covered by neighbourhoods of some Lipschitz surfaces and Lipschitz curves. Copy proofs covered in lectures (they are not in this pdf currently).)

2. Baire category and Besicovitch sets

2.1. Hausdorff metric.

Definition 2.1. (G_δ and F_σ sets) In a metric space a set is called G_δ if it is the intersection of countably many open sets.

A set is called F_σ if it is the union of countably many closed sets.

Remark 2.2. In a metric space, every closed set is G_δ (proof: take the intersection of the open $1/n$-neighbourhoods of the closed set), and every open set is F_σ. The complement of a G_δ set is F_σ.

Remark 2.3. Let (X, d) be a complete metric space, and $A \subset X$. Then

- $(A, d|_A)$ is complete if and only if $A \subset X$ is closed;
- $A \subset X$ is completely metrizable (that is, there is a metric d' on A generating the same topology as d) if and only if A is G_δ.

The first one is easy, the second one is a non-trivial theorem.

Definition 2.4. Let (X, d) be any metric space. For (non-empty compact) sets $E, F \subset X$ we define the Hausdorff distance as

$$d_H(E, F) = \inf\{\delta > 0 \,:\, E \subset B(F, \delta) \text{ and } E \subset B(E, \delta)\}$$

where for a set S,

$$B(S, \delta) = \{x \in X \,:\, \exists s \in S \; d(s, x) < \delta\}$$

is the open δ-neighbourhood. Equivalently,

$$d_H(E, F) = \max\Big(\sup_{x\in E}\inf_{y\in F} d(x, y), \quad \sup_{y\in F}\inf_{x\in E} d(x, y)\Big).$$

Actually, d_H is a metric on the space of non-empty compact subsets of X.

Definition 2.5. For a metric space (X, d), let $(\mathcal{K}(X), d_H)$ denote the metric space of non-empty compact subsets of X in the Hausdorff metric.

Theorem 2.6. *If (X, d) is complete, then $(\mathcal{K}(X), d_H)$ is complete. If (X, d) is compact, then $(\mathcal{K}(X), d_H)$ is compact.*

Proof. Nontrivial. Reference will be given later. □

2.2. **Upper semi-continuity of Lebesgue measure wrt Hausdorff distance.**

Theorem 2.7. *Let $d \geq 1$. Let K_n be a sequence of non-empty compact sets in $\mathbb{R}^d$ converging to a compact set K in the Hausdorff metric. Then*

$$\lambda(K) \geq \limsup_{n\to\infty} \lambda(K_n)$$

where λ denotes the d-dimensional Lebesgue measure.

Proof. Let

$$U_k = B(K, 1/k).$$

Then, since K is compact,

$$K = \bigcap_{k=1}^{\infty} U_k.$$

Since $\lambda(U_1) < \infty$, by the so-called continuity of measures we have

$$\lambda(K) = \lim_{k\to\infty} \lambda(U_k).$$

Let $\varepsilon > 0$ be arbitrary. Fix k such that

$$\lambda(U_k) < \lambda(K) + \varepsilon.$$

If n is large enough, then $d_H(K_n, K) < 1/k$. For such n, we have $K_n \subset B(K, 1/k) = U_k$, implying $\lambda(K_n) \leq \lambda(U_k) \leq \lambda(K) + \varepsilon$. This proves that

$$\limsup_{n\to\infty} \lambda(K_n) \leq \lambda(K).$$

□

We can't have an inequality in the other way around.

Example 2.8. Let

$$K_n = \{0,\ 1/n,\ 2/n, \ldots,\ (n-1)/n\}.$$

Then $K_n \to [0, 1]$ in the Hausdorff metric, but $\lambda([0, 1]) = 1$ and $\lambda(K_n) = 0$.

2.3. **Baire category.**

Definition 2.9. Let (X,d) be a complete metric space (or a completely metrizable space). A set $A \subset X$ is

- dense if $\overline{A} = X$;
- nowhere dense if $\operatorname{int} \overline{A} = \emptyset$;
- of the first category (or meagre) if it is the union of countably many nowhere dense sets;
- of the second category if it is not of the first category;
- residual (or co-meagre) if its complement $X \setminus A$ is of the first category;

Clearly, the closure of a nowhere dense set is nowhere dense. So a set if of the first category if it can be covered by countably many nowhere dense closed sets.

The complement of a nowhere dense closed set is a dense open set. (And the complement of any dense open set is a nowhere dense closed set.) Therefore a set is residual if and only if it contains the intersection of countably many dense open sets.

