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DISCRETE GEVREY REGULARITY, ATTRACTORS AND UPPER–SEMICONTINUITY FOR A FINITE DIFFERENCE APPROXIMATION TO THE GINZBURG–LANDAU EQUATION

Gabriel J. Lord [1]
School of Mathematical Sciences,
University of Bath,
Bath, Avon, BA2 7AY, U.K.

Andrew M. Stuart [2]
SCCM Program,
Division of Applied Mechanics,
Durand 252, Stanford University,
Stanford, California 94305-4040, U.S.A.

Abstract

A semi–discrete spatial finite difference approximation to the complex Ginzburg–Landau equation with cubic non–linearity is considered. Using the fractional powers of a sectorial operator, discrete versions of the Sobolev spaces H^s, and Gevrey classes of regularity τ, G_τ, are introduced. Discrete versions of some standard Sobolev space norm inequalities are proved.

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Current Address: Department of Engineering Mathematics, University of Bristol, Bristol, BS8 1TR, U.K. Email: G.J.Lord@bristol.ac.uk.
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The semi–discrete system is shown to form a continuous semi–group in the discrete L^2 space and absorbing balls in discrete L^2 and H^1 spaces, with radii independent of the spatial step size Δx, are constructed; the existence of a discrete global attractor $\mathcal{A}_{\Delta x}$ follows. It is shown that solutions to the semi–discrete equations lie in a discrete Gevrey regularity class. Using this, upper–semicontinuity of the semi–discrete global attractor $\mathcal{A}_{\Delta x}$ is proved. In contrast to existing techniques, non–smooth initial data error estimates are not required to prove this upper–semicontinuity result; instead the discrete Gevrey regularity is used to enable the use of smooth initial data estimates.

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1 Introduction

The complex Ginzburg–Landau equation is an important equation in a number of scientific fields – it models the evolution of the amplitude of perturbations to steady state solutions at the onset of instability. In fluid dynamics it is found, for example, in the study of Poiseuille flow, Rayleigh–Bénard convection and Taylor–Couette flow [44, 34, 41, 42, 9]. The equation is also used to model the transition to turbulence in chemical mediums [26, 27]. The equation derives the name used here from the study of super–conductivity where it models the phase transition of the material from a superconducting phase to a non–superconducting phase [6, 12]. As a phase transition equation it is closely related to other phase transition equations such as the Allen–Cahn or the Chafee–Infante equation (see [2] and [5]) or the Cahn–Hilliard equation [4].

Like these other phase transition equations the Ginzburg–Landau equation is an example of a dissipative equation whose long time dynamical behaviour is captured by a global attractor [46]. However, unlike the Allen–Cahn or Cahn–Hilliard equations, the Ginzburg–Landau equation does not have a gradient structure except for certain special parameter combinations. Indeed the existence of periodic solutions is easily shown; furthermore Takáč [45] has proved existence of invariant tori and there is good numerical evidence for the existence of chaotic solutions [34], [37]. Another important property that the complex Ginzburg–Landau equation shares with a number of other dissipative equations is that solutions become C^∞ smooth after a finite time. This was shown by Bartuccelli et al [3] by proving a sequence of estimates in Sobolev spaces: so called "ladder estimates". A stronger result has been proved by Doelman and Titi [10] for the Ginzburg–Landau equation with cubic non–linearity and by Duan et al [13] for higher order non–linear terms: the solutions lie in a Gevrey class of regularity which implies analyticity. Further results on Gevrey regularity for a class of parabolic equations may be found in Ferrari and Titi [16].

We consider in this paper a semi–discrete finite difference approximation to the Ginzburg–Landau equation, found by a spatial discretization, which reduces the problem to a system of ordinary differential equations. In [37], fully discrete approximations and numerical results are treated.

In order to consider the semi–discrete equation we start by defining relevant norms on the vector space $\mathbb{C}^J$ and prove discrete versions of standard Sobolev space inequalities.

We prove that the resulting set of ordinary differential equations forms a dynamical system and that there exists absorbing balls in discrete L^2 and H^1 spaces of radii independent of initial data and the spatial mesh size Δx. We conclude the existence of a global attractor bounded independently of Δx in the discrete L^2 and H^1 spaces. It is then shown that solutions to the semi–discrete approximations lie in a discrete Gevrey class of regularity; the method of proof is adapted to the discrete case from that of [10]. We use this result to prove upper–semicontinuity of the semi–discrete global attractor to the global attractor of the Ginzburg–Landau equation. The Gevrey regularity enables us to avoid the non–smooth initial data error estimates which are typically required for such problems. Proofs of such upper–semicontinuity results for numerical methods originate in the work of [21].

The only other work we are aware of proving upper–semicontinuity for finite difference methods is due to Yan in [49]. Our method differs to that of Yan who uses a piecewise linear interpolation to set the analysis in the space L^2 and then derives non–smooth data error estimates. Upper–semicontinuity of a Legendre–Galerkin approximation to reaction–diffusion equations was considered in [43]. For more general systems of ordinary differential equations the existence of global attractors and the question of upper–semicontinuity is considered for example by Kloeden and Lorenz [35] and Humphries and Stuart [30] for one–step methods and by Hill and Süli in [25] for linear multistep methods. Currently lower–semicontinuity of attractors has only been proved for gradient systems [22] or systems whose attractor is the union of unstable manifolds of fixed points [29, 33]. Hence such techniques are not applicable to the Ginzburg–Landau equation.

There are other properties of the Ginzburg–Landau equation which could be examined under discretization. In [11] the dimension of the global attractor was considered and the existence of an inertial manifold for the continuous equation was shown. In [37] the existence of an inertial manifold for a semi–discrete approximation was shown and the dimensionality of the discrete global attractor was considered. Other work in this area for finite difference approximations is by Yan [50, 48] who considers the dimension of attractors and by Jones [32] who proves existence of a discrete inertial manifold for a finite difference approximation to the Kuramoto–Sviashinsky equation and shows C^1 convergence.

The paper is organized as follows. We commence by reviewing some results for the continuous complex Ginzburg–Landau equation along with describing the mathematical setting. In section 3 the semi–discrete finite difference approximation is presented and some discrete Sobolev space results are discussed. We prove in section 4 that the semi–discrete approximation defines a dynamical system and that there exists absorbing balls in the discrete spaces L^2 and H^1. The section concludes with a proof of the existence of a global attractor. Section 5 is devoted to proving that solutions to the semi–discrete approximations are in a discrete Gevrey class of regularity. This is applied in section 6 to establish upper–semicontinuity of the global attractor.

2 The Complex Ginzburg–Landau Equation

Consider the complex Ginzburg–Landau equation with cubic non–linearity and periodic boundary conditions on the interval $\Omega = [0, 1]$, namely:

$$\begin{aligned} U_t = RU - (1+i\nu)A_0 U - (1+i\mu)|U|^2 U,\\ U(0) = U^0. \end{aligned} \tag{2.1}$$

Here R, ν and μ are all real parameters and A_0 is the linear operator given by

$$A_0 := -\Delta \quad \text{with domain } D(A_0) := \left\{ V \in L^2_{\text{per}} : |A_0 V|_{L^2} < \infty \right\}. \tag{2.2}$$

Then

$$U(x,t) : \Omega \times \mathbb{R}^+ \to \mathbb{C}\,.$$

Henceforth we use L^2 to denote L^2_{per}, the standard Lebesgue space of periodic functions on Ω with norm given for $V \in L^2$ by

$$|V|_{L^2} = \left\{ \int_\Omega |V(x)|^2 \, dx \right\}^{\frac{1}{2}}.$$

We make use of the semi–group formulation such as in Henry [24] or Pazy [39]. Since the operator A_0 defined by (2.2) is not positive definite we introduce the linear operator $\widetilde{A}_0$ defined by

$$\widetilde{A}_0 := I + A_0, \tag{2.3}$$

with domain, $D(\widetilde{A}_0) = D(A_0)$, and reformulate (2.1) as

$$U_t + (1+i\nu)\widetilde{A}_0 U = F_0(U), \tag{2.4}$$

$$U^0 = U(0),$$

where

$$F_0(V)(x) := \widetilde{R}V(x) - (1+i\mu)|V(x)|^2 V(x) \tag{2.5}$$

and $\widetilde{R} = R + (1+i\nu)$. The following result is straightforward to prove.

Lemma 2.1 *The linear operator $\widetilde{A}_0$ defined by (2.3) is a sectorial operator. That is $\widetilde{A}_0$ is closed, densely defined , and $\exists \theta \in (0,\pi/2)$, $M > 1$, and $a \in \mathbb{R}$ such that if $\Sigma_{a,\theta}$ is the sector given by $\Sigma_{a,\theta} := \{\lambda \in \mathbb{C} \backslash \{a\} : |\arg(\lambda - a)| < \theta\}$, then the spectrum of $\widetilde{A}_0$, $\sigma(\widetilde{A}_0)$, satisfies*

$$\sigma(\widetilde{A}_0) \subset \Sigma_{a,\theta}$$

and

$$\left|(\lambda I - \widetilde{A}_0)^{-1}\right|_{L^2} \leq \frac{M}{|\lambda - a|} \quad \forall\, \lambda \in \mathbb{C} \backslash \Sigma_{a,\theta}.$$

Furthermore the eigenvalues of $\widetilde{A}_0$ are given by $\left\{\widetilde{\Lambda}_k\right\}$, $k \in \mathbb{Z}$, where

$$\widetilde{\Lambda}_k = 1 + 4k^2\pi^2, \quad k \in \mathbb{Z}, \tag{2.6}$$

and the corresponding eigenfunctions $\{\Psi_k\}_{-\infty}^{\infty}$, form a complete orthonormal set in L^2 given by

$$\Psi_k = e^{2\pi i k x}, \quad k \in \mathbb{Z}. \tag{2.7}$$

Since $\widetilde{A}_0$ is sectorial, fractional powers of $\widetilde{A}_0$ are defined and these may be used to define new Banach spaces, see for example [1] or [39]. If we let $V \in L^2$, then by Lemma 2.1, V may be defined by a Fourier series

$$V = \sum_{k=-\infty}^{\infty} a_k \Psi_k, \qquad \text{where } a_k = \langle V, \Psi_k \rangle . \tag{2.8}$$

The *Sobolev space* $\{H^{2s}, \|\bullet\|_{H^{2s}}\}$ is then defined as

$$H^{2s} := D(\widetilde{A}_0^s) = \left\{ V \in L^2 : |\widetilde{A}_0^s V|_{L^2} < \infty \right\}, \tag{2.9}$$

with

$$\|V\|_{H^{2s}} := |\widetilde{A}_0^s V|_{L^2} = \left\{ \sum_{k=-\infty}^{\infty} \widetilde{\Lambda}_k^{2s} |a_k|^2 \right\}^{\frac{1}{2}} . \tag{2.10}$$

Definition (2.10) is norm equivalent to the standard norm on H^{2s} defined through distributional derivatives.

We may also use the linear operator $\widetilde{A}_0$ to define a special class of Gevrey spaces. The definition of Gevrey class and regularity which we give here is the same as that used in [17, 10] and [13]. First we note that functions of unbounded operators (and so functions of the operator $\widetilde{A}_0^s$, $s > 0$) are defined in the work Dunford and Taylor [14].

Let $V \in L^2$ have Fourier expansion given by (2.8). Then we define the *Gevrey space* $\{G_{\tau,s}, \|\bullet\|_{G_{\tau,s}}\}$ by

$$G_{\tau,s} := D\left(\widetilde{A}_0^s e^{\tau \widetilde{A}_0^s}\right) = \left\{ V \in L^2 : |\widetilde{A}_0^s e^{\tau \widetilde{A}_0^s} V|_{L^2}^2 < \infty \right\}, \tag{2.11}$$

with

$$\|V\|_{G_{\tau,s}} := |\widetilde{A}_0^s e^{\tau \widetilde{A}_0^s} V|_{L^2} = \left\{ \sum_{k=-\infty}^{\infty} \widetilde{\Lambda}_k^{2s} e^{2\tau \widetilde{\Lambda}_k^s} |a_k|^2 \right\}^{\frac{1}{2}} .$$

In this paper we only consider the case when $s = \frac{1}{2}$ and $\tau > 0$. Hence-forward we use G_τ to denote $G_{\tau,1/2}$ with corresponding norm $\|\bullet\|_{G_\tau}$. As is standard practice, we call τ the order of the Gevrey regularity of G_τ and we speak of $V \in G_\tau$ being of Gevrey class of regularity τ. ¿From the definition of the Gevrey spaces it is clear that if $V \in G_\tau$ then the Fourier coefficients of V decay exponentially in k

and hence V is a smooth analytic function. In fact, for all $\alpha \in \mathbb{R}^+$, and for all $\tau > 0$ the following inclusion is true :

$$G_\tau \subset H^\alpha. \tag{2.12}$$

We refer the reader to [40, p12] or [8, p240] for the more general definitions and to [40, 18, 8] for related results and further references.

We use the following notation to denote balls in the spaces defined above. The balls of radius ρ centre 0 in L^2 $(= H^0)$, H^1 and G_τ are defined respectively by

$$\mathcal{B}_0(\rho) := \left\{ V \in L^2 : |V|_{L^2} \leq \rho \right\},$$

$$\mathcal{B}_1(\rho) := \left\{ V \in H^1 : \|V\|_{H^1} \leq \rho \right\},$$

$$\mathcal{B}_{G_\tau}(\rho) := \{ V \in G_\tau : \|V\|_{G_\tau} \leq \rho \}.$$

Before stating results for the Ginzburg–Landau equation we recall some definitions useful in the theory of dynamical systems, which may be found, for example, in Temam [46].

