Reading group on Dirichlet Polynomials   Week 5   Large Values of $D(it)$

Notation:

- $D(s) = \sum_{n=1}^{N} a_n n^{-s}$, $a_n \in \mathbb{C}$
- $\|\underline{a}\|_2^2 = \sum_{n=1}^{N} |a_n|^2$ (Joseph notation?)
- S = finite set of points $t_1 < \dots < t_r$ in the interval $[0,T]$, $T \geq 1$
- $R = \# S$

★ $\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_m \frac{\mu(m)}{m^s}$, $Re(s) > 1$

∴ the partial sum $\sum_{m=1}^{M} \frac{\mu(m)}{m^s} \approx \frac{1}{\zeta(s)}$ if $Re(s) > \frac{1}{2}$

⇒ good for detecting zeros.

($\mu(n) = (-1)^k$ for n = prod. of k distinct prime factors, 0 otw) or $\mu(1) = 1$

⓪ Last time / Recap

2 different discrete mean-value estimates were given:

Theorem

Let S be a set of well-spaced points ($|t_i - t_j| < 1 \Leftrightarrow t_i = t_j$) in $[0,T]$, $T \geq 1$.

Let a_n be any complex numbers, then

This is weak for "smaller sets" since it is independent on R, the size of S.

(I) $\sum_{t \in S} |D(it)|^2 \ll (T+N) \|\underline{a}\|_2^2 \cdot \log(2N)$ (Iwaniec, Kowalski book, Theorem 9.4)
(proof: comparing with L^2 estimates)

improve

(II) $\sum_{t \in S} |D(it)|^2 \ll (N + RT^{\frac{1}{2}}) \|\underline{a}\|_2^2 \cdot \log(2T)$ (Montgomery, Theorem 8)
if $R \ll T^{\frac{1}{2}}$, $N \ll RT^{\frac{1}{2}}$
(proof: using operator theory & C-S ineq.)

Goal: find an estimate/bound for large values of a Dirichlet polynomial at a given set of well-spaced points.

① Non-conditional estimates

Let S be a finite set of points in $[0,T]$ s.t. for all $t \in S$, $|D(it)| \geq V$

Estimate 1 using (I) (Markov Ineq. idea)

$$R = \#\{t \in [0,T] : |D(it)| \geq V\} \ll \frac{(T+N)\|\underline{a}\|_2^2}{V^2} \log(2N) \quad (\dagger), \quad V > 0$$

Estimate 2 using (II) (Montgomery)

If $V \geq \|\underline{a}\|_2 T^{\frac{1}{4}} \log(2T)$, then $R \ll \frac{\|\underline{a}\|_2^2 N}{V^2} \log(2T)$

Proof

If $T \ll N$, this is "trivial": estimate 1 tells us

$$\frac{(T+N)\|\underline{a}\|_2^2}{V^2} \log(2N) \ll \frac{N\|\underline{a}\|_2^2}{V^2} \log(2N)$$

↑ of the form we want

($\dagger$): trivial bound if $V \ll \|\underline{a}\|_2 \log 2N$ (any well-spaced set has $\lesssim T+1$ points)

If $T \gg N$, Substitute $|D(it)|^2 \geq V^2$ into (I) gives

$$RV^2 \ll \|\underline{a}\|_2^2 (N + RT^{\frac{1}{2}}) \log 2T$$
$$\Rightarrow R \overset{(*)}{\ll} V^{-2} \|\underline{a}\|_2^2 (T + RT^{\frac{1}{2}}) \log 2T$$
$$\ll \frac{\|\underline{a}\|_2^2 T}{V^2} \log 2T \quad \text{since } R \ll T^{\frac{1}{2}} \Rightarrow RT^{\frac{1}{2}} \ll T$$

$(*)$ holds only if $V \geq \|\underline{a}\|_2 T^{\frac{1}{4}} \log(2T)$ $\left(\frac{\|\underline{a}\|_2^2 RT^{\frac{1}{2}} \log 2T}{\|\underline{a}\|_2^2 T^{\frac{1}{2}} \log^2 2T} \leq \frac{R}{\log 2T}, \text{ good enough} \right)$ □

$\boxed{\text{Estimate 3}}$ ("Improvement" by removing the restriction on V) (Huxley)

For $V > 0$, $R \ll \{ \|\underline{a}\|_2^2 N V^{-2} + \|\underline{a}\|_2^6 N T V^{-6} \} (\log 2T)^6$

proof: Subdivision method (will see this again in Montgomery Theorem 9)

First assume $V \geq \|\underline{a}\|_2 \log(2T)$ (same reason as in Estimate 1), otw trivial.

Now let T_0 be determined by $V \geq \|\underline{a}\|_2 T_0^{\frac{1}{4}} \log(2T_0)$, since $1 \leq T_0 \leq T$,

$$T_0 = \min\{ T, \|\underline{a}\|_2^{-4} V^4 (\log 2T)^{-4} \}$$

★ Key trick: divide $[0, T]$ into smaller intervals that satisfy $V \geq \|\underline{a}\|_2 T^{\frac{1}{4}} \log(2T)$

so Estimate 2 applicable

Let S_ℓ be the subset of points of S in the interval $[\ell T, (\ell+1) T_0]$, $0 \leq \ell \leq \frac{T}{T_0}$

(so $\ll \frac{T}{T_0}$ intervals of length T_0)

Then each subinterval, Estimate 2 is applicable so

$$R = \sum_\ell R_\ell \ll \left(\frac{T}{T_0} \right) \frac{\|\underline{a}\|_2^2 N}{V^2} \cdot \log 2T$$