Is X itself not meagre (not of the first category)? Is it true that a residual set is not of the first category? Yes, in every *complete* metric space, by the following theorem:

Theorem 2.10 (Baire's category theorem)**.** *In a complete metric space, the intersection of countably many dense open sets is non-empty (in fact, it is dense).*

(Equivalently, the union of countably many nowhere dense sets cannot have non-empty interior.)

Corollary 2.11. *In a complete metric space, the intersection of countably many dense G_δ sets is again dense G_δ.*

Proof. The union of countably many countable sets is still countable. □

To some extent one can draw an analogy between (Lebesgue) measure and Baire category. Meagre sets are like sets of zero measure (union of countably many is still of the same type). Category is a topological notion; it is invariant under taking homeomorphisms.

2.4. **Typical properties.**

Question 2.12. *Is it true that a* typical *continuous function $f : [0,1] \to \mathbb{R}$ is nowhere differentiable?*

The precise meaning of this question is given by the following definition.

Definition 2.13. Let (X,d) be a complete metric space. Let P be a property of the elements of this space (essentially $P \subset X$). We say that P is typical if the elements satisfying it form a residual subset of X.

In the (informal) question above, the complete metric space is the Banach space $C[0,1]$ of the continuous functions $[0,1] \to \mathbb{R}$ in the supremum norm.

Actually, it is true that a typical continuous function (in $C[0,1]$) is nowhere differentiable.

What does a typical compact set look like? Of course, it depends which space of compact sets we look at. We will conside non-empty compact subsets of $[0,1]$ in the Hausdorff metric.

Theorem 2.14. *A typical compact set in $(\mathcal{K}([0,1], d_H)$ has Lebesgue measure zero.*

Proof. Let

$$\mathcal{F}_n = \{E \in \mathcal{K}([0,1]) \,:\, \lambda(E) < 1/n.\}$$

We claim that $\mathcal{F}_n$ is an open set. Indeed, its complement consists of sets that have measure at least $1/n$. By the upper semi-continuity (Theorem 2.7), the limit of compact sets of measure at least $1/n$ (if exists) is a compact set of measure at least $1/n$. So the complement of $\mathcal{F}_n$ is closed, so $\mathcal{F}_n$ is open.

We now claim that $\mathcal{F}_n$ is dense in $\mathcal{K}([0,1])$. Let $E \in \mathcal{K}([0,1])$ be arbitrary, let $\delta > 0$ be arbitrary. Fix a finite set $E' \subset E$ such that

$$B(E', \delta) \supset E.$$

Then E' has Lebesgue measure zero (so it is in $\mathcal{F}_n$), and

$$d_H(E, E') \leq \delta.$$

This proves that $\mathcal{F}_n$ is dense.

Now let

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n.$$

This consists of exactly those non-empty compact subsets of $[0,1]$ that have Lebesgue measure zero. Since $\mathcal{F}$ is the intersection of countably many dense open sets, it is residual, which finishes the proof. (And by Baire's category theorem, we know that $\mathcal{F}$ is a 'large' subset of $\mathcal{K}([0,1])$.) ☐

Remark 2.15. (Just a random remark that won't be used.) Every Borel set is the symmetric difference of an open set and a meagre set. (Non-trivial.) This open set is unique. (Uniqueness is an exercise.)

2.5. **Besicovitch (Kakeya) sets.**

Definition 2.16. A set $E \subset \mathbb{R}^2$ is called a Besicovitch set (or Kakeya set) if it contains a unit line segment in every direction.

Theorem 2.17 (Besicovitch)**.** *There is a compact Besicovitch set in the plane of Lebesgue measure zero.*

Usual proofs (which we won't consider here): Perron tree, or projection theorems and duality.

Question 2.18 (Kakeya)**.** *What is the minimum area of a set in the plane in which a unit line segment can be continuously rotated by* $180°$*?*

The answer is that for every $\varepsilon > 0$, there is a set of measure ε in which the unit line segment can be rotated. (Actually, this follows from the existence of a Besicovitch set and the so called Pál joints.)

Remark 2.19. (Not so well-known folklore result[1]) $\varepsilon = 0$ is not possible: it is not possible to rotate a unit line segment in a set of zero measure. (Non-trivial exercise!)