Let X be a Banach space with norm $\| \bullet \|$ and identity operator $I : X \to X$.

Definition 2.1 Let $A, B \subset X$ and let $u \in X$, then the *distance of a point to a set* is defined by

$$\text{dist}_X(u, A) := \inf_{v \in A} \|u - v\|$$

and the *distance between two sets* A, B is defined by

$$\text{dist}_X(B, A) := \sup_{u \in B} \text{dist}_X(u, A).$$

The *Hausdorff distance* $d_H(A, B)$ between two sets $A, B \subset X$ is defined by

$$d_H(A, B) := \max \{ dist_X(A, B), dist_X(B, A) \}.$$

We define the ϵ*–neighbourhood* of a set $A \subset X$ by

$$\mathcal{N}(A, \epsilon) := \{ u \in X : dist_X(u, A) < \epsilon \}. \quad \square$$

Now suppose we are given a semi–group of continuous operators $\{S(t)\}_{t \geq 0}$, where $S(\cdot) : X \to X$, for each $t \geq 0$.

Definition 2.2 A set $A \subset X$ is *positively invariant* for the semi–group $S(t) : X \to X$ if

$$S(t)A \subseteq A, \quad \forall t \geq 0.$$

A set $A \subset X$ is said to be *invariant* if

$$S(t)A \equiv A, \quad \forall t \geq 0. \quad \square$$

Definition 2.3 A set $\mathcal{A} \subset X$ is said to *attract* a set $B \subset X$ if for any $\epsilon > 0$ there exists $t_0 = t_0(\epsilon, \mathcal{A}, B)$ such that

$$S(t)B \subseteq \mathcal{N}(\mathcal{A}, \epsilon) \quad \forall t \geq t_0.$$

A set $\mathcal{A} \subset X$ is said to be an *attractor* if $\mathcal{A}$ is a compact, invariant set which attracts an open neighbourhood of itself.

We say that $\mathcal{A} \subset X$ is a *global attractor* for the semi–group $\{S(t)\}_{t\geq 0}$ if $\mathcal{A}$ is an attractor that attracts all the bounded sets of X. $\square$

Note that convergence to the attractor may be arbitrarily slow; thus we are also interested in finding sets into which bounded sets are mapped by the semigroup after a finite time; these are now defined:

Definition 2.4

A closed bounded subset $\mathcal{B}$ of E an open set in X is said to be *absorbing* in E if for each bounded set $B \subset E$ there exists $t_0(B) \geq 0$ such that

$$S(t)B \subseteq \mathcal{B} \; \forall \; t \geq t_0.$$

A dynamical system possessing an absorbing set is said to be *dissipative.* $\square$

Since all solutions of a dissipative system eventually enter the absorbing set, it is important to understand what happens to the absorbing set under forward evolution. Recall the following definition:

Definition 2.5 We define the ω*–limit set* of B, $\omega(B)$, by

$$\omega(B) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B}. \quad \square$$

Under suitable compactness conditions, a dissipative system has a global attractor which is the $\omega-$ limit set of any absorbing set. The following theorem summarizes known results for the Ginzburg-Landau equation:

Theorem 2.1 *The following properties hold for the Ginzburg–Landau equation* (2.1)*:*

C1 *Given $U^0 \in L^2$ there exists a unique solution*

$$U(t) \in C([0,T];L^2)\bigcap L^2((0,T);H^1)\ \forall T < \infty,$$

and so there exists a semi–group

$$S(\bullet)\bullet \in C\left(\mathbb{R}^+ \times L^2, L^2\right),$$

defined by $S(t)U^0 = U(t)$.

C2 *There exists a constant $\rho_0 = \rho_0(R) > 0$ such that the ball $\mathcal{B}_0(\rho_0)$ is absorbing and positively invariant for the semi–group $\{S(t)\}_{t\geq 0}$.*

C3 *There exists a constant $\rho_1 = \rho_1(R) > 0$ such that $\mathcal{B}_1(\rho_1)$ is absorbing and positively invariant for the semi–group $\{S(t)\}_{t\geq 0}$.*

C4 *The dynamical system given by the complex Ginzburg–Landau equation* (2.1) *possesses a global attractor $\mathcal{A}$,*

$$\mathcal{A} = \omega\left(\mathcal{B}_1(\rho_1)\right).$$

C5 *If $U^0 \in H^1$ then there exists $T_* = T_*(\|U^0\|_{H^1})$ such that*

$$U(t) \in G_t = D(\widetilde{A}_0^{\frac{1}{2}} e^{t\widetilde{A}_0^{\frac{1}{2}}}), \quad t \in (0,T_*).$$

Furthermore $\exists\ T,\sigma > 0$ independent of t such that

$$U(t) \in G_\sigma, \quad \forall t \geq T.$$

Proof C1: This may be proved using a Faedo-Galerkin approach, details may be found in Temam [46]. **C2,C3,C4**: These were first shown for the Ginzburg–Landau equation in Ghidaglia and Héron [19] but may also be found in [46] or Doering et al [11]. **C5**: This was proved by Doelman and Titi [10] for the Ginzburg–Landau equation with cubic non–linearity. □

3 Semi–Discrete Approximation and Setting

In the previous section we discussed the complex Ginzburg–Landau equation (2.1) with periodic boundary conditions. We introduce in this section the semi–discrete approximation and some discrete Sobolev space results. Similar discrete Sobolev results may be found in the works of Mokin [38], Yan, [48, 49, 50] or Yulin [51].

Consider the interval $[0,1]$ divided into J uniformly distributed mesh points a distance $\Delta x := 1/J$ apart. We let subscripts indicate the spatial mesh point, so that $u_j(\bullet)$ is our approximation to $U(j\Delta x, \bullet)$. Define δ_+ to be the usual forward difference approximation to the derivative and δ_- to be the backward difference approximation. ¿From this we find the standard approximation for second derivatives

$$\delta^2 u_j = \delta_+\delta_- u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}. \tag{3.1}$$

Let $\widetilde{v} \in \ell^2$,

$$\widetilde{v} = (\cdots, v_{-1}, v_0, v_1, \cdots, v_{J-1}, v_J, v_{J+1}, \cdots)^T,$$

be such that

$$\widetilde{v}_k = \widetilde{v}_{k+J}, \ \forall\, k \in \mathbb{Z}. \tag{3.2}$$

Then $\widetilde{v}$ is said to satisfy periodic boundary conditions.

Hence-forward we let

$$v = (v_0, \cdots, v_{J-1}) \in \mathbb{C}^J_{\mathrm{per}}$$

denote $v \in \ell^2$ satisfying (3.2) and make free use of the periodicity. Thus, for example,

$$v_{-1} = v_{J-1}, \quad v_J = v_0, \quad v_{J+1} = v_1.$$

Let M be the $J \times J$ diagonal matrix with entries Δx down the diagonal:

$$M = \mathrm{diag}(\Delta x, \cdots, \Delta x).$$

In finite element terms M corresponds to a mass matrix constructed through mass lumping. We denote the $J \times J$ matrix arising from the forward difference approximation to the derivative by

$$D = \Delta x^{-1} \begin{pmatrix} -1 & 1 & \cdots & 0 \\ 0 & -1 & 1 & \\ & \ddots & \ddots & \\ 1 & & & -1 \end{pmatrix}, \tag{3.3}$$

where the non–zero corner element arises from the periodicity. If we define the $J \times J$ matrix A by

$$A := \Delta x^{-1} \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ 0 & & -1 & 2 & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix}, \tag{3.4}$$

then the matrix $M^{-1}A$ is the matrix representation of the standard finite difference approximation to $-\Delta$ for a problem with periodic boundary conditions arising from (3.1).

As in the continuous case we define a linear operator which allows us to define the mathematical setting for the problem. This new linear operator corresponds to shifting the spectrum to make it positive. In this discrete setting define

$$\widetilde{A} = I + M^{-1}A. \tag{3.5}$$

All that remains is to introduce notation for the non–linear term. For $v \in \mathbb{C}^J_{\text{per}}$ we introduce the diagonal matrix $G(v)$ with entries v_j on the diagonal. The non–linear term is then written using

$$G(|v|^2) := \begin{pmatrix} |v_0|^2 & & 0 \\ & \ddots & \\ 0 & & |v_{J-1}|^2 \end{pmatrix}. \tag{3.6}$$

where it is understood that $|v|^2$ is the element of $\mathbb{C}^J_{\text{per}}$ given by

$$|v|^2 = \left(|v_0|^2, \cdots, |v_{J-1}|^2\right)^T.$$

Thus we are in a position to consider the spatial discretization. This reduces the partial differential equation to a finite system of J ordinary differential equations.

Semi–Discrete Problem (SD) :

Find $u(t) = (u_0(t), \cdots, u_{J-1}(t))^T \in \mathbb{C}^J_{\text{per}}$ satisfying

$$\mathbf{SD} \quad \begin{cases} u_t = Ru - (1+i\nu)M^{-1}Au - (1+i\mu)G(|u|^2)u \\ u(0) = u^0 \in \mathbb{C}^J_{\text{per}} \end{cases} . \tag{3.7}$$

Equivalently we may write **SD** to look more like (2.4) :

$$u_t + (1+i\nu)\widetilde{A}u = F(u), \qquad u(0) = u^0 \tag{3.8}$$

where

$$F(v) := \widetilde{R}v - (1+i\mu)G(|v|^2)v \tag{3.9}$$

and $\widetilde{R} = (R + (1+i\nu))$.

Next we introduce the mathematical setting for our spatially discrete equation and prove some discrete Sobolev inequalities. For $s \in \mathbb{R}$, we define

$$s\rceil := \text{ smallest integer} > s \ , \quad s\rfloor := \text{ greatest integer} \le s.$$

The first lemma uses the eigenvalues and eigenvectors of the finite difference approximation to $-\Delta$ from, for example, Conte and de Boor [7].

Lemma 3.1 *The linear operator $M^{-1}A$ has eigenvalues*

$$\lambda_k = \frac{4}{\Delta x^2}\sin^2(k\pi\Delta x) \quad k = -J/2\rceil, \cdots, 0, \cdots, J/2\rfloor \tag{3.10}$$

and the linear operator $\widetilde{A} = I + M^{-1}A$ has eigenvalues

$$\widetilde{\lambda_k} = 1 + \lambda_k \quad k = -J/2\rceil, \cdots, 0, \cdots, J/2\rfloor. \tag{3.11}$$

The corresponding orthonormal eigenvectors ψ_k for both $M^{-1}A$ and $\widetilde{A}$ are given by

$$\psi_k = \left(1, e^{2\pi ik\Delta x}, \ldots, e^{2\pi ik(J-1)\Delta x}\right) \quad k = -J/2\rceil, \cdots, 0, \cdots, J/2\rfloor. \tag{3.12}$$

The operators $\widetilde{A}$ and $(1+i\nu)\widetilde{A}$ are sectorial operators. This follows immediately from the finite dimensionality and the eigenvalues being bounded away from 0.

The following lemma bounds the ratio of the continuous eigenvalues (see Lemma 2.1) to the discrete eigenvalues (Lemma 3.1).

Lemma 3.2 *The ratio of discrete to continuous eigenvalues*

$$r(k) = \frac{\lambda_k}{\Lambda_k}, \quad k = -J/2\rceil, \cdots, -1, 1, \cdots, J/2\rfloor$$

satisfies

$$\frac{4}{\pi^2} \le r(k) \le 1 \quad k = -J/2\rceil, \cdots, -1, 1, \cdots, J/2\rfloor \tag{3.13}$$

and the ratio

$$\tilde{r}(k) = \frac{\widetilde{\lambda_k}}{\widetilde{\Lambda}_k}, \quad k = -J/2\rceil, \cdots, J/2\rfloor$$

$$\frac{4}{\pi^2} \le \tilde{r}(k) \le 1 \quad k = -J/2\rceil, \cdots, J/2\rfloor. \tag{3.14}$$

Proof

First note that

$$\lim_{k\to 0} r(k) = \lim_{k\to 0} \frac{J^2 \sin^2(k\pi\Delta x)}{k^2\pi^2} = 1$$

and

$$r\left(\frac{J}{2}\right) = \frac{J^2 \sin^2(\pi/2)}{J^2\pi^2/4} = \frac{4}{\pi^2}.$$

Note also that

$$\begin{aligned} r'(k) &= \frac{2J^2\pi\Delta x \sin(k\pi\Delta x)\cos(k\pi\Delta x)}{k^2\pi^2} - \frac{2J^2\sin^2(k\pi\Delta x)}{\pi^2 k^3} \\ &= \frac{2J^2\sin^2(k\pi\Delta x)}{\pi^2k^2}\left\{\frac{k\pi\Delta x - \tan(k\pi\Delta x)}{k\sin(k\pi\Delta x)(\cos(k\pi\Delta x))^{-1}}\right\} \\ &< 0. \end{aligned}$$

Thus we have established (3.13). The bound (3.14) follows from (3.13). □

We now define discrete analogues of the complexified Sobolev spaces of section 2, [39] or [46, page 273]. First we define the discrete L^p spaces : $L^p_{\Delta x}$. Let $u = (u_0, u_1, \ldots, u_{J-1})^T \in \mathbb{C}^J_{\text{per}}$ and $v = (v_0, v_1, \ldots, v_{J-1})^T \in \mathbb{C}^J_{\text{per}}$. The **discrete $L^p_{\Delta x}$ space** on $[0,1]$ is defined to be the normed linear space $\{\mathbb{C}^J_{\text{per}}, |\bullet|_{L^p_{\Delta x}}\}$ where

$$|v|_{L^p_{\Delta x}} = \begin{cases} \left\{\sum_{j=0}^{J-1} \Delta x |v_j|^p\right\}^{1/p} & \text{for } 1 \le p < \infty \\ \sup_{0\le j\le J-1} |v_j| & \text{for } p = \infty. \end{cases} \tag{3.15}$$