$T_0 = T$ gives the first main term, $T_0 = \|\underline{a}\|_2^{-4} V^4 (\log 2T)^{-4}$ gives the second. □

② Conditional Estimates by Montgomery

Montgomery: the bound (II), $\sum_{t \in S} |D(it)|^2 \ll (N + RT^{\frac{1}{2}}) \log T \cdot \|\underline{a}\|_2^2$ is perhaps far from the truth, since the bound for $Z(it) = \sum_{n=1}^{N} n^{-it}$ isn't good enough. $(Z(it) \ll \frac{N}{t} + \sqrt{t} \log t)$

$(*)$ use Lindelof Hypothesis ($Z(it) \ll \frac{N}{t} + \sqrt{N} t^{\epsilon}$ for $t > t_0(\epsilon)$) will give

$N \ll T$, so this is better

$$\sum_{t \in S} |D(it)|^2 \ll (N + R\underline{\underline{N^{\frac{1}{2}}}}) T^{\epsilon} \|\underline{a}\|_2^2$$

He wanted to remove this term

But there's a counterexample by Bourgain.

Further conjectured the following:

$\boxed{\text{Conjecture}}$

If $|a_n| \leq 1$ for all n (so $\|a_n\|_\infty \leq 1$), then

$$\sum_{t \in S} |D(it)| \ll (N + R) N T^{\epsilon}$$

$(*)$: see last page

$\boxed{\text{Estimate 4}}$ (Montgomery Theorem 9)

Using Montgomery's conjecture (so assume $\|a\|_\infty \leq 1$ as well), then

$$R \ll N^2 V^{-2} T^{\varepsilon} \quad \text{if } V \geq N^{\frac{1}{2}} T^{\varepsilon}$$

(Proof) Same argument as before

Conjecture tells us $RV^2 \ll (N+R) N T^{\varepsilon}$

$$\text{So } R \ll N^2 V^{-2} T^{\varepsilon} + \frac{RNT^{\varepsilon}}{V^2} \ll N^2 V^{-2} T^{\varepsilon} \text{ if } V \geq N^{\frac{1}{2}} T^{\varepsilon}$$

$\square$

Actually, the original conjecture was

$$\sum_{t \in S} |D(it)|^2 \ll (N+R) T^{\varepsilon} \sum_{n=1}^{N} |a_n|^2$$

Due to Bourgain

Counterexample: $T^{\frac{1}{2}} < N < T$, $H = [\frac{CN}{\sqrt{T}}]$, $a_n = \begin{cases} 1 & N < n \leq N+H \\ 0, & \text{otw} \end{cases}$

$$J = [\tfrac{T}{2\pi N}], \quad K = [\sqrt{T}]$$

Assume $t = 2\pi j N - k$ where $j, k \in \mathbb{Z}$, $1 \leq j \leq J$, $1 \leq k \leq K$, so $0 \leq t \leq T$
is an integer

$$\begin{aligned}
t \log n &= t \log N + t \log\left(1 + \frac{n-N}{N}\right) \\
&= t \log N + \underline{\frac{t(n-N)}{N} + O\left(\frac{TH^2}{N^2}\right)} \quad \text{taylor expand } \log(1+x) \\
&= t \log N + \underbrace{2\pi j (n-N) - \frac{k(n-N)}{N}}_{(n-N)(2\pi j - \frac{k}{N}) = \frac{t}{N}(n-N)} + O\left(\frac{TH^2}{N^2}\right) \\
&= t \log N + 2\pi j (n-N) + O(c), \quad 0 < c < 1
\end{aligned}$$

Hence, for such a t, $\left| \sum_{n=N+1}^{N+H} n^{-it} \right| \gg H$ (H terms there, for trivial upper bound) if c small

$\#$ of such $t = JK \gg \underline{\frac{T^{\frac{3}{2}}}{N}}$, so

$$\sum_{r=1}^{R} \left| \sum_{n=N+1}^{N+H} n^{-it_r} \right|^2 \gg \frac{T^{\frac{3}{2}}}{N} \cdot H \gg NT^{\frac{1}{2}} \text{ --- } (\dagger\dagger)$$

But $\sum |a_n|^2 = H \ll \frac{N}{\sqrt{T}}$

Now substitute $R \gg \frac{T^{\frac{3}{2}}}{N}$, $\|a\|_2^2 \ll \frac{N}{\sqrt{T}}$ into the wrong conjecture,
then if N is slightly bigger than $T^{\frac{1}{2}}$, $NT^{\frac{1}{2}}$ (in $(\dagger\dagger)$) will dominate

(*): In the proof of (II): $\sum_{t\in S} |D(it)|^2 \ll (N+RT^{\frac{1}{2}})\,\|\underline{a}\|_2^2 \log(2T)$, $R \ll T^{\frac{1}{2}}$, $N \ll RT^{\frac{1}{2}}$,

we said $\mathrm{LHS} \overset{\text{operator}}{\underset{\text{theory}}{=}} N\|\underline{y}\|_2^2 + \sum_{\substack{r_1,r_2\le R \\ r_1\ne r_2}} y_{r_1}\overline{y_{r_2}}\, Z(itr_1 - itr_2)$

where $Z(s) = \sum_{n\le N} n^{-s}$

we could use Lindelöf Hypothesis

$$\le N\|\underline{y}\|_2^2 + \boxed{\sup_{r_1\le R} \sum_{\substack{r_2\le R \\ r_1\ne r_2}} |Z(itr_1 - itr_2)|} \cdot \|\underline{y}\|_2^2 \quad \text{(C-S ineq.)}$$

this bilinear form should be smaller (conjectured/expected)