Our aim here is to prove Besicovitch's theorem by showing that a *typical* Besicovitch set (in a suitable sense) is of Lebesuge measure zero. This is Tom Körner proof, with some modifications.

First we define a complete metric space consisting of certain "almost Besicovitch sets".

Definition 2.20. Let X consist of those compact sets $E \subset [-1,1] \times [0,1]$ which satisfy that

- E is the union of line segments joining a point $(x_1, 0)$ to a point $(x_2, 1)$, where $x_1, x_2 \in [-1,1]$;
- For every $u \in [-1,1]$, there are $x_1, x_2 \in [-1,1]$ such that E contains the line segment joining $(x_1, 0)$ to $(x_2, 1)$ and $u = x_2 - x_1$.

X is non-empty since $E = [-1,1] \times [0,1]$ is in X.

Clearly, any $E \in X$ contains line segments in all those directions that form an angle at most $45°$ with the vertical line. Therefore if we could show that there is an $E \in X$ that has measure zero, then by taking the union of E and a copy of it rotated by $90°$, we would obtain a Besicovitch set of measure zero.

We will show that a typical set $E \in X$ has measure zero. But to make sure this makes sense, first we have to show that X is completely metrizable. We do this by showing that X is a closed subset of a complete metric space.

Lemma 2.21. *X is a closed subset of* $\mathcal{K}([-1,1] \times [0,1])$ *(in the Hausdorff metric).*

Proof. Consider a sequence $E_n \in X$ such that $E_n \to E \in \mathcal{K}([-1,1] \times [0,1])$ in the Hausdorff metric. We have to show that $E \in X$.

First we show that E is the union of line segments as required. Let $z \in E$ be arbitrary. Fix $\varepsilon_n \to 0$ such that

$$d_H(E_n, E) < \varepsilon_n.$$

[1] Actually, I heard this from Imre Ruzsa around 2005.

Then, in particular,

$$E_n \subset B(E, \varepsilon_n) \tag{2.1}$$

and

$$E \subset B(E_n, \varepsilon_n).$$

Therefore there is $z_n \in E_n$ such that $d(z, z_n) \leq \varepsilon_n$. Since $E_n \in X$, it is the union of line segments (joining the bottom and top edge of the rectangle $[-1,1] \times [0,1]$). Therefore there are $x_0^n, x_1^n \in [-1,1]$ such that the line segment L_n joining

$$(x_0^n, 0) \text{ and } (x_1^n, 1)$$

is in E_n and goes through the point z_n. By (2.1),

$$L_n \subset B(E, \varepsilon_n). \tag{2.2}$$

Since $[-1,1]$ (and its square) is compact, we may choose a subsequence $n_k \to \infty$ such that

$$x_0^{n_k} \to x_0 \text{ and } x_1^{n_k} \to x_1$$

for some $x_0, x_1 \in [-1,1]$. Let L be the line segment joining $(x_0, 0)$ and $(x_1, 1)$. Clearly, L is the limit of L_{n_k} is the Hausdorff metric. By (2.2),

$$L \subset E.$$

Since $z_n \to z$ and $z_{n_k} \in L_{n_k}$, we must have $z \in L$. This finishes the proof that E is the union of line segments joining the bottom and top edge of the rectangle.

Now we have to prove that E contains the necessary line segments. Let $u \in [-1,1]$ be arbitrary. Since $E_n \in X$, we can choose x_0^n and x_1^n such that

$$u = x_0^n - x_1^n$$

and that the line segment L_n joining $(x_0^n, 0)$ and $(x_1^n, 1)$ is in E_n. We can choose a subsequence $n_k \to \infty$ such that

$$x_0^{n_k} \to x_0 \text{ and } x_1^{n_k} \to x_1.$$

Clearly, $u = x_1 - x_0$. Let L be the line segment joining $(x_0, 0)$ to $(x_1, 1)$. Then

$$d_H(L, L_{n_k}) \to 0.$$

As before, we have

$$L_{n_k} \subset B(E, \varepsilon_{n_k}).$$

Therefore

$$L \subset E,$$

and it has the right slope. □

Now we can state the theorem that "typical Besicovitch sets have measure zero".

Theorem 2.22 (T. Körner). *A typical compact set $E \in X$ has Lebesgue measure zero. That is,*

$$\{E \in X \,:\, E \text{ has positive Lebesgue measure }\}$$

is of the first category (meagre).

In fact, we show the following stronger result (from which the previous follows by Fubini).