The **discrete $L^2_{\Delta x}$ inner product** is defined by

$$\begin{aligned}\langle u, v\rangle &= u^T M\overline{v} \\ &= \sum_{j=0}^{J-1} \Delta x u_j \overline{v_j}.\end{aligned} \tag{3.16}$$

The discrete Sobolev spaces, $H^{2s}_{\Delta x}$, are defined in an analogous manner to section 2. Consider the expansion of $v = (v_0, \ldots, v_{J-1})^T \in L^2_{\Delta x}$ as a Fourier series based on the eigenvectors of $\widetilde{A}$ given in Lemma 3.1, so that

$$v = \sum_{k=-J/2\rceil}^{J/2\rfloor} a_k \psi_k. \tag{3.17}$$

For $s > 0$, we define the **discrete Sobolev space** $H^{2s}_{\Delta x}$ as the normed linear space $\left\{\mathbb{C}^J_{\text{per}}, \|\bullet\|_{H^{2s}_{\Delta x}}\right\}$ where

$$\|v\|_{H^{2s}_{\Delta x}} = |\widetilde{A}^s v|_{L^2_{\Delta x}} = \left\{\sum_{k=-J/2\rceil}^{J/2\rfloor} \widetilde{\lambda}_k^{2s}|a_k|^2\right\}^{\frac{1}{2}}. \tag{3.18}$$

Note that for $s = 0$ we recover the $L^2_{\Delta x}$ norm, and for $s = \frac{1}{2}$ we find the space $H^1_{\Delta x}$ which approximates H^1. We can also formulate our discrete Sobolev spaces, $H^{2s}_{\Delta x}$, in terms of discrete approximations to the derivatives.

We define the **discrete Dirichlet inner–product** by

$$\langle u, v\rangle_A := u^T A\overline{v} \tag{3.19}$$

$$\begin{aligned}&= -\sum_{j=0}^{J-1} \Delta x u_j \frac{\overline{v_{j+1}} - 2\overline{v_j} + \overline{v_{j-1}}}{\Delta x^2} \\ &= -\sum_{j=0}^{J-1} \Delta x u_j \delta^2 \overline{v}_j,\end{aligned} \tag{3.20}$$

where $j = 0$ and $j = J-1$ are dealt with by the periodicity.

The inner–product allows us to define the semi-norm $\|\bullet\|_1$ by

$$\|v\|_1^2 = \langle v, v\rangle_A = v^T A\overline{v} \geq 0. \tag{3.21}$$

An alternative definition for the **discrete $H^1_{\Delta x}$ space** is the normed vector space $\{\mathbb{C}^J, \|\bullet\|_{H^1_{\Delta x}}\}$, where

$$\|v\|_{H^1_{\Delta x}} = \left\{|v|^2_{L^2_{\Delta x}} + \|v\|_1^2\right\}^{1/2}. \tag{3.22}$$

The definitions of the space $H^1_{\Delta x}$ through the operator $\widetilde{A}$ in (3.18) and through discrete derivatives in (3.22) are equivalent: see for example [37]. It is a straightforward application of the Cauchy–Schwarz inequality to deduce the following interpolation inequality

$$\|v\|_1^2 \leq |v|_{L^2_{\Delta x}} |M^{-1}Av|_{L^2_{\Delta x}}. \tag{3.23}$$

The next few results concern the relation between the continuous spaces H^{2s} and the discrete spaces $H^{2s}_{\Delta x}$. We relate the discrete space $H^1_{\Delta x}$ and the conventional Sobolev space H^1 in the theorem, following this definition.

Definition 3.1 Let $\mathbf{V} \subset H^1$ be the space of piecewise–linear functions and let $\{\Phi_j\}$ be the standard basis of "hat" functions given by

$$\Phi_k(x) = \begin{cases} (x - x_{k-1})/\Delta x & \text{for } x_{k-1} \leq x < x_k \\ (x_{k+1} - x)/\Delta x & \text{for } x_k \leq x < x_{k+1} \\ 0 & \text{otherwise;} \end{cases} \quad k = 1, \cdots, J-2$$

with

$$\Phi_0(x) = \begin{cases} (x_1 - x)/\Delta x & \text{for } x_0 \leq x < x_1 \\ 0 & \text{otherwise;} \end{cases}$$

$$\Phi_{J-1}(x)) = \begin{cases} (x - x_{J-2})/\Delta x & \text{for } x_{J-2} \leq x < x_{J-1} \\ 0 & \text{otherwise;} \end{cases}$$

and $x_j = j\Delta x$. Then we define the *prolongation of* $H^1_{\Delta x}$ *into* H^1, $P_L : H^1_{\Delta x} \to \mathbf{V} \subset H^1$, for $v \in \mathbb{C}^J_{\text{per}}$ by

$$P_L v = \sum_{j=0}^{J-1} v_j \Phi_j. \quad \Box$$

Theorem 3.1 *Let* $\mathbf{V} \subset H^1$ *be the space of piecewise–linear functions, let* $v \in H^1_{\Delta x}$ *and define* $V \in \mathbf{V}$ *by*

$$V := P_L v,$$

where P_L *is the prolongation defined in Definition 3.1. Then* $\exists$ *a constant* $\kappa > 1$ *such that*

$$\kappa^{-1}\|V\|_{H^1} \leq \|v\|_{H^1_{\Delta x}} \leq \kappa\|V\|_{H^1} \quad \forall \Delta x > 0.$$

Proof Note that this proof is set in dimension $p = 1$. ¿From (2.10) we have that $\|V\|_{H^1} = \left\{|V|_{L^2}^2 + |A_0^{1/2}V|_{L^2}^2\right\}^{\frac{1}{2}}$ and let $\{\Phi_j\}$ denote the standard basis for **V**. Now by standard finite element analysis,

$$|A_0^{1/2}V|_{L^2}^2 = \left\langle A_0^{1/2}V, A_0^{1/2}V\right\rangle = v^T K \overline{v}$$

where K is the so called *stiffness matrix* given by $K_{i,j} = \left\langle A_0^{1/2}\Phi_i, A_0^{1/2}\Phi_j\right\rangle$. Evaluating the inner–products we find in 1 dimension that

$$K = A.$$

The proof is completed by noting another standard result (for example Hackbusch [20, Theorem 8.8.1]) that $\exists$ constant $C > 1$ independent of Δx such that

$$C^{-1}|V|_{L^2} \leq |v|_{L^2_{\Delta x}} \leq C|V|_{L^2}.$$

This is proved by looking at the mass matrix. Estimates for the constant C, and hence the constant κ, may be found in [47]. □

The following definition enables us to relate the discrete spaces $H^{2s}_{\Delta x}$ and the conventional Sobolev spaces H^{2s}.

Definition 3.2 Let $s \geq \frac{1}{2}$ and define $P_{\Delta x} : H^{2s} \to H^{2s}_{\Delta x}$ to be the operator which evaluates the continuous function $V(x) \in H^{2s}$ at the grid points. Thus

$$P_{\Delta x}V(x) = (V(0), V(\Delta x), \ldots, V((J-1)\Delta x))^T.$$

Let $\mathcal{W} \subset H^{2s}$ be a bounded set , then

$$P_{\Delta x}\mathcal{W} = \bigcup_{W \in \mathcal{W}} P_{\Delta x}W. \quad \square$$

We now turn our attention to proving some results about the discrete spaces.

Lemma 3.3 *The $H^2_{\Delta x}$ space as defined by* (3.18) *is norm equivalent to the space defined by discrete approximations to the appropriate distributional derivatives. Specifically we have*

$$\frac{1}{2}(|v|_{L^2_{\Delta x}}^2 + \|v\|_1^2 + |M^{-1}Av|_{L^2_{\Delta x}}^2) \leq |\widetilde{A}v|_{L^2_{\Delta x}}^2 \leq 2(|v|_{L^2_{\Delta x}}^2 + \|v\|_1^2 + |M^{-1}Av|_{L^2_{\Delta x}}^2).$$

Furthermore we have the inequality

$$\|v\|_1^2 + |M^{-1}Av|^2_{L^2_{\Delta x}} \le 2\{|v|^2_{L^2_{\Delta x}} + |M^{-1}Av|^2_{L^2_{\Delta x}}\}.$$

Proof A proof of this Lemma may be found in [37]. □

Lemma 3.4 *The following two equalities hold :*

$$i) \qquad |Dv|^2_{L^2_{\Delta x}} = \|v\|_1^2;$$

$$ii) \qquad \|Dv\|_1^2 = |M^{-1}Av|^2_{L^2_{\Delta x}}.$$

Proof By equation (3.20)

$$\|v\|_1^2 = \sum_{j=0}^{J-1} \Delta x v_j \frac{\overline{v_{j+1}} - 2\overline{v_j} + \overline{v_{j-1}}}{\Delta x^2}.$$

Now by summation by parts and the periodic boundary conditions we see that $|Dv|^2_{L^2_{\Delta x}} = \|v\|_1^2$ and we have proved *i*). To prove *ii*) note that by (3.3) we have that

$$(Dv)_j = \frac{(v_{j+1} - v_j)}{\Delta x} = \delta_+ v_j,$$

so that by (3.20)

$$\|Dv\|_1^2 = -\sum_{j=0}^{J-1} \Delta x(\delta_+ v_j)\delta^2(\delta_+\overline{v_j}).$$

By summation by parts we find

$$\|Dv\|_1^2 = \sum_{j=0}^{J-1} \Delta x \left|\delta_+(v_{j+1} - v_j)\right|^2.$$

Now note

$$(v_{j+2} - v_{j+1}) - (v_{j+1} - v_j) = v_{j+2} - 2v_{j+1} - v_j,$$

expand $\delta_+(v_{j+1} - v_j)$, use summation by parts and the boundary conditions to get

$$\begin{aligned}\|Dv\|_1^2 &= \sum_{j=0}^{J-1} \Delta x \left|\frac{(v_{j+2} - v_{j+1}) - (v_{j+1} - v_j)}{\Delta x}\right|^2 \\ &= \sum_{j=0}^{J-1} \Delta x \left|\frac{v_{j+2} - 2v_{j+1} + v_j}{\Delta x^2}\right|^2.\end{aligned}$$

By the periodicity of the boundary conditions:

$$\begin{aligned} \|Dv\|_1^2 &= \sum_{j=0}^{J-1} \Delta x \left| \frac{v_{j+1} - 2v_j + v_{j-1}}{\Delta x^2} \right|^2 \\ &= \left\langle M^{-1}Av, M^{-1}Av \right\rangle . \qquad \square \end{aligned}$$

We now present some lemmas which are discrete versions of well known continuous results.

Lemma 3.5 (Bounds on $L^\infty_{\Delta x}$)

The discrete space $H^1_{\Delta x}$ is embedded in the discrete space $L^\infty_{\Delta x}$ independently of Δx. Precisely

$$i)\quad |v|^2_{L^\infty_{\Delta x}} \leq 3\|v\|^2_{H^1_{\Delta x}}, \qquad \forall v \in H^1_{\Delta x}.$$

Also, for any $\epsilon > 0$, we have

$$ii)\quad |v|^2_{L^\infty_{\Delta x}} \leq (1 + \frac{2}{\epsilon^2})|v|^2_{L^2_{\Delta x}} + \frac{1}{2}\epsilon^2\|v\|_1^2 \qquad \forall v \in H^1_{\Delta x}.$$

Finally,

$$iii)\quad |v|^2_{L^\infty_{\Delta x}} \leq |v|^2_{L^2_{\Delta x}} + 2|v|_{L^2_{\Delta x}}\|v\|_1 \qquad \forall v \in H^1_{\Delta x}.$$

Proof

i) It is easily verified that for any j

$$\begin{aligned} \delta_+(v_j\overline{v_j}) &= \frac{(v_{j+1}\overline{v_{j+1}} - v_j\overline{v_j})}{\Delta x} \\ &= v_j\delta_+\overline{v_j} + \overline{v_{j+1}}\delta_+ v_j . \end{aligned}$$

Summing over j from k_0 to $k-1$ we get

$$|v_k|^2 - |v_{k_0}|^2 = \sum_{j=k_0}^{k-1} \Delta x(v_j\delta_+\overline{v}_j + \overline{v}_{j+1}\delta_+ v_j),$$

and taking the real part yields

$$\begin{aligned} |v_k|^2 &= |v_{k_0}|^2 + \frac{1}{2}\sum_{j=k_0}^{k-1} \Delta x \left((v_j + v_{j+1})\delta_+\overline{v_j} + (\overline{v_j} + \overline{v_{j+1}})\delta_+ v_j\right) \\ &\leq |v_{k_0}|^2 + \sum_{j=k_0}^{k-1} \Delta x \left(|v_j + v_{j+1}|\,|\delta_+ v_j|\right). \end{aligned} \tag{3.24}$$

