Assignment 2 (part 1) for MA914 'Topics in PDEs'

Submission deadline: 17/03/2014

Problem 1: Consider the elliptic problem

$$-\nabla \cdot A(\nabla u) = f \ , \quad \text{in } \Omega \ , \qquad u = 0 \ , \quad \text{on } \partial\Omega$$

with $f \in L^2(\Omega)$ and with an operator A which satisfies the conditions for the existence of a unique solution given in the lecture. Prove that under the same conditions the general Dirichlet problem

$$-\nabla \cdot A(\nabla u) = f \ , \quad \text{in } \Omega \ , \qquad u = g \ , \quad \text{on } \partial\Omega$$

with $g \in H^1(\Omega)$ also has a unique solution in a suitable space V.

Hint: First consider the problem with an operator depending on x and verify that the existence proof from the lecture carries over to this case.

Problem 2: Given a continuous operator A and a bounded function f, rewrite the elliptic problem

$$-\nabla \cdot A(\nabla u(x)) = f(x)$$

as a minimization problem with $W = W(x, u, \chi)$ under the assumption that there exists a $G \in C^1$ such that

$$A(\chi) = \nabla G(\chi) \ .$$

Show that the simplified conditions given for existence of a minimizer used in the proof of Theorem 2.3.1 are equivalent to the assumptions made for existence of a solution to the non-linear PDE given in Theorem 3.4.4. Also show the equivalence of the corresponding conditions for uniquness.

Problem 3: Consider the following *control problem*: minimize J_λ over the spsace $L^2(\Omega)$ with

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |S(u) - y_d|_{L^2}^2 + \frac{\lambda}{2}\|u\|_{L^2}^2$$

with a continuous linear operator $S\colon L^2(\Omega) \to H_0^1(\Omega)$, a given target function $y_d \in L^2(\Omega)$ and $\lambda \geq 0$.

Part 1: prove that the problem has a unique solution for $\lambda > 0$.

Hint: show that J_λ is strictly convex and satisfies

$$J_\lambda(u_n) \to \infty \ , \quad u_n \to \infty \quad \text{in } \Omega$$

then follow the ideas from the *direct method from the calculus of variations.*

Part 2:[*Source control problem for elliptic pdes*]:

Consider the solution operator $S\colon u \in L^2(\Omega) \mapsto y \in H_0^1(\Omega)$ of the elliptic problem

$$-\triangle y(x) = \beta u(x) \ , \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega \ ,$$

with $\beta > 0$ fixed.

Show that the existence result from part one can be applied here.

Show that the solution to the optimal control problem is given by

$$u = -\frac{1}{\lambda} S^*\big(S(u) - y_d\big)$$

where S^* is the dual operator of S w.r.t. the L^2 scalar product. What is the pde that $p = S^*(y - y_d)$ satisfies?

Problem 4: (**Minty-Browder theory in classical spaces**):

Consider the problem

$$F(\nabla^2 u) = f \ , \quad \text{in } \Omega \ , \qquad u = 0 \ , \quad \text{on } \partial\Omega \tag{1}$$

where $F\colon S^{d\times d} \to \mathbb{R}$ is a given function from the space of symmetric matrices.

Now consider a sequence of smooth solutions

$$F(\nabla^2 u_k) = f_k \ , \quad \text{in } \Omega \ , \qquad u_k = 0 \ , \quad \text{on } \partial\Omega$$

with $f_k \to f$ uniformly. Assume that $(u_k)_k$ satisfies a uniform a-priori bound in $W^{2,\infty}(\Omega)$. We want to show that $u_k \to u$ uniformly and that u satisfies (1) under "monotonicity" assumptions on F:

Assumption (M)*:*

$$[F(\nabla^2 u) - F(\nabla^2 v), u - v] \geq 0 \quad \forall u, v \in C_0^2(\Omega)$$

where we define $C_0^2(\bar\Omega) = \{v \in C^2(\bar\Omega)\colon u = 0 \text{ on } \partial\Omega\}$*. For a Banach space* $(X, \|\cdot\|_X)$ *we define for all* $f, g \in X$*:*

$$[f, g] := \lim_{\lambda\to 0_+} \frac{\|g + \lambda f\|_X^2 - \|g\|_X^2}{2\lambda} \ .$$

To prove the result we need to show the following (taking $X = C^0(\Omega)$):

(1) $[f, g]$ is well defined.

(2) $[f, g]$ is upper semicontinuous: for all $f, g \in X$, $f_n \to f, g_n \to g$ in X:

$$\lim_{n\to\infty} [f_n, g_n] \leq [f, g]$$

(3) $[f, g] = \max\{f(x_0)g(x_0)\colon x_0 \in \bar\Omega \ , |g(x_0)| = \|g\|_X\}$

(4) Under the given assumptions there is a $u \in X$ which is a.e. in C^2 and $u = 0$ on $\partial\Omega$ so that $u_k \to u$ uniformly and $\nabla^2 u \overset{*}{\rightharpoonup} \nabla^2 u$ in $L^\infty(\Omega, S^{d\times d})$.

(5) u solves (1) almost everywhere.

You need only show the final point and can use that for $x_0 \in \Omega$ so that $\nabla^2 u(x_0)$ exists, there are functions $v, w \in C_0^2$ so that $|u - v|$ and $|u - w|$ have a unique maximum at x_0 and for x close enough to x_0 we have

$$v(x) = u(x_0) + \nabla u(x_0)\cdot(x - x_0) + \frac{1}{2}(x - x_0)\nabla^2 u(x_0)(x - x_0) + \varepsilon|x - x_0|^2 - 1 \ ,$$

$$w(x) = u(x_0) + \nabla u(x_0)\cdot(x - x_0) + \frac{1}{2}(x - x_0)\nabla^2 u(x_0)(x - x_0) - \varepsilon|x - x_0|^2 + 1 \ ,$$

for all small enough $\varepsilon > 0$.