Theorem 2.23 (T. Körner). *Every horizontal section*

$$E_y = \{x \in [-1,1] \,:\, (x,y) \in E\} \quad (y \in [0,1])$$

of a typical $E \in X$ has (one-dimensional) Lebesgue measure zero. (In particular, there is at least one such $E \in X$.)

Proof. First we restrict our attention to a narrow horizontal strip in the rectangle $[-1,1] \times [0,1]$. For a height $h \in [0,1]$ and an $\varepsilon > 0$ we define

$$\mathcal{G}(h, \varepsilon) = \{E \in X \,:\, \forall y \in [h-\varepsilon, h+\varepsilon] \;\; \lambda(E_y) < 4\varepsilon\}$$

and its complement

$$\mathcal{F}(h, \varepsilon) = \{E \in X \,:\, \exists y \in [h-\varepsilon, h+\varepsilon] \;\; \lambda(E_y) \geq 4\varepsilon\}.$$

Claim 2.24. *$\mathcal{G}(h, \varepsilon)$ is open.*

Proof. We show that $\mathcal{F}(h,\varepsilon)$ is closed. Let $E^k \in \mathcal{F}(h,\varepsilon)$ be a sequence converging in the Hausdorff metric to $E \in X$. In particular,

$$E^k \subset B(E,\delta_k) \tag{2.3}$$

for some $\delta_k \to 0$. Since $E^k \in \mathcal{F}(h,\varepsilon)$, there are $y_k \in [h-\varepsilon, h+\varepsilon]$ such that

$$\lambda(E^k_{y_k}) \geq 1/n \tag{2.4}$$

for every k. By passing to a subsequence, we may assume that y_k converges to some $y \in [h-\varepsilon, h+\varepsilon]$, and we still have $E^k \to E$ and (2.3) still holds. We would like to show that $\lambda(E_y) \geq 1/n$. Suppose to the contrary that $\lambda(E_y) < 1/n$. Then E_y can be covered by an open set U such that $\lambda(U) < 1/n$. Comparing this to (2.4), we see that for every k, there is a point x_k such that

$$x_k \in E^k_{y_k} \setminus U.$$

By passing to a subsequence again we may assume that $x_k \to x$, and $x \in [-1,1] \setminus U$. We have

$$(x_k, y_k) \in E^k.$$

By (2.3), we also have

$$(x_k, y_k) \in B(E, \delta_k).$$

Since $x_k \to x$, $y_k \to y$, $\delta_k \to 0$, we must have $(x,y) \in E$. Then $x \in E^y$, which is a contradiction, as $x \notin U$ but $U \supset E^y$. □

Claim 2.25. *$\mathcal{G}(h,\varepsilon)$ is dense.*

Proof. Fix any $E \in X$ and an integer N. We show that there is $E''' \in \mathcal{G}(h,\varepsilon)$ such that $d_H(E,E') \leq 5/N$.

By line segments we will always mean a line segment connecting the bottom and top edge of the rectangle $[-1,1] \times [0,1]$. The idea is to 'move' the line segments in E such that they intersect the horizontal line at height h in finitely many points only.

For a line segment L connecting $(x_0, 0)$ and $(x_1, 1)$, let us call $u(L) = u = x_1 - x_0$ the *slope* of L. Also, let $L_h \in [-1,1]$ denote the point for which (L_h, h) lies on the line segment L.

Consider the arithmetic progression

$$S = \left\{ -\frac{N-1}{N}, -\frac{N-3}{N}, \ldots, \frac{N-3}{N}, \frac{N-1}{N} \right\}.$$

Now let E' be the union of all those line segments L' of $[-1,1] \times [0,1]$ which satisfy the following

- $L'_h \in S$;
- there is $L \subset E$ such that $u(L) = u(L')$ and
$$L'_h - \frac{1}{N} \leq L_h \leq L'_h + \frac{1}{N}.$$

It is easy to see that $E' \subset [-1,1] \times [0,1]$, E' is closed, and every line segment in E can be moved by a small horizontal translation into E. Some line segments can be moved both left and right into E' (this is necessary to make it compact). So, in fact, $E' \in X$.

Unfortunately for us, E' might contain many line segments of the same slope. Before we can rectify this, we have to make E' a bit nicer: we will consider E'' which contains line segments with slope $1/N$-close to line segments in E'. To be more precise, E'' should be the union of all those line segments L'' of $[-1,1] \times [0,1]$ for which there is a line segment $L' \subset E'$ such that

$$L''_h = L'_h \in S$$

and

$$\lceil Nu(L') - 1 \rceil \leq Nu(L'') \leq \lfloor Nu(L') + 1 \rfloor$$

(this awkward condition is to make sure E'' is still closed).