To get the embedding we use the generalized Cauchy–Schwarz inequality ($2ab \leq \epsilon^2 a^2 + \frac{1}{\epsilon^2}b^2$), complete the square and use the periodic boundary conditions:

$$
\begin{aligned}
|v_k|^2 &\le |v_{k_0}|^2 + \frac{1}{2}\sum_{j=k_0}^{k-1} \Delta x \left(\frac{1}{\epsilon^2}|v_j + v_{j+1}|^2 + \epsilon^2|\delta_+ v_j|^2\right) \\
&\le |v_{k_0}|^2 + \frac{1}{2}\sum_{j=0}^{J-1} \Delta x \left(\frac{4}{\epsilon^2}|v_j|^2 + \epsilon^2|\delta_+ v_j|^2\right) \qquad (3.25) \\
&\le |v_{k_0}|^2 + 2\sum_{j=0}^{J-1} \Delta x \left(\frac{1}{\epsilon^2}|v_j|^2 + \epsilon^2|\delta_+ v_j|^2\right). \qquad (3.26)
\end{aligned}
$$

Taking $\epsilon^2 = 1$ in (3.26) and applying Lemma 3.4 we find

$$|v_k|^2 \le |v_{k_0}|^2 + 2\left(|v|^2_{L^2_{\Delta x}} + \|v\|_1^2\right).$$

Now if we sum over k_0 we get:

$$|v_k|^2 \le |v|^2_{L^2_{\Delta x}} + 2\left(|v|^2_{L^2_{\Delta x}} + \|v\|_1^2\right).$$

Since this is true for any k, by (3.22)

$$|v|^2_{L^\infty_{\Delta x}} \le 3\|v\|^2_{H^1_{\Delta x}}$$

and we have proved part $i)$ of the Lemma. Part $ii)$ follows from (3.25) by summing over k_0. Part $iii)$ is found by applying Cauchy–Schwarz to (3.24) :

$$
\begin{aligned}
|v_k|^2 &\le |v_{k_0}|^2 + \left\{\sum_{j=0}^{J-1} \Delta x|v_j + v_{j+1}|^2\right\}^{1/2} \left\{\sum_{j=0}^{J-1} \Delta x\, |\delta_+ v_j|^2\right\}^{1/2} \\
&\le |v_{k_0}|^2 + \left\{2\sum_{j=0}^{J-1} \Delta x \left(|v_j|^2 + |v_{j+1}|^2\right)\right\}^{1/2} \left\{\sum_{j=0}^{J-1} \Delta x\, |\delta_+ v_j|^2\right\}^{1/2} \\
&\le |v_{k_0}|^2 + \sqrt{2}\left\{\sum_{j=0}^{J-1} \Delta x|v_j|^2 + \sum_{j=0}^{J-1} \Delta x|v_{j+1}|^2\right\}^{1/2} \left\{\sum_{j=0}^{J-1} \Delta x\, |\delta_+ v_j|^2\right\}^{1/2}.
\end{aligned}
$$

Thus, using the periodic boundary conditions,

$$|v_k|^2 = |v_{k_0}|^2 + 2|v|_{L^2_{\Delta x}}\|v\|_1.$$

Summing over k_0 and noting that the choice of k was arbitrary yields the final inequality.□

Lemma 3.6 *For $1 \le q \le p < \infty$ we have the following bounds on the $L^p_{\Delta x}$ norm.*

$$|v|^p_{L^p_{\Delta x}} \le |v|^p_{L^{2p}_{\Delta x}}; \tag{3.27}$$

$$|v|^p_{L^p_{\Delta x}} \le |v|^{p-q}_{L^\infty_{\Delta x}} |v|^q_{L^q_{\Delta x}}; \tag{3.28}$$

$$|v|^p_{L^p_{\Delta x}} \le 3^{(p-q)/2} \|v\|^{p-q}_{H^1_{\Delta x}} |v|^q_{L^q_{\Delta x}} \tag{3.29}$$

and finally

$$|v|^p_{L^p_{\Delta x}} \le \left\{ |v|^2_{L^2_{\Delta x}} + 2|v|_{L^2_{\Delta x}} \|v\|_1 \right\}^{(p-q)/2} |v|^q_{L^q_{\Delta x}}. \tag{3.30}$$

Proof The first inequality (3.27) is found by an application of Hölder's inequality. To find (3.28) use the definition of the $L^p_{\Delta x}$ norm (3.15) to get:

$$|v|^p_{L^p_{\Delta x}} = \sum_{j=0}^{J-1} \Delta x |v_j|^{p-q} |v_j|^q \le \sup_{0 \le j \le J-1} |v_j|^{p-q} \sum_{j=0}^{J-1} \Delta x |v_j|^q = |v|^{p-q}_{L^\infty_{\Delta x}} |v|^q_{L^q_{\Delta x}},$$

hence we have (3.28). To get (3.29) we simply apply Lemma 3.5 *i*) to bound the $L^\infty_{\Delta x}$ norm in (3.28). The final inequality is found by applying Lemma 3.5 *iii*) to bound the $L^\infty_{\Delta x}$ norm in (3.28). □

We note that inequality (3.29) is a discrete version of the Gagliardo–Nirenberg inequality in one spatial dimension.

Lemma 3.7 *The $L^4_{\Delta x}$ norm of Dv satisfies*

$$|Dv|^4_{L^4_{\Delta x}} \le 6 \left\{ |v|^2_{L^2_{\Delta x}} + |M^{-1}Av|^2_{L^2_{\Delta x}} \right\} \|v\|^2_1.$$

Proof Applying inequality (3.29) to Dv we get:

$$\begin{aligned} |Dv|^4_{L^4_{\Delta x}} &\le 3\|Dv\|^2_{H^1_{\Delta x}} |Dv|^2_{L^2_{\Delta x}} \\ &= 3\left\{ \|Dv\|^2_1 + |Dv|^2_{L^2_{\Delta x}} \right\} |Dv|^2_{L^2_{\Delta x}}. \end{aligned}$$

Combining this and Lemma 3.4 gives:

$$|Dv|^4_{L^4_{\Delta x}} \le 3\{\|v\|^2_1 + |M^{-1}Av|^2_{L^2_{\Delta x}}\}\|v\|^2_1.$$

By Lemma 3.3 the term in brackets is norm equivalent to the natural norm on $H^2_{\Delta x}$, hence

$$|Dv|^4_{L^4_{\Delta x}} \le 6\{|v|^2_{L^2_{\Delta x}} + |M^{-1}Av|^2_{L^2_{\Delta x}}\}\|v\|^2_1. \qquad \square$$

In Section 2 the concept of Gevrey class and regularity was introduced for the continuous Ginzburg–Landau equation. Here we introduce the notion of discrete Gevrey class and regularity.

Given a function $v \in \mathbb{C}^{J}_{\text{per}}$ with Fourier expansion given by (3.17) we define the **discrete Gevrey class of regularity** τ, $G_{\tau,s,\Delta x}$, to be the normed linear space $\left\{\mathbb{C}^{J}_{\text{per}}, \|\bullet\|_{G_{\tau,s,\Delta x}}\right\}$ where

$$\|v\|_{G_{\tau,s,\Delta x}} = |\widetilde{A}e^{\tau \widetilde{A}^{s}} v|_{L^2_{\Delta x}} = \left\{ \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \widetilde{\lambda}_k^{2s} e^{2\tau \widetilde{\lambda}_k^{s}} |a_k|^2 \right\}^{\frac{1}{2}} \tag{3.31}$$

and $\widetilde{\lambda}_k$ is the k^{th} eigenvalue of $\widetilde{A}$ given in Lemma 3.1. As in the continuous case we consider $s = 1/2$ only and use $\|\bullet\|_{G_{\tau,\Delta x}}$ to denote the corresponding norm. Our first lemma is a discrete version of the continuous result that $G_\tau \subset H^{2s}$ for any $s > 0$.

Lemma 3.8 *Let $v \in \mathbb{C}^{J}_{\text{per}}$ and suppose $\exists$ constants K and $\tau > 0$ such that*

$$\|v\|^2_{G_{\tau,\Delta x}} \leq K. \tag{3.32}$$

Then, for any $\alpha > 0$, there exists a constant $C = C(\alpha, \tau, K) > 0$ such that

$$\|v\|^2_{H^{2\alpha}_{\Delta x}} = |\widetilde{A}^{\alpha} v|^2_{L^2_{\Delta x}} \leq C.$$

Proof By elementary calculus there exists a constant $C_1 = C_1(\alpha, \tau) > 0$ such that

$$x^{2\alpha} \leq C_1 x e^{2\tau x^{1/2}} \; \forall x \geq 1.$$

We apply this to $|\widetilde{A}^{\alpha} v|^2_{L^2_{\Delta x}}$ as follows

$$|\widetilde{A}^{\alpha} v|^2_{L^2_{\Delta x}} = \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \lambda_k^{2\alpha} |a_k|^2 \leq \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} C_1 \lambda_k e^{2\tau \lambda_k^{1/2}} |a_k|^2 \leq C,$$

and the lemma is proved. □

We now prove that given $u \in G_{\tau,\Delta x}$, we can find a corresponding V in a continuous Gevrey class which equals u on the grid.

Lemma 3.9 *Suppose we are given* $u \in \mathcal{C}^J_{\text{per}}$ *and constants* $C, \tau > 0$ *so that* u *satisfies*

$$\|u\|^2_{G_{\tau,\Delta x}} \leq C < \infty.$$

Then there exists $V \in G_\sigma$, $\sigma \in (0, 2\tau/\pi)$, *such that*

$$\|V\|^2_{G_\sigma} \leq \frac{\pi^2}{4} C \text{ and } P_{\Delta x} V = u.$$

Furthermore $V \in H^\alpha$, $\forall \alpha \geq 0$.

Proof Consider the Fourier expansion of $u \in L^2_{\Delta x}$:

$$u = \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} a_k \psi_k$$

and define $V(x)$ by

$$V(x) := \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} a_k e^{2\pi i k x}. \tag{3.33}$$

Clearly we have that $P_{\Delta x} V(x) = u$ and

$$\begin{aligned} |V|^2_{L^2} &= \int_0^1 \left(\sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} a_k e^{2\pi i k x} \right) \left(\sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \overline{a_k} e^{-2\pi i k x} \right) dx \\ &= \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} |a_k|^2 < \infty. \end{aligned}$$

So certainly V is bounded in L^2 independently of Δx. In order to prove V is Gevrey of regularity σ consider

$$\sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \widetilde{\Lambda}_k e^{2\widetilde{\Lambda}_k^{1/2}\sigma} |a_k|^2 = \sum_{k=-\lceil J/2 \rceil, k\neq 0}^{\lfloor J/2 \rfloor} \widetilde{\lambda}_k e^{2\widetilde{\lambda}_k^{1/2}\tau} |a_k|^2 \left\{ \frac{\widetilde{\Lambda}_k}{\widetilde{\lambda}_k} e^{-2\widetilde{\lambda}_k^{1/2}\tau + 2(\widetilde{\Lambda}_k^{1/2})\sigma} \right\} + \widetilde{\Lambda}_0 e^{2\widetilde{\Lambda}_0^{1/2}\sigma} |a_0|^2.$$

Re–writing the right–hand side so we can apply Lemma 3.2 we find, using $\widetilde{\Lambda}_0 = \widetilde{\lambda}_0$ and the choice of $\sigma \in (0, 2\tau/\pi)$, that

$$\begin{aligned} \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \widetilde{\Lambda}_k e^{2\widetilde{\Lambda}_k^{1/2}\sigma} |a_k|^2 &= \sum_{k=-\lceil J/2 \rceil, k\neq 0}^{\lfloor J/2 \rfloor} \widetilde{\lambda}_k e^{2\widetilde{\lambda}_k^{1/2}\tau} |a_k|^2 \left\{ \frac{1}{r(k)} e^{2(-r(k)^{1/2}\tau + \sigma)\widetilde{\Lambda}_k^{1/2}} \right\} \\ &\quad + \widetilde{\Lambda}_0 e^{2\widetilde{\Lambda}_0^{1/2}\sigma} |a_0|^2. \end{aligned}$$

$$\leq \sum_{k=-\lceil J/2\rceil, k\neq 0}^{\lfloor J/2\rfloor} \widetilde{\lambda}_k e^{2\widetilde{\lambda}_k^{1/2}\tau} |a_k|^2 \left\{ \frac{\pi^2}{4} e^{2(-\frac{2\tau}{\pi}+\sigma)\widetilde{\lambda}_k^{1/2}} \right\} + \widetilde{\lambda}_0 e^{2\widetilde{\lambda}_0^{1/2}\sigma} |a_0|^2.$$

$$\leq \frac{\pi^2}{4} \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \widetilde{\lambda}_k e^{2\widetilde{\lambda}_k^{1/2}\tau} |a_k|^2.$$

By the hypothesis we have a uniform bound on the right hand side and thus $V \in G_\sigma$. The rest is an easy consequence of the Gevrey regularity and follows from (2.12).□

We now consider the smoothing action of the linear semi–groups generated by $\widetilde{A}$ in the semi–discrete case. We obtain discrete analogs of the well known smoothing properties for analytic semigroups [39].