For $i \in S$, let

$$U_i = \{u(L'') : L'' \subset E'' \text{ and } L''_h = i\},$$

the set of slopes represented by lines going through (i,h). We know that $\cup_i U_i \supset [-1,1]$.

Since the sets U_i are nice closed sets (union of finitely many intervals), we may choose $V_i \subset U_i$ such that

- each V_i is closed;

- $\cup_i V_i \supset [-1,1]$;
- $\sum_i \lambda(V_i) \leq 3$.

For each i, by adding finitely many points of U_i to V_i we can make sure that $d_H(U_i, V_i) < 1/N$.

Now let E''' be the union of all those line segments L''' for which there is a line segment $L'' \in E''$ such that $L'''_h = L''_h = i \in S$ and $u(L''') \in V_i$. We still have that $E''' \in X$. By the construction, we also have

$$d_H(E''', E) \leq 5/N.$$

Now we look at the area of E''' in the strip $[-1,1] \times [h-\varepsilon, h+\varepsilon]$. In this strip, the measure of the set covered by line segments going through (i,h) is exactly

$$\varepsilon\lambda(V_i).$$

Moreover, every horizontal section has one-dimensional measure at most $\varepsilon\lambda(V_i)$. Therefore, for every $y \in [h-\varepsilon, h+\varepsilon]$,

$$\lambda(E'''_y) \leq \varepsilon \sum_i \lambda(V_i) \leq 3\varepsilon.$$

□

Now we finish the proof of the theorem. Let

$$\mathcal{G} = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{n-1} \mathcal{G}(\tfrac{m}{n}, \tfrac{1}{n}).$$

This $\mathcal{G}$ is an intersection of countably many dense open sets, so $\mathcal{G}$ is residual. Clearly, $E \in \mathcal{G}$ if and only if every horizontal section of E has Lebesgue measure zero. This proves that for a typical $E \in X$, every horizontal section of E has Lebesgue measure zero. □

3. Using the same proof

Talagrand: centers on a line segment.

4. Nikodym sets

This is between the easy and the difficult case. A bit better geometric argument is needed here. This argumen is similar to usual arguments constructing Kakeya needle sets of arbitrarily small measure.

It is enough to construct a Nikodym set that contains punctured lines trough every point of $[0,1]^2$.

Davies's theorem.

Davies, Falconer, Chang–Csörnyei–Héra–Keleti proof.

Also: there is a nullset such that for every $x \in \mathbb{R}^2$, there is a punctured line through x in E, and for every $r > 0$, there is a line in E with distance r from x.

Here: only Nikodym in plane; existence of Nikodym sets in $\mathbb{R}^n$ can be proved with minor modifications.

Definition 4.1. Coding. Let $K \subset [0,1]^2 \times S^1$ be arbitrary (compact) set. Consider

$$\varphi(K) = \bigcup_{(x,v)\in K} (x + \mathbb{R}v) \setminus \{x\}.$$

This is a subset of the plane, containing a punctured line through every point x that is in the projection of K.

Definition 4.2. Let $\mathcal{K}$ denote the space of all compact sets

$$K \subset [0,1]^2 \times S^1$$

that have full projection onto $[0,1]^2$.

Claim 4.3. *$\mathcal{K}$ in the Hausdorff metric is a complete metric space.*

Proof. Enough to show that $\mathcal{K}$ is a closed subset of $\mathcal{K}([0,1]^2 \times S^1)$. We show that the complement is open. If $K \subset [0,1]^2 \times S^1$ is non-empty compact but its projection onto $[0,1]^2$ is not everything, then there is an open interval of length δ that does not appear in the projection. Then, for any $K' \subset B(K, \delta/3)$, (in particular, if $d_H(K,K') < \delta/3$, the projection of K' is not full. □

Claim 4.4. *For a compact set $K \in \mathcal{K}$, $\varphi(K)$ is an F_σ set (this measurable) in the plane that contains a punctured line through every point of $[0,1]^2$.*

Our aim is the following.

Theorem 4.5. *For a typical $K \in \mathcal{K}$, $\varphi(K)$ has Lebesgue measure zero. So $\varphi(K)+\mathbb{Z}^2$ is a Nikodym set.*

affine invariance, it does not matter which cone we take

Lemma 4.6 (Lemma 1)**.** *Let D be a double cone centred at zero not containing the horizontal direction, let $R > 0$, and $\varepsilon > 0$. Then we can partition D into double cones D_i centred at zero and find translates*

$$D'_i = t_i + D_i$$

$$D' = \cup_i D'_i$$

so that

- *The measure of $D' \cap \{(x,y) : 0 \le y \le R\}$ is less than ε.*
- *D' covers the bottom half of D. In particular, for every point p in the bottom half of D' there is a line through p in D'.*
- *$D' \cap \{(x,y) : y \le 0\}$ is contained in the ε-neighbourhood of $D \cap \{(x,y) : y \le 0\}$.*

This is based on the following.

here we should use all cones or prove that it does not matter

Lemma 4.7 (Lemma 0)**.** *D centred at zero, $\delta > 0$, $\varepsilon > 0$. The we can choose a partiion and translates $D'_i = v_i + D_i$ such that*

- *$\lambda(D \cap \{(x,y) : 0 \le y \le \delta\} \le c\delta^2$ where $c = \lambda(D \cap \{0 \le y \le 1\})$. (c is like the angle of the cone)*
- *bottom half of D is covered (so there are lines through them)*
- *$v_i \in \{(x,y) : y = \delta\}$*
- *$D' \cap \{y \le 0\}$ is contained in the ε neighbourhood of bottom half of D.*

Proof. We choose $v_i \in D_i$. □

Proof. Lemma 1 from Lemma 0. Let N be large, apply Lemma 0 with $\delta = R/N$. Until vertex of each double cone lies in $\{y = R\}$.

Lebesgue regularity. Small neighbourhood implies that we don't increase the measure too much on previous levels. □

Lemma 4.8 (Lemma 2)**.** *Let $(x_0, v_0) \subset \mathbb{R}^2 \times S^1$, let $B \subset \mathbb{R}^2$ be a closed ball, and suppose that $x_0 \notin B$. Let $\eta > 0$ be arbitrary. Then there is an open neighbourhood (disc) U of x_0 in $\mathbb{R}^2$ such that:*

for each $\varepsilon > 0$ there is a set $D \subset \mathbb{R}^2$ such that for all $x \in U$, there is a line $L \subset D$ through x such that the angle between L and v_0 is at most η, and

$$\lambda(B \cap D) < \varepsilon$$

Proof. Apply an affine copy of the construction Lemma 1. □

We can restate this using the coding space.

For a closed ball B, let $2B$ denote the *open* ball of the same centre but twice the radius.

Definition 4.9. For a closed ball B, let $\mathcal{A}(B)$ be the set of all those $K \in \mathcal{K}$ such that

$$B \cap \bigcup_{(x,v)\in K,\ x \notin 2B} x + \mathbb{R}v$$

has measure zero.

Lemma 4.10. *$\mathcal{A}(B)$ is residual in $\mathcal{K}$.*

Proof. For $\varepsilon > 0$, let $\mathcal{A}(B, \varepsilon)$ be the set of all those $K \in \mathcal{K}$ such that

$$B \cap \bigcup_{(x,v)\in K,\ x\notin 2B} x + \mathbb{R}v$$

has measure less than ε.

Since $\mathcal{A}(B) = \bigcap_{m=1}^{\infty} \mathcal{A}(B, 1/m)$, it is enough to show that $\mathcal{A}(B, \varepsilon)$ is open and dense in $\mathcal{K}$.

To see that $\mathcal{A}(B, \varepsilon)$ is open, let $K \in \mathcal{A}(B, \varepsilon)$. Then

$$B \cap \bigcup_{(x,v)\in K,\ x\notin 2B} x + \mathbb{R}v$$

is compact, has measure less than ε, so, for some $\delta > 0$, its open δ-neighbourhood has measure less than ε. (Add the small annulus around boundary to this open set.) For a suitable $\delta' > 0$, if $d_H(K, K') < \delta'$, then

$$B \cap \bigcup_{(x,v)\in K',\ x\notin 2B} x + \mathbb{R}v$$

lies in the δ-neighbourhood.