Lemma 3.10 *Consider the linear homogeneous problem given by*

$$u_t = -(1+i\nu)\widetilde{A}u.$$

Then the linear operator $-(1+i\nu)\widetilde{A}$ is the infinitesimal generator of an analytic semigroup $E_{\Delta x}(t)$ defined by

$$E_{\Delta x}(t) := e^{-(1+i\nu)\widetilde{A}t}.$$

Furthermore for any α, β: $0 \leq \alpha \leq \beta$ there exists a constant $C = C(\alpha,\beta)$ independent of Δx such that the discrete smoothing property holds, that is

$$\|E_{\Delta x}(t)v\|_{H^\beta_{\Delta x}} \leq Ct^{-(\beta-\alpha)/2}\|v\|_{H^\alpha_{\Delta x}}, \quad \forall\, t > 0.$$

Proof Since $(1+i\nu)\widetilde{A}$ is a sectorial operator, it is the infinitesimal generator of the analytic semigroup $E_{\Delta x}$. For further details see for example [39, 24]. Let v have Fourier expansion as in (3.17), then

$$\|E_{\Delta x}(t)v\|^2_{H^\beta_{\Delta x}} = \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \lambda_k^\beta e^{-2\lambda_k t}|a_k|^2.$$

Elementary calculus shows that there exists $C = C(\alpha,\beta)$ such that for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}^+$

$$x^\beta e^{-2xt} \leq \frac{Cx^\alpha}{t^{\beta-\alpha}},$$

and the result follows. □

The next lemma gives a Lipschitz inequality for the non–linear term.

Lemma 3.11

With $G(\cdot)$ defined in (3.6), for all $u, v \in L^2_{\Delta x}$, $\mu \in I\!R$, we have that

$$Re\left\{(1+i\mu)\left\langle G(|u|^2)u - G(|v|^2)v, u-v\right\rangle\right\} \le (1+\mu^2)^{1/2}(|u|^2_{L^\infty_{\Delta x}} + |v|^2_{L^\infty_{\Delta x}})|u-v|^2_{L^2_{\Delta x}}.$$

Proof The lemma follows by writing the inner–product as a summation, taking the real part and noting that for any $a, b \in \mathbb{C}$

$$\begin{aligned}
&\left(|a|^2a - |b|^2b\right)\left(\overline{a}-\overline{b}\right) + \left(|a|^2\overline{a} - |b|^2\overline{b}\right)(a-b) \\
&= |a|^4 - |a|^2a\overline{b} - |b|^2b\overline{a} + |b|^4 + |a|^4 - |a|^2b\overline{a} - |b|^2a\overline{b} + |b|^4 \\
&= |a|^2\left(|a|^2 - a\overline{b} - b\overline{a} + |b|^2\right) - |b|^2|a|^2 + |b|^2\left(|a|^2 - b\overline{a} - a\overline{b} + |b|^2\right) - |b|^2|a|^2 \\
&\le \left(|a|^2 + |b|^2\right)\left(|a-b|^2\right). \qquad \square
\end{aligned}$$

4 The Semi-Discrete Problem

We define the ball centre 0, radius ρ in $L^2_{\Delta x}(=H^0_{\Delta x})$, $H^1_{\Delta x}$ and $G_{\tau,\Delta x}$ respectively by

$$\mathcal{B}_0(\rho) := \left\{v \in \mathbb{C}^J_{\text{per}} \ : \ |v|_{L^2_{\Delta x}} \le \rho\right\},$$

$$\mathcal{B}_1(\rho) := \left\{v \in \mathbb{C}^J_{\text{per}} \ : \ \|v\|_{H^1_{\Delta x}} \le \rho\right\},$$

$$\mathcal{B}_{G_\tau}(\rho) := \left\{v \in \mathbb{C}^J_{\text{per}} \ : \ \|v\|_{G_{\tau,\Delta x}} \le \rho\right\}.$$

The immediate aim is to prove that the set of ordinary differential equations (3.7) which arise from the spatial discretisation of (2.1), satisfies semi–discrete versions of **C1 - C5** stated in Theorem 2.1. These semi–discrete versions will be denoted **SD1 - SD5** and will be made precise in the statement of the Theorems. The proof of **SD2 - SD4** is adapted from the continuous analysis in [46] and the proof of **SD5** the analysis of [10]. Our aim is to ensure that the constants in **SD1 - SD5** are uniformly bounded as $\Delta x \to 0$.

This first lemma forms the backbone of the proof of both **SD1** and **SD2**.

Lemma 4.1 *If $u(t)$ is a solution of (3.7) defined on $t \in [0,T)$ then the $L^2_{\Delta x}$ norm of $u(t)$ satisfies*

$$\frac{1}{2}\frac{d}{dt}|u|^2_{L^2_{\Delta x}} = R|u|^2_{L^2_{\Delta x}} - \|u\|^2_1 - |u|^4_{L^4_{\Delta x}}, \forall t \in [0,T) \tag{4.1}$$

and hence

$$|u(t)|^2_{L^2_{\Delta x}} \leq |u(0)|^2_{L^2_{\Delta x}} + 2|R| \quad \forall t \in [0,T). \tag{4.2}$$

Proof Take the $L^2_{\Delta x}$ inner product of (3.7) with u to get :

$$\left\langle \frac{du}{dt}, u \right\rangle = R\langle u,u\rangle - (1+i\nu)\left\langle M^{-1}Au, u\right\rangle - (1+i\mu)\left\langle G(|u|^2)u, u\right\rangle .$$

Taking the real part, using (3.21) and noting that

$$\left\langle G(|u|^2)u, u\right\rangle = \sum_{j=0}^{J-1} \Delta x |u_j|^2 u_j \overline{u_j} = \sum_{j=0}^{J-1} \Delta x |u_j|^4 = |u|^4_{L^4_{\Delta x}},$$

yields equation (4.1).

In order to get (4.2) we have three possible cases to consider: $R < 0$, $R = 0$, $R > 0$.

If $R < 0$ then

$$\frac{d}{dt}|u|^2_{L^2_{\Delta x}} \leq 2R|u|^2_{L^2_{\Delta x}}.$$

This we can solve to find

$$|u(t)|^2_{L^2_{\Delta x}} \leq |u(0)|^2_{L^2_{\Delta x}} \exp(2Rt). \tag{4.3}$$

Since $R < 0$ the result is trivially true.

If $R = 0$ then inequality (3.27) applied to (4.1) yields:

$$\frac{1}{2}\frac{d}{dt}|u|^2_{L^2_{\Delta x}} \leq -|u|^4_{L^2_{\Delta x}}$$

and integration gives

$$\frac{1}{|u(0)|^2_{L^2_{\Delta x}}} + t \leq \frac{1}{|u(t)|^2_{L^2_{\Delta x}}},$$

from which we find

$$|u(t)|^2_{L^2_{\Delta x}} \leq \frac{|u(0)|^2_{L^2_{\Delta x}}}{1 + t|u(0)|^2_{L^2_{\Delta x}}}. \tag{4.4}$$

Since $t \geq 0$

$$|u(t)|^2_{L^2_{\Delta x}} \leq |u(0)|^2_{L^2_{\Delta x}}$$

and (4.2) is satisfied.

If $R > 0$ then we use that $\forall s \in \mathbb{R}$

$$\frac{1}{2}s^4 - 2Rs^2 \geq -2R^2 \tag{4.5}$$

to bound the non–linear term in equation (4.1). This yields

$$\begin{aligned}\frac{d}{dt}|u|^2_{L^2_{\Delta x}} &\leq 2R|u|^2_{L^2_{\Delta x}} - 2\|u\|_1^2 - |u|^4_{L^4_{\Delta x}} - 4R|u|^2_{L^2_{\Delta x}} + 4R^2 \\ &= -2R|u|^2_{L^2_{\Delta x}} - 2\|u\|_1^2 - |u|^4_{L^4_{\Delta x}} + 4R^2,\end{aligned} \tag{4.6}$$

and so

$$\frac{d}{dt}|u|^2_{L^2_{\Delta x}} \leq -2R|u|^2_{L^2_{\Delta x}} + 4R^2.$$

Now apply the standard Gronwall inequality (see for example [46, p88]) to get

$$|u(t)|^2_{L^2_{\Delta x}} \leq |u(0)|^2_{L^2_{\Delta x}} \exp(-2Rt) + 2R(1 - \exp(-2Rt)) \quad \forall t \geq 0. \tag{4.7}$$

Hence the Lemma is proved. □

We now prove **SD1**.

Theorem 4.1 (SD1) *For each $u^0 \in L^2_{\Delta x}$ there exists a unique solution $u \in C^1([0,T]; L^2_{\Delta x})$ of (3.7) for all $T > 0$. Hence there exists a semi–group $S_{\Delta x}(\bullet)\bullet \in C(\mathbb{R}^+ \times L^2_{\Delta x}, L^2_{\Delta x})$ defined by $S_{\Delta x}(t)u^0 = u(t)$.*

Proof

Consider (3.7) as a set of ordinary differential equations in $\mathbb{R}^{2J}$. Then the right hand side of (3.7) is locally Lipschitz with constant K_1, and local existence and uniqueness is immediate from the standard theory such as in [23] .

To prove existence for any $T > 0$ we note that Lemma 4.1 gives an a priori bound on the norm of $u(t)$; since the problem is finite dimensional global existence follows. Continuity in time also follows from the standard theory in [23]. □

Theorem 4.2 (SD2) *There exists a constant $\rho_0 = \rho_0(R) > 0$, independent of Δx, such that the ball $\mathcal{B}_0(\rho_0)$ is absorbing and positively invariant for the semi–group $\{S_{\Delta x}(t)\}_{t\geq 0}$.*

Proof Recall (4.1) and consider the three possible cases for R: $R < 0$, $R = 0$ and $R > 0$.

For $R < 0$ Lemma 4.1, equation (4.3), immediately yields trivial dynamics. Thus for any $\epsilon > 0$ we have $\rho_0 = \epsilon$, and for $B \subset \mathcal{B}_0(\rho), \rho > 0$, we have that $S_{\Delta x}(t)B \subset \mathcal{B}_0(\rho_0)$ $\forall t \geq t_0$ where

$$t_0 = \begin{cases} \frac{1}{2R}\log\left(\frac{\rho_0}{\rho}\right) & \rho > \rho_0 \\ 0 & \rho \leq \rho_0 \end{cases}.$$

For $R = 0$ equation (4.4) immediately gives algebraic decay to zero. For any $\epsilon > 0$ we have $\rho_0 = \epsilon$ for any $\epsilon > 0$ and for $B \subset \mathcal{B}_0(\rho), \rho > 0$, we have that $S_{\Delta x}(t)B \subset \mathcal{B}_0(\rho_0)$ $\forall t \geq t_0$ where

$$t_0 = \begin{cases} \frac{1}{\rho_0^2} - \frac{1}{\rho^2} & \rho > \rho_0 \\ 0 & \rho \leq \rho_0 \end{cases}.$$

For $R > 0$ note that (4.7) gives

$$\limsup_{t\to\infty} |u(t)|_{L^2_{\Delta x}} \leq \rho'$$

where $\rho' := \sqrt{2R}$.

Therefore the ball $\mathcal{B}_0(\rho_0)$, $\rho_0 > \rho'$ is positively invariant and is absorbing for the semi-group $S_{\Delta x}(t)$. Thus for $B \subset \mathcal{B}_0(\rho), \rho > 0$, we have that $S_{\Delta x}(t)B \subset \mathcal{B}_0(\rho_0)$ $\forall t \geq t_0$. For $\rho \leq \rho_0$, we have $t_0 = 0$, whereas for $\rho > \rho_0$ we find that t_0 is given by

$$t_0 = \frac{1}{2R}\log\left\{\frac{\rho^2}{\rho_0^2 - \rho'^2}\right\}.$$

Hence the theorem is proved. □

For t_0 given in Theorem 4.2 we can integrate (4.6) between t and $t + r$ for $t > t_0$ to get

$$\int_t^{t+r} \frac{d}{dt}|u(t)|^2_{L^2_{\Delta x}}\,dt \leq -\int_t^{t+r} \left(2R|u|^2_{L^2_{\Delta x}} + |u|^4_{L^4_{\Delta x}} + 2\|u\|_1^2 - 4R^2\right)\,dt;$$

which becomes using the uniform bound on the $L^2_{\Delta x}$ norm from Theorem 4.2

$$\int_t^{t+r} \left\{2R|u|^2_{L^2_{\Delta x}} + 2\|u\|_1^2 + |u|^4_{L^4_{\Delta x}}\right\} dt \leq \rho_0^2 + 4R^2 r, \quad \forall t > t_0. \tag{4.8}$$

This will be of use when applying the uniform Gronwall lemma to prove the existence of an absorbing set in $H^1_{\Delta x}$.

We recall that since we are in finite dimensions, the existence of the $L^2_{\Delta x}$ absorbing set immediately yields the existence of a global attractor and, by inverse inequalities, the existence of absorbing sets in $H^1_{\Delta x}$. However since we wish to prove convergence of the global attractors as $\Delta x \to 0$ we seek bounds in $H^1_{\Delta x}$ which hold uniformly as $\Delta x \to 0$.

Since for $R \leq 0$ both the continuous equation and the semi-discrete system have trivial dynamics it is assumed for the remainder of this section that R is strictly positive.