Now we show that $\mathcal{A}(B, \varepsilon)$ is dense. Let $K \in \mathcal{K}$ be arbitrary. For every $x \in [0, 1]^2 \setminus 2B$ (compact set), there is v such that $(x, v) \in K$. Choose a small open neighbourhood given by Lemma 2. Finitely many cover. Now choose ε_i such that total sum is at most ε. Add a δ-net. Done. □

Proof of theorem. Let B_i be a countable collection of balls such that every point in $\mathbb{R}^n$ is covered by a ball of arbitrarily small diameter. Suppose $K \in \bigcap \mathcal{A}(B_i)$. (Residual.)

The is a nullset

$$E = \bigcup_i \left(B_i \cap \bigcup_{(x,v)\in K,\ x\notin 2B_i} (x + \mathbb{R}v) \setminus \{x\} \right)$$

and it contains our Nikodym set. □

Corollary 4.11 (Davies's theorem). *Let $E \subset \mathbb{R}^2$ be Borel. Then we can cover E by lines (or even non-vertical lines) such that the union of these lines minus the set E is Lebesgue null.*

(exercise from the existence of Nikodym sets)

5. Falconer's digital sundial theorem

See Falconer: Fractal Geometry (Mathematical foundations and applications) and en.wikipedia.org/wiki/Digital_sundial and Falconer's original paper: Falconer K. J. (1986a) Sets with prescribed projections and Nikodym sets, Proc. Lond. Math. Soc. (3), 53, 48?64.

If we have a set in $\mathbb{R}^3$, we can look at all possible projections onto a fixed plane $\mathbb{R}^2$, like if we are looking at a sundial, where the direction of projection is determined by where the Sun is on the sky. Digital sundial: we would like to have the projection (the shadow) of a set in $\mathbb{R}^3$ to be the exact time written in digital format. Is there a set in $\mathbb{R}^3$ with those projections? Yes, actaully the projections (shadows) can be arbitrary prescribed sets.

Let's look at the planar version of this problem (originally due to Davies).

Theorem 5.1. *Consider the projections*

$$proj_t : (x, y) \mapsto x + yt \qquad (t \in \mathbb{R})$$

from the plane to the real line. We are given sets (prescribed projections) $A_t \subset \mathbb{R}$, and assume that

$$A = \{(t, x) : t \in \mathbb{R},\ x \in A_t\}$$

is Lebesuge measurable. Then there is a Borel set $B \in \mathbb{R}^2$ such that

$$proj_t(B) \supset A_t \text{ for every } t,$$

and

$$A_t \setminus proj_t(B)$$

has Lebesgue measure zero for (almost every) t.

Proof. For $B \subset \mathbb{R}^2$, consider

$$A(B) = \{(t, x) : t \in \mathbb{R},\ x \in proj_t(B).$$

A point in B adds a nonvertical line to this set, and every non-vertical line can be added to this set by adding a point to B. So our task it to cover the set A (as in the theorem) by a family of non-vertical lines without adding to much extra measure. This is Davies's theorem. $\square$

6. Nikodym sets, many directions

One can give a much stronger Nikodym set as well.

Theorem 6.1. *There is a Borel set $E \subset \mathbb{R}^2$ that has Lebesgue measre zero such that for every $p = (x_0, y_0) \in \mathbb{R}^2$ there are "residually many punctured lines through p in E", that is,*

$$\{t \in \mathbb{R} \,:\, \{(x_0 + x, y_0 + tx) \,:\, x \in \mathbb{R} \setminus \{0\}\} \subset E\}$$

is residual, that is, contains a dense G_δ set, for every $(x_0, y_0) \in \mathbb{R}^2$.

Proof sketch. Take the union of countably many affine copies of a Nikodym set, and take an arbitrary G_δ hull that still has Lebesgue measure zero. [Was covered in the lectures.] $\square$

One corollary of this is that there is a Nikodym set such that for every points p, p' there are parallel punctured lines through p and p' inside E.

Exercise 6.2. *Show the $\mathbb{R}^3$ version of the digital sundial theorem holds, using (perhaps) the previous corollary.*

7. Kakeya sets in the plane, dimension

This section is based on the first few pages of Thomas Wolff's survey paper "Recent work connected with the Kakeya problem".

Definition 7.1. Minkowski / box dimension

Lemma 7.2.

$$N_\delta(E)\delta^2 \approx \lambda(B(E, \delta))$$

Theorem 7.3. *Every Kakeya set in the plane has Minkowski (box) dimension* 2.