Theorem 4.3 (SD3) *There exists a constant $\rho_1 = \rho_1(R) > 0$, independent of Δx, such that the ball $\mathcal{B}_1(\rho_1)$ is absorbing and positively invariant for the semi-group $\{S_{\Delta x}(t)\}_{t\geq 0}$.*

Proof To show there exists an absorbing set in the discrete $H^1_{\Delta x}$ norm, we seek a bound on the $H^1_{\Delta x}$ semi-norm $\|\cdot\|_1$ independent of the spatial mesh size Δx. Taking the discrete Dirichlet inner-product (defined in (3.20)) of (3.7) with u and taking the real part we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_1^2 = R\|u\|_1^2 - |M^{-1}Au|^2_{L^2_{\Delta x}} + \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1} \Delta x\ |u_j|^2 u_j \delta^2 \overline{u}_j\right\}.$$

Now use summation by parts and the boundary conditions to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_1^2 &= R\|u\|_1^2 - |M^{-1}Au|^2_{L^2_{\Delta x}} \\
&\quad - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1} \Delta x\ (u_{j+1}|u_{j+1}|^2 - u_j|u_j|^2)\frac{\overline{u_{j+1}} - \overline{u_j}}{\Delta x^2}\right\} \\
&= R\|u\|_1^2 - |M^{-1}Au|^2_{L^2_{\Delta x}} \\
&\quad - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1} \Delta x\ \left(\frac{u_{j+1} - u_j}{\Delta x}(|u_{j+1}|^2 + |u_j|^2)\frac{\overline{u_{j+1}} - \overline{u_j}}{\Delta x}\right.\right. \\
&\qquad \left.\left. + (|u_{j+1}|^2 u_j - u_{j+1}|u_j|^2)\frac{\overline{u_{j+1}} - \overline{u_j}}{\Delta x^2}\right)\right\} \\
&\leq R\|u\|_1^2 - |M^{-1}Au|^2_{L^2_{\Delta x}} \\
&\quad - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1} \Delta x\ \left(\left|\frac{u_{j+1} - u_j}{\Delta x}\right|^2 (|u_{j+1}|^2 + |u_j|^2)\right.\right. \\
&\qquad \left.\left. + u_{j+1}u_j\frac{(\overline{u_{j+1}} - \overline{u_j})^2}{\Delta x^2}\right)\right\}.
\end{aligned} \tag{4.9}$$

Completing the square on the last term gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_1^2 \\ &\le R\|u\|_1^2 - |M^{-1}Au|^2_{L^2_{\Delta x}} + (1+\mu^2)^{1/2}\sum_{j=0}^{J-1}\Delta x\left|\frac{u_{j+1}-u_j}{\Delta x}\right|^2(|u_{j+1}|^2+|u_j|^2) \\ &+(1+\mu^2)^{1/2}\sum_{j=0}^{J-1}\Delta x\frac{|u_{j+1}|^2+|u_j|^2}{2}\left|\frac{\overline{u_{j+1}}-\overline{u_j}}{\Delta x}\right|^2 \\ &\le R\|u\|_1^2 - |M^{-1}Au|^2_{L^2_{\Delta x}} + \frac{3}{2}(1+\mu^2)^{1/2}\sum_{j=0}^{J-1}\Delta x\left|\frac{u_{j+1}-u_j}{\Delta x}\right|^2(|u_j|^2+|u_{j+1}|^2).\end{aligned}$$

Now apply Schwarz's inequality on the last term and use the periodicity to get

$$\frac{1}{2}\frac{d}{dt}\|u\|_1^2 \le R\|u\|_1^2 - |M^{-1}Au|^2_{L^2_{\Delta x}} + 3(1+\mu^2)^{1/2}|u|^2_{L^4_{\Delta x}}|Du|^2_{L^4_{\Delta x}}. \quad (4.10)$$

At this point we call upon Lemma 3.7 which bounds the $L^4_{\Delta x}$ norm of Du so that (4.10) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_1^2 &\le R\|u\|_1^2 - |M^{-1}Au|^2_{L^2_{\Delta x}} \\ &+ 3\sqrt{6}(1+\mu^2)^{1/2}|u|^2_{L^4_{\Delta x}}\|u\|_1\left\{|u|^2_{L^2_{\Delta x}} + |M^{-1}Au|^2_{L^2_{\Delta x}}\right\}^{1/2} \quad (4.11)\end{aligned}$$

Complete the square on the last term to find that

$$\frac{1}{2}\frac{d}{dt}\|u\|_1^2 \le R\|u\|_1^2 - |M^{-1}Au|^2_{L^2_{\Delta x}} + \frac{54}{2}(1+\mu^2)|u|^4_{L^4_{\Delta x}}\|u\|_1^2 + \frac{1}{2}\left\{|u|^2_{L^2_{\Delta x}} + |M^{-1}Au|^2_{L^2_{\Delta x}}\right\};$$

and hence

$$\frac{d}{dt}\|u\|_1^2 \le (2R + 54(1+\mu^2)|u|^4_{L^4_{\Delta x}})\|u\|_1^2 + |u|^2_{L^2_{\Delta x}}. \quad (4.12)$$

All that remains is to apply the uniform Gronwall Lemma [46, p89] with

$$y = \|u\|_1^2, \quad g = 2R + 54(1+\mu^2)|u|^4_{L^4_{\Delta x}}, \quad h = |u|^2_{L^2_{\Delta x}};$$

and use the integral bound (4.8) to find the constants a_1, a_2 and a_3

$$\begin{aligned} a_1 &= 2Rr + 54(1+\mu^2)\left\{\rho_0^2 + 4R^2 r\right\}, \\ a_2 &= \rho_0^2, \\ a_3 &= \frac{1}{2}(\rho_0^2 + 4R^2 r). \end{aligned}$$

Therefore

$$\|u(t)\|_1^2 \leq \left(\frac{a_3}{r} + a_2\right) \exp(a_1) \; \forall t \geq t_0 + r; \tag{4.13}$$

where r is an arbitrary positive number and t_0 is as in Theorem 4.2. Thus we obtain absorbing balls, $\mathcal{B}_1(\rho_1)$, of radius ρ_1 satisfying

$$\rho_1^2 > \rho_0^2 + \left(\frac{a_3}{r} + a_2\right) \exp(a_1)$$

and for $B \subset \mathcal{B}_1(\rho), \rho > \rho_1$, we have that $S_{\Delta x}(t)B \subset \mathcal{B}_1(\rho_1) \; \forall t \geq t_1$, where $t_1 > t_0 + r$. □

We now state the theorem on the existence of a a global attractor for (3.7).

Theorem 4.4 (SD4) *The semi–group $S_{\Delta x}(t)$ for equation* (2.1) *possesses a global attractor $\mathcal{A}_{\Delta x}$ given by*

$$\mathcal{A}_{\Delta x} = \omega(\mathcal{B}_1(\rho_1))$$

and hence is bounded in $H^1_{\Delta x}$, independently of Δx.

Proof Theorems 4.1 and 4.3 give us all we require by applying [46, Theorem 1.1, p.23]. □

Now that we have uniform estimates for the $L^2_{\Delta x}$ and $H^1_{\Delta x}$ norms of solutions $u(t)$ for $t > t_1$, we can apply Lemma 3.5 to obtain $L^\infty_{\Delta x}$ bounds independent of Δx on $u(t)$ inside the $H^1_{\Delta x}$ absorbing set (i.e. for all $t > t_1$). Furthermore it should be noted that we have established in Theorems 4.2–4.4 the existence of $L^2_{\Delta x}$ absorbing balls, $H^1_{\Delta x}$ absorbing balls and a global attractor without imposing a restriction on the spatial mesh size Δx.

The proof presented above of Theorem 4.3 is adapted from the continuous analysis in [46]. However in one spatial dimension there are alternative methods for establishing an absorbing ball in H^1. In particular the analysis in [11] treats the non–linear term in a more delicate fashion to obtain bounds dependent on the parameters. A discrete version of their argument may be found in [37].

5 Discrete Gevrey Class

In Section 2 the concept of Gevrey class and regularity was introduced and we stated that solutions to the complex Ginzburg–Landau equation were of a Gevrey class - see Theorem 2.1. We prove in **SD5** below that solutions to the semi–discrete system (3.7) are in a discrete class of regularity. In addition to being an interesting result itself, the extra regularity will be exploited to simplify the proofs concerning attractors in Section 6. To simplify forthcoming expressions we employ the following notation:

$$\sum_{k,\ell,m}' := \sum_{\substack{k,\ell,m\in[-J/2\rceil,J/2\rfloor] \\ k-\ell+m-\eta J=p,\eta\in\mathbb{N}}} \tag{5.1}$$

Thus $\sum_{k,\ell,m}'$ means the sum over all k,ℓ,m in the appropriate interval $[-J/2, J/2]$ such that $k-\ell+m = p+\eta J$ for any integer η.

We now prove that solutions to the semi–discrete problem **SD** (namely equation (3.7)) lie in a discrete Gevrey class. In the following t_1 and ρ_1 are given in Theorem 4.3 and its proof.

Theorem 5.1 (SD5 Gevrey Regularity)

i) *Consider equation* (3.7) *with initial condition* $u(0) = u^0 \in \mathcal{B}_1(\rho)$. *Then there exists* $T = T(\rho)$ *and* $\rho' = \sqrt{2(1+\rho^2)}$ *such that*

$$u(t) \in \mathcal{B}_{G_t}(\rho'), \quad \forall t \in (0,T].$$

ii) *Consider* (3.7) *with* $u(0) = u^0 \in \mathcal{B}_1(\rho)$. *Then there exists* $\tau = \tau(\rho_1) > 0$ *and* $t_2 = t_2(t_1,\rho_1) > 0$ *such that*

$$u(t) \in \mathcal{B}_{G_\tau}(\rho_2), \quad \forall\, t \geq t_2,$$

where $\rho_2 = \sqrt{2(1+\rho_1^2)}$.

Proof The method of proof is adapted from the analysis in the continuous case employed by Doelman and Titi [10] for (2.1) and Duan et al [13] for the generalized complex Ginzburg–Landau equation.

Let $u(t)$ be given by the Fourier series

$$u(t) = \sum_{k=-J/2\rceil}^{J/2\rfloor} a_k(t)\psi_k$$

and define $v(t)$ by

$$v(t) := e^{t\widetilde{A}^{1/2}} u(t) = \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} e^{t\widetilde{\lambda}_k^{\frac{1}{2}}} a_k(t)\psi_k \,. \tag{5.2}$$

The eigenvectors ψ_k and eigenvalues $\widetilde{\lambda}$ used here are defined in Lemma 3.1. We wish to find a bound for $|\widetilde{A}^{1/2} v|_{L^2_{\Delta x}}$. Now differentiate $v(t)$ with respect to t and substitute in for u_t from (3.8) to get

$$\begin{aligned} v_t &= \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \widetilde{\lambda}_k^{1/2} e^{\widetilde{\lambda}_k^{\frac{1}{2}} t} a_k(t)\psi_k + \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} a_k'(t) e^{\widetilde{\lambda}_k^{\frac{1}{2}} t}\psi_k \\ &= \widetilde{A}^{\frac{1}{2}} v(t) + e^{t\widetilde{A}^{1/2}} u_t \\ &= \widetilde{A}^{\frac{1}{2}} v(t) + e^{t\widetilde{A}^{1/2}} \left\{ \widetilde{R}u - (1+i\nu)\widetilde{A}u - (1+i\mu)G(|u|^2)u \right\} \\ &= \widetilde{A}^{\frac{1}{2}} v(t) + \widetilde{R}v - (1+i\nu)\widetilde{A}v - (1+i\mu)e^{t\widetilde{A}^{1/2}} G(|u|^2)u. \end{aligned} \tag{5.3}$$

Take the $L^2_{\Delta x}$ inner–product of (5.3) with $\widetilde{A}v$ and then take the real part :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\widetilde{A}^{\frac{1}{2}} v|^2_{L^2_{\Delta x}} \leq\ & \mathrm{Re}\left\{ \left\langle \widetilde{A}^{\frac{1}{2}} v, \widetilde{A}v \right\rangle \right\} + |\widetilde{R}||\widetilde{A}^{\frac{1}{2}} v|^2_{L^2_{\Delta x}} - |\widetilde{A}v|^2_{L^2_{\Delta x}} \\ & -\mathrm{Re}\left\{ (1+i\mu)\left\langle e^{t\widetilde{A}^{1/2}} G(|u|^2)u, \widetilde{A}v \right\rangle \right\}. \end{aligned}$$

Apply the generalised Cauchy–Schwarz inequality to get for all $\epsilon > 0$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\widetilde{A}^{\frac{1}{2}} v|^2_{L^2_{\Delta x}} \leq\ & (|\widetilde{R}| + \frac{1}{2\epsilon})|\widetilde{A}^{\frac{1}{2}} v|^2_{L^2_{\Delta x}} + \frac{\epsilon}{2}|\widetilde{A}v|^2_{L^2_{\Delta x}} - |\widetilde{A}v|^2_{L^2_{\Delta x}} \\ & -\mathrm{Re}\left\{ (1+i\mu)\left\langle e^{t\widetilde{A}^{1/2}} G(|u|^2)u, \widetilde{A}v \right\rangle \right\}. \end{aligned} \tag{5.4}$$

Now consider the non–linear term separately, and start by noting that

$$\begin{aligned} \left\{G(|u|^2)u\right\}_j &= \left\{ \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} a_k e^{2\pi i k j\Delta x} \right\} \left\{ \sum_{\ell=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \overline{a_\ell} e^{-2\pi i \ell j\Delta x} \right\} \left\{ \sum_{m=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} a_m e^{2\pi i m j\Delta x} \right\} \\ &= \sum_{p=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \sum_{k,\ell,m}{}' a_k \overline{a_\ell} a_m e^{2\pi i p j\Delta x}. \end{aligned}$$

Thus,

$$e^{t\widetilde{A}^{1/2}} G(|u|^2)u = \sum_{p=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \sum_{k,\ell,m}{}' a_k \overline{a_\ell} a_m e^{t\widetilde{\lambda}_p^{1/2}} \psi_p.$$