Proof. Now let E be Besicovitch set for unit line segments in the plane. Take δ neighbourhood. Choose $1/\delta$ many line segments L_i of δ-different angles in E. Let T_i be their δ-neighbourhood. Then

$$B(E, \delta) \supset \cup_i T_i.$$

Lemma 7.4.

$$\lambda(T_i \cap T_j) \le C \frac{\delta^2}{\theta(T_i, T_j) + \delta}.$$

Proof. Verify. $\square$

So the right hand side, for a fixed i, is about δ times 1, 1/2, 1/3, ..., δ. Their sum is $\delta \log 1/\delta$. Now let f_i denote the characteristic function of T_i. Then $\int f_i \approx \delta$.

Lemma 7.5.

$$\lambda(\cup_i T_i) \ge \frac{\sum_i \lambda(T_i)}{\sum_i \sum_j \lambda(T_i \cap T_j)}.$$

Proof. Cauchy–Scharz for $\sum_i f_i$ and 1_E. $\square$

$$\int (\sum f_i)^2 \int 1_E^2 \ge (\int \sum f_i 1_E)^2$$

$$\int (\sum f_i)^2 \lambda(E) \ge 1$$

$$\int (\sum f_i)^2 \lambda(E) \ge 1$$

$$\sum_i \sum_j \int f_i f_j \cdot \lambda(E) \geq 1$$

$$\sum_i \sum_j \lambda(T_i \cap T_j) \cdot \lambda(E) \geq 1$$

$$\log 1/\delta \cdot \lambda(E) \geq 1$$

□

Theorem 7.6. *Every Kakeya set in the plane has Hausdorff dimension* 2.

Proof. $\dim_H(E)$ is the infimum of those $s > 0$ for which for every $\varepsilon > 0$, we can cover E by discs $B(x_i, r_i)$ such that $\sum_i r_i^s < \varepsilon$.

If $s < \dim_H(E)$, then

$$\sum_i r_i^s \geq c_s > 0$$

for any family of discs $B(x_i, r_i)$ covering E. We will show that if E is a Kakeya set in the plane, then this c_s is positive for $s < 2$. This is enough to conclude that the dimension of E is at least 2 (and in fact, equals 2 as $E \subset \mathbb{R}^2$).

So cover E by balls. We may assume that all balls have radius 2^{-k}.

Let

$$E_k = E \cap \bigcup_{r_i = 2^{-k}} B(x_i, r_i).$$

Then $E = \cup_{k=1}^{\infty} E_k$. (If you use a disc of radius 1, then the sum is at least 1.)

For $t \in \mathbb{R}$, let $L_t \subset E$ be the line segment of slope t.

$$S_k = \{t \in [0,1] \,:\, \lambda_1(E_k \cap L_t) \geq 6/\pi^2 k^2.\}$$

Then

$$\bigcup_{k=1}^{\infty} S_k = [0,1].$$

Chose k such that

$$\lambda(S_k) \geq 6/\pi^2 k.$$

Now we use the L^2 (Cauchy–Schwarz) argument for the lines coded by S_k and the set $E_k' = B(E_k, 2^{-k})$, which is the union of balls $B(x_i, 2 \cdot 2^{-k})$.

Now choose about $\lambda(S_k)/2 \cdot 2^k$ many points $t_j \in S_k$ that are 2^{-k}-separated. Let

$$T_j = E_k' \cap B(L_{t_j}, 2^{-k}).$$

Note that

$$\lambda_2(T_j) \geq c 2^{-k}/k^2$$

and we still have, with $\delta = 2^{-k}$,

Lemma 7.7.

$$\lambda(T_i \cap T_j) \leq C \frac{\delta^2}{\theta(T_i, T_j) + \delta}.$$

Then Cauchy–Schwarz gives us

$$\log 1/\delta \cdot \lambda(E_k') \geq (c/k^2)^2$$

so

$$\lambda(E_k') \geq c/k^4 / \log 2^k = c/k^5$$

Therefore we have at least $4^k c'/k^5$ many balls of radius 2^{-k}. If $s < 2$, then

$$4^k c'/k^5 (2^{-k})^s \geq c_s > 0$$

where c_s does not depend on k. By the remarks above, this implies that the Hausdorff dimension of E is 2. □

8. Bibliography for other topics

(1) Alberti–Csörnyei–Preiss: "Structure of null sets and applications"
(2) Matousek: "On Lipschitz mappings onto a square"
(3) S. V. Khrushchev: "Simple proof of a theorem on removable singularities of analytic functions satisfying a Lipschitz condition"

(you can click on these to get these papers)