We let b_p denote the p^{th} Fourier coefficient for $e^{t\widetilde{A}^{1/2}} G(|u|^2)u$, so that

$$b_p = \left\langle e^{t\widetilde{A}^{1/2}} G(|u|^2)u, \psi_p \right\rangle = \sum_{k,\ell,m}{}' a_k \overline{a_\ell} a_m e^{t\widetilde{\lambda}_p^{1/2}} \tag{5.5}$$

Recall from (3.11) that the eigenvalue $\widetilde{\lambda}_p = 1 + \lambda_p$, where

$$\begin{aligned} \lambda_p^{\frac{1}{2}} &= \left| \frac{2}{\Delta x} \sin(p\pi\Delta x) \right| \\ &= \frac{2}{\Delta x} \left| \sin((k - \ell + m - \eta J)\pi\Delta x) \right|, \qquad \eta \in \mathbb{N}. \end{aligned}$$

Thus, by the trigonometric identity for $\sin(a+b)$, we have

$$\begin{aligned} \lambda_p^{\frac{1}{2}} &\leq \frac{2}{\Delta x} |\sin(k\pi\Delta x)| + \frac{2}{\Delta x} |\sin(\ell\pi\Delta x)| + \frac{2}{\Delta x} |\sin(m\pi\Delta x)| \\ &= \lambda_k^{\frac{1}{2}} + \lambda_\ell^{\frac{1}{2}} + \lambda_m^{\frac{1}{2}} . \end{aligned}$$

Therefore, for every k, ℓ, m such that $k - \ell + m = p + \eta J$, we have

$$\widetilde{\lambda}_p^{\frac{1}{2}} \leq \widetilde{\lambda}_k^{\frac{1}{2}} + \widetilde{\lambda}_\ell^{\frac{1}{2}} + \widetilde{\lambda}_m^{\frac{1}{2}}$$

and from (5.5)

$$|b_p| \leq \sum_{k,\ell,m}{}' e^{t\widetilde{\lambda}_k^{\frac{1}{2}}} |a_k| \, e^{t\widetilde{\lambda}_\ell^{\frac{1}{2}}} |a_\ell| \, e^{t\widetilde{\lambda}_m^{\frac{1}{2}}} |a_m|.$$

Now define $\widehat{v}$ by

$$\widehat{v} := \sum_{k=-J/2\rceil}^{J/2\rfloor} e^{t\widetilde{\lambda}_k^{\frac{1}{2}}} |a_k| \psi_k, \tag{5.6}$$

then

$$|\widehat{v}|_{L^2_{\Delta x}} = |v|_{L^2_{\Delta x}} \, ; |\widetilde{A}^{\frac{1}{2}} \widehat{v}|_{L^2_{\Delta x}} = |\widetilde{A}^{\frac{1}{2}} v|_{L^2_{\Delta x}} \, ; \; |\widetilde{A}\widehat{v}|_{L^2_{\Delta x}} = |\widetilde{A}v|_{L^2_{\Delta x}}. \tag{5.7}$$

Furthermore, the Fourier coefficient of $G(|\widehat{v}|^2)\widehat{v}$, c_p, $(p = -J/2\rceil, \cdots J/2\rfloor)$, is given by

$$c_p = \sum_{k,\ell,m}{}' e^{t\widetilde{\lambda}_k^{\frac{1}{2}}} |a_k| \, e^{t\widetilde{\lambda}_\ell^{\frac{1}{2}}} |a_\ell| \, e^{t\widetilde{\lambda}_m^{\frac{1}{2}}} |a_m|.$$

Hence

$$|b_p| \leq c_p. \tag{5.8}$$

Recalling that $\langle \psi_p, \psi_q \rangle = \delta_{pq}$ and using (5.8) we find that

$$-\text{Re} \left\{ (1 + i\mu) \left\langle e^{t\widetilde{A}^{1/2}} G(|u|^2)u, \widetilde{A}v \right\rangle \right\}$$

$$
\begin{aligned}
&= -\mathrm{Re}\left\{(1+i\mu)\left\langle \sum_{p=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} b_p\psi_p\,,\ \sum_{q=-\lceil J/2\rceil}^{\lfloor J/2\rfloor}\widetilde{\lambda}_q e^{t\widetilde{\lambda}_q^{\frac{1}{2}}}\overline{a_q}\overline{\psi}_q\right\rangle\right\}\\
&= -\mathrm{Re}\left\{(1+i\mu)\sum_{p=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} b_p\overline{a_p}\widetilde{\lambda}_p e^{t\widetilde{\lambda}_p^{1/2}}\right\}\\
&\le (1+\mu^2)^{\frac{1}{2}}\sum_{p=-\lceil J/2\rceil}^{\lfloor J/2\rfloor}|b_p||a_p|\widetilde{\lambda}_p e^{t\widetilde{\lambda}_p^{1/2}}\\
&\le (1+\mu^2)^{\frac{1}{2}}\sum_{p=-\lceil J/2\rceil}^{\lfloor J/2\rfloor}c_p|a_p|\widetilde{\lambda}_p e^{t\widetilde{\lambda}_p^{1/2}}\\
&= (1+\mu^2)^{\frac{1}{2}}\left\langle \sum_{p=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} c_p\psi_p\,,\ \sum_{q=-\lceil J/2\rceil}^{\lfloor J/2\rfloor}\widetilde{\lambda}_q e^{t\widetilde{\lambda}_q^{\frac{1}{2}}}|a_q|\overline{\psi}_q\right\rangle\\
&= (1+\mu^2)^{\frac{1}{2}}\left\langle G(|\widehat{v}|^2)\widehat{v},\widetilde{A}\widehat{v}\right\rangle \qquad (5.9)
\end{aligned}
$$

If we expand the inner–product in (5.9) and then apply Cauchy–Schwarz we find

$$
\begin{aligned}
&-\mathrm{Re}\left\{(1+i\mu)\left\langle e^{t\widetilde{A}^{1/2}}G(|u|^2)u,\widetilde{A}v\right\rangle\right\}\\
&\le (1+\mu^2)^{\frac{1}{2}}\sum_{j=0}^{J-1}\Delta x|\widehat{v}_j|^3\left|\overline{\widehat{v}_j}+\delta^2\overline{\widehat{v}_j}\right|\\
&\le (1+\mu^2)^{\frac{1}{2}}\left\{\sum_{j=0}^{J-1}\Delta x|\widehat{v}_j|^6\right\}^{1/2}\left\{\sum_{j=0}^{J-1}\Delta x\left|\widehat{v}_j+\delta^2\widehat{v}_j\right|^2\right\}^{1/2}\\
&\le (1+\mu^2)^{\frac{1}{2}}|\widehat{v}|^3_{L^6_{\Delta x}}|\widetilde{A}\widehat{v}|_{L^2_{\Delta x}}. \qquad (5.10)
\end{aligned}
$$

By the discrete Gagliardo–Nirenberg inequality (3.29) with $p=6$, $q=2$,

$$
\begin{aligned}
|\widehat{v}|_{L^6_{\Delta x}} &\le 3^{1/3}|\widehat{v}|^{1/3}_{L^2_{\Delta x}}\|\widehat{v}\|^{2/3}_{H^1_{\Delta x}}\\
&= 3^{1/3}|v|^{1/3}_{L^2_{\Delta x}}|\widetilde{A}^{1/2}v|^{2/3}_{L^2_{\Delta x}}, \qquad (5.11)
\end{aligned}
$$

since (5.7) holds. So, after applying the generalized Cauchy–Schwarz inequality we may re–write (5.10) as

$$
\begin{aligned}
&-\mathrm{Re}\left\{(1+i\mu)\left\langle e^{t\widetilde{A}^{\frac{1}{2}}}G(|u|^2)u,\widetilde{A}v\right\rangle\right\}\\
&\le (1+\mu^2)^{1/2}\left\{\frac{1}{2\epsilon}|v|^2_{L^2_{\Delta x}}|\widetilde{A}^{1/2}v|^4_{L^2_{\Delta x}}+9\frac{\epsilon}{2}|\widetilde{A}v|^2_{L^2_{\Delta x}}\right\}. \qquad (5.12)
\end{aligned}
$$

Finally we return to the full equation (5.4), and substitute in (5.12) to get :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{A}^{\frac{1}{2}}v|^2_{L^2_{\Delta x}} &\le (|\widetilde{R}| + \frac{1}{2\epsilon})|\widetilde{A}^{\frac{1}{2}}v|^2_{L^2_{\Delta x}} + (\frac{\epsilon}{2} - 1)|\widetilde{A}v|^2_{L^2_{\Delta x}} \\ &\quad + (1+\mu^2)^{1/2}\left\{\frac{1}{2\epsilon}|v|^2_{L^2_{\Delta x}}|\widetilde{A}^{1/2}v|^4_{L^2_{\Delta x}} + 9\frac{\epsilon}{2}|\widetilde{A}v|^2_{L^2_{\Delta x}}\right\} \\ &\le (|\widetilde{R}| + \frac{1}{2\epsilon})|\widetilde{A}^{\frac{1}{2}}v|^2_{L^2_{\Delta x}} + (1+\mu^2)^{1/2}\frac{1}{2\epsilon}|\widetilde{A}^{1/2}v|^6_{L^2_{\Delta x}} \\ &\quad + \left(\frac{\epsilon}{2}(1 + 9(1+\mu^2)^{1/2}) - 1\right)|\widetilde{A}v|^2_{L^2_{\Delta x}}; \end{aligned} \tag{5.13}$$

where for the final inequality we have used that $|v|_{L^2_{\Delta x}} \le |\widetilde{A}^{1/2}v|_{L^2_{\Delta x}}$. We now make our choice of ϵ to dispose of the $|\widetilde{A}v|^2_{L^2_{\Delta x}}$ term :

$$\epsilon < \frac{2}{1 + 9(1+\mu^2)^{\frac{1}{2}}}.$$

Thus, there exist K_1, K_2 depending only on R, ν and μ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{A}^{\frac{1}{2}}v|^2_{L^2_{\Delta x}} &\le (|\widetilde{R}| + \frac{1}{2\epsilon})|\widetilde{A}^{\frac{1}{2}}v|^2_{L^2_{\Delta x}} + (1+\mu^2)^{1/2}\frac{1}{2\epsilon}|\widetilde{A}^{1/2}v|^6_{L^2_{\Delta x}} \\ &= K_1|\widetilde{A}^{\frac{1}{2}}v|^2_{L^2_{\Delta x}} + K_2|\widetilde{A}^{\frac{1}{2}}v|^6_{L^2_{\Delta x}}. \end{aligned} \tag{5.14}$$

Let $Y(t) = 1 + |\widetilde{A}^{\frac{1}{2}}v(t)|^2_{L^2_{\Delta x}}$ and let $Y^0 = Y(0)$. Then we can re-write (5.14) as

$$\frac{d}{dt}Y \le KY^3 \tag{5.15}$$

with

$$K := K(R,\nu,\mu) = \frac{2}{3}K_1 + 2K_2.$$

Thus

$$Y(t) \le \frac{Y^0}{\sqrt{1 - 2Kt(Y^0)^2}}, \quad \text{for} \quad 0 \le t < \frac{1}{2K(Y^0)^2}. \tag{5.16}$$

Defining $T(\rho)$ by

$$T(\rho) := \frac{3}{8K(1+\rho^2)^2}, \tag{5.17}$$

and noting that $T(\rho) < \frac{1}{2K(Y^0)^2}$, we find from (5.16) that

$$\begin{aligned}\|u(t)\|^2_{G_{t,\Delta x}} &\le \frac{Y^0}{\sqrt{1 - 3/4}} \\ &= 2(1+\rho^2). \end{aligned} \tag{5.18}$$

Hence part i) is proved with $\rho' := \sqrt{2(1+\rho^2)}$, since $Y^0 = 1 + \rho^2$.

To prove (ii), let t_1 and ρ_1 be as in Theorem 4.3 so that for $u_0 \in \mathcal{B}_1(\rho)$, $u(t) \in \mathcal{B}_1(\rho_1)$, $\forall t \geq t_1$. We let $T = T(\rho_1)$ be defined by (5.17). Then application of (i) on the overlapping intervals $[t_1+kT, t_1+(k+1)T]$, $[t_1+(k+\frac{1}{2})T, t_1+(k+\frac{3}{2})T]$ for each non-negative integer k gives the required result with $\tau = T/2$, $t_2 = t_1 + \frac{1}{2}T$ and $\rho_2 = \sqrt{2(1+\rho_1^2)}$. □

The previous theorem states that given any initial data $u^0 \in \mathcal{B}_1(\rho)$ we have a Gevrey ball $\mathcal{B}_{G_\tau}(\rho_2)$ which is absorbing. Since the global attractor of a finite dimensional system is the ω–limit set of any absorbing set this leads us to the following corollary.

Corollary 5.1 *Let τ and ρ_2 be as in Theorem 5.1 part ii) so that ρ_2 is independent of Δx. Then the global attractor $\mathcal{A}_{\Delta x}$ of Theorem 4.4 is given by*

$$\mathcal{A}_{\Delta x} = \omega(\mathcal{B}_{G_\tau}(\rho_2)). \tag{5.19}$$

6 Upper–semicontinuity of Attractors

In this section we prove upper–semicontinuity of the approximate global attractor $\mathcal{A}_{\Delta x}$ (whose existence and properties are established in Theorem 4.4 and Corollary 5.1) to the continuous global attractor $\mathcal{A}$. We examine upper–semicontinuity using the notion of semi–distance defined in Definition 2.1.

In the proof there are two different limiting processes to consider: the limit as $\Delta x \to 0$ and the limit as $t \to \infty$. Standard error estimates alone are not enough to prove a convergence result because they are of the form

$$||u_{\text{true}} - u_{\text{numerical}}|| \leq C_1 \Delta x^p e^{C_2 T} \quad \forall\, t \in (0,T], \tag{6.1}$$

and give no time asymptotic information. In essence we prove the result over any finite time interval by applying the standard type of estimate and use induction to extend to the infinite time interval. It is the attracting property which allows us to perform the induction step. This basic method of proof was introduced by Hale, Lin and Raugel [21].

However, for many evolution problems which are second order in space, $\mathcal{A}$ and $\mathcal{A}_{\Delta x}$ are constructed as the ω–limit set of balls in $H^1(\Omega)$; hence error estimate are required for initial data in that space – see Larsson [36] and Elliott and Larsson

[15] for examples in the context of finite element methods and [21] for other applications. However such error estimates can be complicated to derive for finite difference schemes. In this section we employ the discrete regularity established in section 5 to prove convergence of the attractor only using error estimates for smooth initial data.

Yin Yan [49] proves upper–semicontinuity of the global attractors for finite difference approximations to the Navier–Stokes equations. As far as we are aware this is the only other upper–semicontinuity result for finite difference approximations to partial differential equations. The result in [49] is proved by a piecewise linear interpolation at the grid points to set the analysis in the continuous space L^2; non–smooth data error estimates are then derived.

The following theorem bounds the error in the $H^1_{\Delta x}$ norm between continuous and discrete trajectories starting inside a Gevrey ball.

Theorem 6.1 *Assume that* $\exists\ \tau, \rho > 0$ *such that* $u^0 \in \mathcal{B}_{G_\tau}(\rho) \subset G_{\tau,\Delta x}$ *and that* $V^0 \in \mathcal{B}_{G_\sigma}(\pi\rho/2) \subset G_\sigma, \sigma \in (0, 2\tau/\pi)$, *is given by Lemma 3.9 so that* $P_{\Delta x}V^0 = u^0$. *Then, for any* $T > 0$, $\exists C = C(T, \tau) > 0$ *such that*

$$\|S_{\Delta x}(t)u^0 - P_{\Delta x}S(t)V^0\|_{H^1_{\Delta x}} \leq C\Delta x^2 \quad \forall\ t \in [0, T] \tag{6.2}$$

where $S_{\Delta x}$ *is the semi–group of Theorem 4.1 for the semi–discrete problem* (3.7) *and* $S(t)$ *is the semi–group of* **C1** *Theorem 2.1 for the continuous problem* (2.1).

Proof Using the Gevrey class of our initial data, we follow a standard smooth error analysis proof. The evolution $S_{\Delta x}(t)u^0 = u(t)$ satisfies the semi–discrete equation (3.8) and $v(t) := P_{\Delta x}S(t)V_0$ satisfies :

$$\frac{dv}{dt} = -(1 + i\nu)\tilde{A}v(t) + F(v(t)) + \eta(t) \tag{6.3}$$

where $\eta(t)$ is the truncation error.

Since $V(0) = V^0$ is in a Gevrey class G_σ by (2.12) we have that $V(t) \in H^\alpha$, $\forall\ t > 0$ and $\forall\ \alpha > 0$. For any interval I in one space dimension ($p = 1$), $C^4(I) \supset H^5(I)$ (see for example [31] or [39]) and hence we have the required regularity to apply Taylor's theorem and the mean value theorem to estimate the truncation error $\eta(t)$ by

$$\eta(t) = \frac{\Delta x^2}{12}\left(\frac{\partial^4 V}{\partial x^4}\Big|_{\theta_0}, \frac{\partial^4 V}{\partial x^4}\Big|_{\theta_1+\Delta x}, \dots, \frac{\partial^4 V}{\partial x^4}\Big|_{\theta_{J-1}+(J-1)\Delta x}\right)$$

where $\theta_i \in (0, \Delta x)$, $\Delta x < 1$ and for any $g \in C([0,1], \mathbb{R})$, $g|_\theta = f(\theta)$. Therefore,

$$|\eta(t)|_{L^2_{\Delta x}} \leq C_0(T)\Delta x^2 \quad \forall t \in (0, T). \tag{6.4}$$

Now let the error $e(t) := v(t) - u(t)$. Then $e(t)$ satisfies :-

$$\frac{de}{dt} = -(1 + i\nu)\widetilde{A}e + F(v) - F(u) + \eta(t). \tag{6.5}$$

Let $E_{\Delta x}(t)$ denote the linear semi–group generated by $(1 + i\nu)\widetilde{A}$. Then, by Duhamel's principle, equation (6.5) becomes

$$e(t) = E_{\Delta x}(t)e(0) + \int_0^t E_{\Delta x}(t-s)\left(F(v(s)) - F(u(s))\right) ds + \int_0^t E_{\Delta x}(t-s)\eta(t)\, ds. \tag{6.6}$$

Taking the $H^1_{\Delta x}$ norm of (6.6), using the smoothing property of the linear semi–group $E(t)$ (see Lemma 3.10) and that $e(0) = 0$ we get

$$\|e(t)\|_{H^1_{\Delta x}} \leq \int_0^t C(t-s)^{-\frac{1}{2}} |F(v(s)) - F(u(s))|_{L^2_{\Delta x}} ds + \int_0^t (t-s)^{-\frac{1}{2}} |\eta(t)|_{L^2_{\Delta x}} ds. \tag{6.7}$$

By the regularity of the initial data, both $u(t)$ and $v(t)$ lie in $H^1_{\Delta x}$ for all $t \geq 0$ and hence, by Lemma 3.5, we have uniform bounds on the $L^\infty_{\Delta x}$ norms of these quantities. Thus, by Lemma 3.11 we find

$$\begin{aligned} |F(u(t)) - F(v(t))|_{L^2_{\Delta x}} &\leq \left(|\widetilde{R}| + (1+\mu^2)^{\frac{1}{2}}(|u(t)|^2_{L^\infty_{\Delta x}} + |v(t)|^2_{L^\infty_{\Delta x}})\right) |u(t) - v(t)|^2_{L^2_{\Delta x}} \\ &\leq C_2(T)\|u - v\|^2_{H^1_{\Delta x}}, \quad \forall\, t \in (0, T). \end{aligned} \tag{6.8}$$

Therefore (6.7) becomes

$$\begin{aligned} \|e(t)\|_{H^1_{\Delta x}} &\leq C_1 \int_0^t (t-s)^{-\frac{1}{2}} \|e(s)\|_{H^1_{\Delta x}} ds + C_0 \Delta x^2 \int_0^t (t-s)^{-\frac{1}{2}} ds \\ &\leq C_1 \int_0^t (t-s)^{-\frac{1}{2}} \|e(s)\|_{H^1_{\Delta x}} ds + C_0 \Delta x^2 t^{1/2}. \end{aligned} \tag{6.9}$$

By an application of the version of Gronwall's lemma in [24, Lemmas 6.3 and 7.1] we obtain

$$\|e(t)\|_{H^1_{\Delta x}} \leq C(T)\Delta x^2 \quad \forall t \in (0, T]. \quad \square \tag{6.10}$$

We are now in a position to prove our main theorem of this section on the upper–semicontinuity of the attractors.

Theorem 6.2 (Upper–Semicontinuity) *Let $\mathcal{A}$ denote the global attractor for the semi-group $S(t)$ and $\mathcal{A}_{\Delta x}$ denote the global attractor for the semi-group $S_{\Delta x}(t)$ with $0 < \Delta x < 1$. Then*

$$dist_{H^1_{\Delta x}}(\mathcal{A}_{\Delta x}, P_{\Delta x}\mathcal{A}) \rightarrow 0 \text{ as } \quad \Delta x \rightarrow 0. \tag{6.11}$$

Proof Recall the result of Corollary 5.1, namely there exists ρ_2 independent of Δx and $\tau > 0$ such that

$$\mathcal{A}_{\Delta x} = \omega\left(\mathcal{B}_{G_\tau}(\rho_2)\right) = \bigcap_{s\geq 0} \overline{\cup_{t\geq s} S_{\Delta x}(t)\mathcal{B}_{G_\tau}(\rho_2)}. \tag{6.12}$$

Thus it is sufficient to prove that $\forall \epsilon > 0$, $\exists\ \Delta x_0$, T^* such that $\forall\ \Delta x < \Delta x_0$,

$$S_{\Delta x}(t)\mathcal{B}_{G_\tau}(\rho_2)\ \in\ \mathcal{N}(P_{\Delta x}\mathcal{A}, \epsilon)\ \ \forall\, t > T^*. \tag{6.13}$$

If we can prove that (6.13) holds, then by (6.12) we have that

$$\mathcal{A}_{\Delta x}\ \in\ \mathcal{N}(P_{\Delta x}\mathcal{A}, \epsilon) \tag{6.14}$$

and the theorem is proved.

We now proceed to prove (6.13) by induction. Let $\epsilon > 0$ be given.
Note that by Lemma 3.9 for every $u^0 \in \mathcal{B}_{G_\tau}(\rho_2) \subset G_{\tau,\Delta x}$ there exists $\sigma > 0$ and $V^0\ \in \mathcal{B}_{G_\sigma} \subset G_\sigma$ such that $P_{\Delta x}V^0 = u^0$. By the attracting property of $\mathcal{A}$, Theorem 4.4, $\exists\ T = T(\epsilon, \rho_2) > 0$ such that

$$\text{dist}_{H^1}\left(S(t)V(0), \mathcal{A}\right)\ \leq\ \frac{\epsilon}{2\kappa},\ \ \forall\, t \geq T$$

where κ is the constant from Theorem 3.1 on norm equivalence. Then by Theorem 3.1 we have that

$$\text{dist}_{H^1_{\Delta x}}\left(P_{\Delta x}S(t)V(0), P_{\Delta x}\mathcal{A}\right)\ \leq\ \frac{\epsilon}{2},\ \ \forall\, t \geq T. \tag{6.15}$$

Furthermore, by the error estimate of Theorem 6.1, for all $u^0 \in \mathcal{B}_{G_\tau}(\rho_2) \subset G_{\tau,\Delta x}$

$$\|S_{\Delta x}(t)u(0) - P_{\Delta x}S(t)V(0)\|_{H^1_{\Delta x}} \leq\ \frac{\epsilon}{2},\quad \forall\ t\ \in (0, 2T] \tag{6.16}$$

provided $\Delta x^2 \leq \Delta x_0^2 := \frac{\epsilon}{2C(2T)}$.

Combining the two properties (6.15) and (6.16) we get that $\forall u^0 \in \mathcal{B}_{G_\tau}(\rho_2)$,

$$\begin{aligned}
&\text{dist}_{H^1_{\Delta x}}(S_{\Delta x}(t)u^0, P_{\Delta x}\mathcal{A}) \\
&= \inf_{U\in\mathcal{A}} \|S_{\Delta x}(t)u^0 - P_{\Delta x}U\|_{H^1_{\Delta x}} \\
&\leq \inf_{U\in\mathcal{A}} \|P_{\Delta x}S(t)V^0 - P_{\Delta x}U\|_{H^1_{\Delta x}} + \|S_{\Delta x}(t)u^0 - P_{\Delta x}S(t)V^0\|_{H^1_{\Delta x}} \\
&= \text{dist}_{H^1_{\Delta x}}(P_{\Delta x}S(t)V^0, P_{\Delta x}\mathcal{A}) + \|S_{\Delta x}(t)u^0 - P_{\Delta x}S(t)V^0\|_{H^1_{\Delta x}} \\
&\leq \epsilon/2 + \epsilon/2 \qquad \forall t \in [T, 2T].
\end{aligned}$$

and so

$$u(t) \in \mathcal{N}(\mathcal{A}, \epsilon) \quad \forall\, t \in [T, 2T].$$

To complete the inductive step note that

$$S_{\Delta x}(t)\mathcal{B}_{G_\tau}(\rho_2) \subset \mathcal{B}_{G_\tau}(\rho_2), \;\; \forall t > 0 \tag{6.17}$$

and so in particular $S_{\Delta x}(T)\mathcal{B}_{G_\tau}(\rho_2) \subset \mathcal{B}_{G_\tau}(\rho_2)$.

Thus the above argument may be repeated for $t \in [2T, 3T]$, yielding $\forall u^0 \in \mathcal{B}_{G_\tau}(\rho_2)$

$$u(t) \in \mathcal{N}(\mathcal{A}, \epsilon)\; \forall t \in [T, 3T].$$

By property (6.17) we may repeat the argument again for the intervals $[3T, 4T]$, $[4T, 5T] \cdots$. Hence (6.13) holds by induction and the theorem is proved.□

7 Conclusion

In summary we have shown the existence of a semi–discrete global attractor $\mathcal{A}_{\Delta x}$ (Theorem 4.3) and proved convergence in the sense of upper–semicontinuity to the true global attractor $\mathcal{A}$ (Theorem 6.2). This result is obtained by using the Gevrey regularity of the true solution and proving a discrete Gevrey regularity result for the semi–discrete approximation (Theorem 5.1). The approaches used in this paper should be applicable to other dissipative partial differential equations with solutions of Gevery class.

Note that a further discretization in time is required to obtain a numerical scheme which can be implemented on a computer. The existence of a global attractor in these cases is considered in [37] for the complex Ginzburg–Landau equation. The results here and the results in [37], combined with the results of either [28] or [25] to incorporate the effect of time-discretization, yield understanding about the manner in which long–time finite difference simulations of the complex Ginzburg–Landau equation should be interpreted.

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